Infinite Symplectic Groups over Rings

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§0. Introduction

Let A be a commutative ring, M a free A-module $A^{(I)}$, for some infinite set I, and $R = \mathrm{End}_A(M)$. In a previous paper [4], we proved that normal subgroups of the group $U(R)$ of units of R must lie in congruence layers determined by the ideals of A. We now suppose that M possesses a nondegenerate alternate bilinear form $(\cdot, \cdot)$ and prove a similar result for the infinite symplectic group

$$\mathrm{Sp}(R) = \{u \in U(R) \mid (u(x), u(y)) = (x, y) \quad \text{for all} \quad x, y \in M\},$$

at least when $\frac{1}{2} \in A$ and the form $(\cdot, \cdot)$ is "locally hyperbolic". The strategy of the proof is again the one mapped out by Bass in [2] and [3]. When A is a field, our results coincide with those of Spiegel [6]. One should also note that Bak [1], Vaserstein [7] and Vaserstein and Mihalev [8] have recently studied the orthogonal analogue of Bass' results in the "stable" finite case.

§1. Locally Hyperbolic forms

A submodule N of M is called hyperbolic if $M = N \oplus N$ and $N = N_1 \oplus N_2$, where N_1 and N_2 are totally isotropic and have bases $\{e_j\}_{j \in J}$ and $\{e^j\}_{j \in J}$ such that $(e_j, e^j) = 1$ for all $j \in J$. The basis $\{e_j, e^j\}_{j \in J}$ is then called a hyperbolic basis of N. The form $(\cdot, \cdot)$ is called locally hyperbolic if every finitely generated submodule of M is contained in a hyperbolic submodule. When A is a field, this condition is automatically satisfied. In general, it may be satisfied by assuming a priori the existence of a hyperbolic basis for all of M.

(1.1.) *Remark.* If $(\cdot, \cdot)$ is locally hyperbolic, then for all unimodular $x \in M$ there exists a unimodular $y \in M$ such that $(y, x) = 1$. For suppose N is a hyperbolic submodule of M containing x and $\{e_j, e^j\}_{j \in J}$ is a hyperbolic basis of N. If $x = \sum (e_j a_j + e^j a^j)$, there must exist a relation $\sum (b_j a_j + b^j a^j) = 1$ since x is unimodular. It suffices to take $y = \sum (e_j b^j - e^j b_j)$.

If $\mathfrak{q}$ is an ideal of A, the form $(\cdot, \cdot)$ induces, as usual, an alternate bilinear form $(\cdot, \cdot)_\mathfrak{q}$ on the free $A/\mathfrak{q}$-module $M \otimes_A A/\mathfrak{q} \cong (A/\mathfrak{q})^{(I)}$ which, in general, need not be nondegenerate. However, if $(\cdot, \cdot)$ is locally hyperbolic, then $(\cdot, \cdot)_\mathfrak{q}$ is clearly locally hyperbolic and is furthermore nondegenerate. For suppose $x \otimes 1 \in M \otimes_A A/\mathfrak{q}$ is such that $(x \otimes 1, y \otimes 1)_\mathfrak{q} = 0$ for all $y \in M$. Let N be a hyperbolic submodule of M containing

x and suppose $\{e_j, e^j\}_{j\in J}$ is a hyperbolic basis of N. If $x=\sum(e_j a_j + e^j a^j)$, we have $(x, e^j)=a_j$ and $(x, e_j)=-a^j$ so that all a_j and a^j must be in $\mathbf{q}$ i.e. $x\otimes 1=0$.

We have $\mathrm{End}_{A/\mathbf{q}}(M\otimes_A A/\mathbf{q})\cong \mathrm{End}_{A/\mathbf{q}}((A/\mathbf{q})^{(I')})\cong R/(\mathbf{q})$, where $(\mathbf{q})=\{u\in U(R) \mid u(M)\subset M\cdot\mathbf{q}\}$ is an ideal of R. The projection $R\to R/(\mathbf{q})$ induces a homomorphism $U(R)\to U(R/(\mathbf{q}))$ and, if we regard $M\otimes_A A/\mathbf{q}$ as being equipped with the form $(\cdot,\cdot)_{\mathbf{q}}$, a homomorphism $\mathrm{Sp}(R)\to \mathrm{Sp}(R/(\mathbf{q}))$. The kernel of this homomorphism is denoted by $\mathrm{Sp}(\mathbf{q})$ and the inverse image of the center of $\mathrm{Sp}(R/\mathbf{q})$ by $\mathrm{Sp}'(\mathbf{q})$.

§2. Preliminary Results

From now on, the form $(\cdot,\cdot)$ is assumed to be locally hyperbolic. For every unimodular $x\in M$ and every $a\in A$, the mapping $\tau(a, x)(m)=m+x\cdot a(m, x)$ belongs to $\mathrm{Sp}(R)$ and is called a transvection. The subgroup generated by all transvections is denoted by $\mathrm{ESp}(R)$. If $\mathbf{q}$ is an ideal of A and $a\in\mathbf{q}$, $\tau(a, x)$ is called a $\mathbf{q}$-transvection: the subgroup generated by all $\mathbf{q}$-transvections is denoted by $\mathrm{ESp}(\mathbf{q})$. If $\sigma\in\mathrm{Sp}(R)$, the formula

$$\sigma\tau(a, x)\,\sigma^{-1}=\tau(a, \sigma(x)) \tag{2.1}$$

shows that $\mathrm{ESp}(\mathbf{q})$, and in particular $\mathrm{ESp}(\mathbf{R})$, is a normal subgroup of $\mathrm{Sp}(R)$ for all ideals $\mathbf{q}$. It is clear that $\mathrm{ESp}(\mathbf{q})=\mathrm{ESp}(\mathbf{q}')$ only if $\mathbf{q}=\mathbf{q}'$.

(2.2) PROPOSITION. *The orbits of* $\mathrm{ESp}(\mathbf{q})$ *operating on the unimodular elements of* M *are the congruence classes* $\bmod M\cdot\mathbf{q}$. *In particular,* $\mathrm{ESp}(R)$ *operates transitively.*

Proof. Suppose x and y are unimodular elements of M congruent $\bmod M\cdot\mathbf{q}$. Since $(\cdot,\cdot)$ is locally hyperbolic, there exists a hyperbolic submodule N containing both x and y. Let $\{e_j, e^j\}_{j\in J}$ be a hyperbolic basis of N. It is sufficient to show that a fixed $e_i\in N$ can be mapped into any unimodular element $z\equiv e_i \bmod M\cdot\mathbf{q}$ by an element of $\mathrm{ESp}(\mathbf{q})$. For then, applying this with $\mathbf{q}=A$, we first see that $\beta(x)=e_i$ for some $\beta\in\mathrm{ESp}(R)$. Since $\beta(y)\equiv e_i \bmod M\cdot\mathbf{q}$, the same argument shows that $\gamma(e_i)=\beta(y)$ for some $\gamma\in\mathrm{ESp}(\mathbf{q})$. Therefore $\beta^{-1}\gamma\beta(x)=y$ and $\beta^{-1}\gamma\beta\in\mathrm{ESp}(\mathbf{q})$.

By enlarging N if necessary, we may assume that for a certain index $k\in J (k\neq i)$, both e_k and e^k occur with coefficient zero in z. Suppose

$$z=e_i(1+q_i)+e^i q^i+\sum_{j\neq i}(e_j q_j+e^j q^j).$$

Since z is unimodular, there exists a relation

$$a_i(1+q_i)+a^i q^i+\sum_{j\neq i}(a_j q_j+a^j q^j)=1.$$

Let

$$\alpha_j = \tau(-q_j - q^j - q_jq^j, e^i)\,\tau(-q^j, e^j)\,\tau(q^j, e^i + e^j)\,\tau(q_j, e^i + e_j)$$
$$\beta = \tau(-q_i, e^k - e_i + e^i - e_k)\,\tau(q_i, e^k - e_i)\,\tau(q_i(1 + q^i), e^i - e_k)\,\tau(q^i, e^i)$$
$$\gamma_j = \tau(q_ia_j, e^j + e^k)\,\tau(-q_ia_j, e^j)\,\tau(-q_ia^j, e_j + e^k)\,\tau(q_ia^j, e_j)$$
$$\delta = \Big(\prod_{j\neq k}\gamma_j\Big)\,\beta\,\Big(\prod_{j\neq i}\alpha_j\Big)$$

Then $\delta(e_i)=z$ since $\prod_{j\neq i}\alpha_j$ adds $\sum_{j\neq i}(e_jq_j+e^jq^j)$ to e_i, β adds $e_iq_i+e^iq^i$ at the expense of subtracting e^kq_i and $\prod_{j\neq k}\gamma_j$ removes the e^kq_i. ‖

(2.3) COROLLARY. *The natural homomorphism* $\mathrm{ESp}(R)\to\mathrm{ESp}(R/(\mathfrak{q}))$ *is surjective.*

Proof. Let $\tau(\bar a, \bar x)$ be a transvection in $\mathrm{ESp}(R/(\mathfrak{q}))$: $\bar x$ is unimodular in $M\otimes_A A/\mathfrak{q}$, but x need not be unimodular in M. Suppose N is a hyperbolic submodule of M containing x with a hyperbolic basis $\{e_j, e^j\}_{j\in J}$. Applying (2.2) to $M\otimes_A A/\mathfrak{q}$, we see that $\bar x=\bar\delta(\bar e_i)$ for some $i\in J$ and $\bar\delta$ constructed as above; hence $\tau(\bar a, \bar x)=\bar\delta\tau(\bar a, \bar e_i)\bar\delta^{-1}$. However, each of the unimodular elements of $M\otimes_A A/\mathfrak{q}$ occurring in the transvections composing $\bar\delta$ clearly comes from a unimodular element of M. ‖

(2.4) PROPOSITION.

$$\mathrm{ESp}(\mathfrak{q}) = [\mathrm{ESp}(R), \mathrm{ESp}(\mathfrak{q})].$$

Proof. In view of (2.2) and (2.1), it is sufficient to prove that all $\mathfrak{q}$-transvections $\tau(a, x)$ for some particular unimodular $x\in M$ are in $[\mathrm{ESp}(R), \mathrm{ESp}(\mathfrak{q})]$. Choose a hyperbolic submodule N with a hyperbolic basis $\{e_i, e^i\}_{1\leqslant i\leqslant 3}$. The easily verified identity

$$\tau(-a, e_1+e_2+e_3)\,\tau(a, e_1+e_2)\,\tau(a, e_1+e_3)\,\tau(a, e_2+e_3)\,\tau(-a, e_1)\,.$$
$$\tau(-a, e_2)\,\tau(-a, e_3) = 1$$

can be written in the form

$$\tau(a, e_1) = [\beta, \tau(-a, e_2+e_3)\,\tau(a, e_3)]\,[\gamma, \tau(a, e_2)],$$

where $\beta=\tau(-1, e^3)\,\tau(1, e^3+e_1)$ and $\gamma=\tau(-1, e^2)\,\tau(1, e^2+e_1)$ are in $\mathrm{ESp}(R)$ and have the effect, respectively, of sending e_3 to e_3+e_1 and e_2 to e_2+e_1. ‖

(2.5) PROPOSITION.

$$[\mathrm{ESp}(R), \mathrm{Sp}'(\mathfrak{q})] = \mathrm{ESp}(\mathfrak{q}).$$

Proof. We first show that $[\mathrm{ESp}(R), \mathrm{Sp}(\mathfrak{q})] \subset \mathrm{ESp}(\mathfrak{q})$. If $\tau(a, x)$ is any transvection and $\sigma \in \mathrm{Sp}(\mathfrak{q})$, then by (2.2) $\sigma(x)=\beta(x)$ for some $\beta \in \mathrm{ESp}(\mathfrak{q})$. Hence $[\tau(a, x), \sigma] = \tau(a, x)\, \tau(-a, \sigma(x)) = [\tau(a, x), \beta] \in \mathrm{ESp}(\mathfrak{q})$. Reducing mod $\mathfrak{q}$, we see that $[\mathrm{ESp}(R), \mathrm{Sp}'(\mathfrak{q})] \subset \mathrm{Sp}(\mathfrak{q})$; therefore $[\mathrm{ESp}(R), [\mathrm{ESp}(R), \mathrm{Sp}'(\mathfrak{q})]] \subset \mathrm{ESp}(\mathfrak{q})$. The "3-subgroups" lemma [5, p. 59] now implies that $[[\mathrm{ESp}(R), \mathrm{ESp}(R)], \mathrm{Sp}'(\mathfrak{q})] \subset \mathrm{ESp}(\mathfrak{q})$. However, $[\mathrm{ESp}(R), \mathrm{ESp}(R)] = \mathrm{ESp}(R)$ by (2.4) so that $[\mathrm{ESp}(R), \mathrm{Sp}'(\mathfrak{q})] \subset \mathrm{ESp}(\mathfrak{q})$; the opposite inclusion follows from (2.4). $\|$

§3. The Main Theorem

From now on, we assume that $\frac{1}{2} \in A$. In the following propositions, G is a subgroup of $\mathrm{Sp}(R)$ normalised by $\mathrm{ESp}(R)$.

(3.1) PROPOSITION. *If $(x, \sigma(x))=0$ for all $\sigma \in G$ and all unimodular $x \in M$, then G is contained in the center of* $\mathrm{Sp}(R)$.

Proof. Linearising the identity $(x, \sigma(x))=0$, we conclude that if x, y and $x+y$ are all unimodular, then $(x, \sigma(y)) + ((y, \sigma(x))=0$. Since every $x \in M$ can be written in the form $\sum e_i a_i$ for some basis $(e_i)_{i \in I}$ of M, we conclude that $(x, \sigma(x)) = \sum (e_i, \sigma(e_i)) a_i^2 + \sum_{i \neq j} (e_i, \sigma(e_j)) + (e_j, \sigma(e_i)) a_i a_j = 0$ for all $x \in M$.

Therefore, for all $x, y \in M$, we have $(x, \sigma(y)) = -(y, \sigma(x)) = (\sigma(x), y) = (x, \sigma^{-1}(y))$. Since $(\cdot, \cdot)$ is nondegenerate, we conclude that $\sigma = \sigma^{-1}$ for all $\sigma \in G$, i. e. G is an abelian group consisting of involutions.

If $\sigma \in G$ and $x \in M$ is unimodular, $[\sigma, \tau(1, x)] = \tau(1, \sigma(x))\, \tau(-1, x) \in G$ and is therefore an involution. Moreover, $\tau(1, \sigma(x))$ commutes with $\tau(-1, x)$ since $(x, \sigma(x))=0$. We conclude that $\tau(2, \sigma(x)) = \tau(2, x)$ i. e. $2(y, \sigma(x))\, \sigma(x) = 2(y, x)\, x$ for all $y \in M$. In view of (1.1), we can choose y such that $(y, \sigma(x))=1$; since $\frac{1}{2} \in A$, it follows that $\sigma(x) = x a_x$ for some $a_x \in A$. If $(e_i)_{i \in I}$ is a basis of M and $\sigma(e_i) = e_i a_i$, then $\sigma(e_i + e_j) = (e_i + e_j) a_{ij} = e_i a_i + e_j a_j$, so that $a_i = a_{ij} = a_j$ for $i \neq j$. Hence a_x is independent of x and σ is in the center. $\|$

(3.2) PROPOSITION. *If G is not contained in the center of* $\mathrm{Sp}(R)$, *then G contains a transvection $\tau \neq 1$.*

Proof. By (3.1), $(x, \sigma(x)) = a \neq 0$ for some $\sigma \in G$ and some unimodular $x \in M$. Then $\sigma_1 = [\sigma, \tau(1, x)] = \tau(1, \sigma(x)).\ \tau(-1, x) \in G$. Let N be a hyperbolic submodule of M containing both x and $\sigma(x)$; suppose $\{e_j, e^j\}_{j \in J}$ is a hyperbolic basis of N. Enlarging N if necessary, we may assume that for some $k \in J$, both e_k and e^k occur with zero coefficient in x and $\sigma(x)$. Then G contains $\sigma_2 = [\tau(-1, \sigma(x))\, \tau(1, e_k + \sigma(x)), \sigma_1] = \tau(1, \sigma(x))\, \tau(-1, x + e_k a)\, \tau(1, x)\, \tau(-1, \sigma(x))$ and hence $\sigma_3 = \sigma_1^{-1} \sigma_2 \sigma_1 = \tau(1, x) \cdot \tau(-1, x + e_k a)$.

The construction of (1.1) produces an element $y \in N$ such that $(y, x)=1$ and both

e_k and e^k occur with zero coefficient in y. Thus G contains $[\tau(1, y)\,\tau(-1, e_k+y), \sigma_3]$ $=\tau(1, x+e_k)\,\tau(-1, x+(a+1)e_k)\,\tau(1, x+ae_k)\,\tau(-1, x)=\tau(-2a, e_k)$. ‖

(3.3) PROPOSITION *If G contains a transvection $\tau\neq 1$, then $G\supset \mathrm{ESp}(\mathfrak{q})$ for some $\mathfrak{q}\neq 0$.*

Proof. Suppose $\tau(a, x)\in G$ for some $a\neq 0$ and some unimodular $x\in M$. In view of (2.1), (2.2) implies that $\tau(a, x)\in G$ for all unimodular $x\in M$. To prove that $\mathrm{ESp}(aA)\subset G$, it is therefore sufficient to show that $\tau(ab, x)\in G$ for a particular unimodular $x\in M$ and all $b\in A$.

Let N be a hyperbolic submodule of M with a hyperbolic basis $\{e_j, e^j\}_{1\leqslant j\leqslant 3}$. As in the proof of (2.4), the identity

$$\tau(-a, e_2+e_3+be_1)\,\tau(a, e_2+be_1)\,\tau(a, e_3+be_1)\,\tau(a, e_2+e_3)\,\tau(-a, e_2)\,.$$
$$\tau(-a, e_3)\,\tau(-ab^2, e_1)=1$$

can be written as

$$\tau(ab^2, e_1)=[\beta, \tau(-a, e_2+e_3)\,\tau(a, e_3)]\,[\gamma, \tau(a, e_2)]$$

where $\beta=\tau(-b, e^3)\,\tau(b, e^3+e_1)$ and $\gamma=\tau(-b, e^2)\,\tau(b, e^2+e_1)$ are in $\mathrm{ESp}(R)$ and $\tau(-a, e_2+e_3)=\tau(a, e_2+e_3)^{-1}$, $\tau(a, e_3)$ and $\tau(a, e_2)$ belong to G in view of the initial remarks. Hence $\tau(ab^2, e_1)\,\tau(ac^2, e_1)^{-1}=\tau(a(b^2-c^2), e_1)\in G$ for all a, $b\in A$. Since $\frac{1}{2}\in A$, any element in A can be written in the form b^2-c^2, proving the assertion. ‖

We now come to our principal result.

(3.4) THEOREM. *Suppose $\frac{1}{2}\in A$ and the form $(\cdot,\cdot)$ is locally hyperbolic. The following assertions are equivalent:*

(i) *G is a subgroup of $\mathrm{Sp}(R)$ normalised by $\mathrm{ESp}(R)$.*

(ii) *There exists a unique ideal $\mathfrak{q}$ in A such that $\mathrm{ESp}(\mathfrak{q})\subset G\subset \mathrm{Sp}'(\mathfrak{q})$.*

Proof. Choose $\mathfrak{q}$ maximal w.r.t. the property $\mathrm{ESp}(\mathfrak{q})\subset G$. Suppose $G\not\subset \mathrm{Sp}'(\mathfrak{q})$; then the image $\bar{G}$ of G in $\mathrm{Sp}(R/(\mathfrak{q}))$ will not be in the center. Since the homomorphism $\mathrm{ESp}(R)\to \mathrm{ESp}(R/(\mathfrak{q}))$ is surjective by (2.3), we may apply (3.2) and (3.3) to $\bar{G}$ and conclude that $\bar{G}\supset \mathrm{ESp}(\mathfrak{q}'/\mathfrak{q})$ for some ideal $\mathfrak{q}'\supsetneq\mathfrak{q}$; lifting to A, we have $\mathrm{ESp}(\mathfrak{q}')\subset$ $\subset \mathrm{Sp}(\mathfrak{q})\cdot G$. Now by (2.4) and (2.5), $\mathrm{ESp}(\mathfrak{q}')=[\mathrm{ESp}(R), \mathrm{ESp}(\mathfrak{q}')]\subset[\mathrm{ESp}(R), \mathrm{Sp}(\mathfrak{q})\cdot G]\subset G$, contradicting the maximality of $\mathfrak{q}$. Therefore $G\subset \mathrm{Sp}'(\mathfrak{q})$.

If $\mathrm{ESp}(\mathfrak{q})\subset G\subset \mathrm{Sp}'(\mathfrak{q})$ then by (2.4) and (2.5) we have $\mathrm{ESp}(\mathfrak{q})=[\mathrm{ESp}(R), \mathrm{ESp}(\mathfrak{q})]\subset$ $\subset[\mathrm{ESp}(R), G]\subset[\mathrm{ESp}(R), \mathrm{Sp}'(\mathfrak{q})]\subset \mathrm{ESp}(\mathfrak{q})\subset G$. This shows that $\mathfrak{q}$ is unique and that (ii)$\Rightarrow$(i). ‖

(3.5) COROLLARY. *The following are equivalent:*

(i) *G is a normal subgroup of $\mathrm{ESp}(R)$.*

(ii) *There exists a unique ideal* $\mathbf{q}$ *such that*

$$\mathrm{ESp}(\mathbf{q}) \subset G \subset \mathrm{ESp}(R) \cap \mathrm{Sp}(\mathbf{q}).$$

The groups $\boldsymbol{\delta}(\mathbf{q}) = \mathrm{ESp}(R) \cap \mathrm{Sp}(\mathbf{q})/\mathrm{ESp}(\mathbf{q})$ are all abelian.

Proof. Suppose G is normal in $\mathrm{ESp}(R)$; (3.4) provides a unique ideal $\mathbf{q}$ such that $\mathrm{ESp}(\mathbf{q}) \subset G \subset \mathrm{ESp}(R) \cap \mathrm{Sp}'(\mathbf{q})$. To show (i)$\Rightarrow$(ii), it suffices to prove that $\mathrm{ESp}(R) \cap \mathrm{Sp}'(\mathbf{q}) = \mathrm{ESp}(R) \cap \mathrm{Sp}(\mathbf{q})$. However, it is easy to see that the center of $\mathrm{Sp}(R/(\mathbf{q}))$ consists of homotheties, of which only 1 can lie in $\mathrm{ESp}(R/(\mathbf{q}))$.

Both (ii)$\Rightarrow$(i) and the commutativity of $\boldsymbol{\delta}(\mathbf{q})$ are implied by (2.5). $\|$

(3.6) COROLLARY. *If* $\mathbf{q}$ *is a maximal ideal of* A, *the group* $\mathrm{ESp}(R)/\mathrm{ESp}(R) \cap \mathrm{Sp}(\mathbf{q})$ *is simple.* $\|$

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