Parapolicity and existence of bounded biharmonic functions

Parabolicity and existence of bounded biharmonic functions[1]

by LEO SARIO and CECILIA WANG

The existence of bounded biharmonic functions has exhibited interesting dependence on the dimension of the base manifold. Typically, such functions exist on the punctured Euclidean N-space $E^N: 0<|x|<\infty$ for $N=2$ and for $N=3$, but not for any $N \geqslant 4$ (Sario-Wang [17]). In the present paper we are interested in the problem: Is there any relation between the parabolicity of a manifold and the existence of bounded biharmonic functions, and does the dimension of the manifold have any bearing on the question.

Denote by H^2B the class of bounded biharmonic functions. In contrast with the case of bounded harmonic functions, which are known not to exist on any parabolic manifold (see e.g. Sario-Nakai [14]), it is possible to endow even the Euclidean plane with a metric which allows H^2B-functions (Nakai-Sario [8]). The process relies on the fact that harmonicity on a Riemann surface, and hence parabolicity, are not affected by a conformal metric ,which thus can be freely chosen to bring in H^2B-functions. For manifolds of dimension $N \geqslant 3$ this process is no longer possible. We shall show that, nevertheless, there exist parabolic manifolds of any dimension which carry H^2B-functions.

That there exist hyperbolic manifolds with H^2B-functions is trivial in view of the Euclidean N-ball. We shall prove that there also exist hyperbolic manifolds of any dimension which do not possess H^2B-functions.

Our study is completed by giving parabolic manifolds of any dimension which do not tolerate H^2B-functions. Thus the totality of Riemannian manifolds for any N is decomposed into four disjoint nonempty classes,

$$O_G^N \cap \tilde{O}_{H^2B}^N, \quad \tilde{O}_G^N \cap \tilde{O}_{H^2B}^N, \quad \tilde{O}_G^N \cap O_{H^2B}^N, \quad O_G^N \cap O_{H^2B}^N,$$

where O_G^N is the class of parabolic N-manifolds, $O_{H^2B}^N$ the class of N-manifolds which do not carry nonharmonic H^2B-functions, and $\tilde{O}$ stands for the complement of a given class O.

1. Consider the punctured N-space

$$M_\alpha^N = \{0 < r < \infty\}, \quad r = |x|, \quad x = (x_1, \ldots, x_N),$$

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with the metric

$$ds^2 = r^\alpha dr^2 + r^{\alpha+2} d\theta_1^2 + \sum_{i=2}^{N-1} \varphi_i(\theta)\, d\theta_i^2,$$

α a constant. Here we have utilized the global polar coordinates $(r, \theta_1, \dots, \theta_{N-1})$ of the punctured N-space, and $\varphi_2(\theta), \dots, \varphi_{N-1}(\theta)$ are positive (periodic) functions of $\theta_2, \dots, \theta_{N-1}$ only.

LEMMA 1. *$M_\alpha^N \in O_G^N$ for every α.*

Proof. Set $\varphi_0 = \Pi_{i=2}^{N-1} \varphi_i$. The metric tensor (g_{ij}) is diagonal, with $g^{rr} = r^{-\alpha}$, $g^{\theta_1\theta_1} = r^{-\alpha-2}$, and $\sqrt{g} = r^{\alpha+1} \varphi_0^{\frac{1}{2}}$. For $f(r) \in C^2$, the Laplace-Beltrami operator $\Delta = d\delta + \delta d$ gives

$$\Delta f = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial r} (\sqrt{g} g^{rr} f') = -r^{-\alpha-1} \varphi_0^{-\frac{1}{2}} \frac{\partial}{\partial r} (r^{\alpha+1} \varphi_0^{\frac{1}{2}} r^{-\alpha} f'),$$

which vanishes if and only if $d(rf')/dr = 0$. Thus every radial harmonic function on M_α^N has the form $f(r) = a \log r + b$. For $b = 0$, and a suitable $a, f(r)$ is the harmonic measure ω_R of the region bounded by $r = 1$ and $r = R$, say. As $R \to \infty$ or $R \to 0$, $\omega_R \to 0$, and $M_\alpha^N \in O_G^N$.

LEMMA 2. *For every α, $\cos(\alpha+2)\theta_1 \in H^2B(M_\alpha^N)$.*

Proof. Since the φ_i's are independent of θ_1,

$$\begin{aligned} \Delta \cos(\alpha+2)\theta_1 &= -r^{-\alpha-1} \varphi_0^{-\frac{1}{2}} \frac{\partial}{\partial \theta_1} \left[r^{\alpha+1} \varphi_0^{\frac{1}{2}} r^{-\alpha-2} \frac{d}{d\theta_1} \cos(\alpha+2)\theta_1 \right] \\ &= (\alpha+2)^2 r^{-\alpha-2} \cos(\alpha+2)\theta_1, \end{aligned}$$

and

$$\begin{aligned} \Delta^2 \cos(\alpha+2)\theta_1 &= -(\alpha+2)^2 \left\{ r^{-\alpha-1} \frac{\partial}{\partial r} \left[r \frac{\partial}{\partial r} (r^{-\alpha-2} \cos(\alpha+2)\theta_1) \right] \right. \\ &\quad \left. + \frac{\partial}{\partial \theta_1} \left[r^{-\alpha-2} \frac{\partial}{\partial \theta_1} (r^{-\alpha-2} \cos(\alpha+2)\theta_1) \right] \right\} \\ &= -(\alpha+2)^2 [(\alpha+2)^2 r^{-2\alpha-4} \cos(\alpha+2)\theta_1 - (\alpha+2)^2 r^{-2\alpha-4} \\ &\quad \times \cos(\alpha+2)\theta_1] = 0. \end{aligned}$$

We have proved:

THEOREM 1. *For every N, $O_G^N \cap \tilde{O}_{H^2B}^N \neq \emptyset$.*

2. Consider the space discussed in [13]

$$E_\alpha^N = \{0 < r < \infty\}, \quad r = |x|, \quad x = (x_1, \dots, x_N),$$

with the metric $ds=r^{\alpha}|dx|$, $\alpha\in\mathbf{R}$.

LEMMA 3. $E_{\alpha}^{N}\in O_{G}^{N}$ *if and only if* $\alpha=-1$.
Proof. The metric tensor is given by

$$ds^2 = r^{2\alpha}dr^2 + r^{2\alpha+2}\varphi_1(\theta)\, d\theta_1^2 + \cdots + r^{2\alpha+2}\varphi_{N-1}(\theta)\, d\theta_{N-1}^2,$$

where $\theta=(\theta_1,\dots,\theta_{N-1})$ and $\varphi_1,\dots,\varphi_{N-1}$ are trigonometric functions of θ. Set $\varphi_0=\Pi_1^{N-1}\varphi_i$. Then $\sqrt{g}=r^{N-1+N\alpha}\varphi_0^{\frac{1}{2}}(\theta)$, $g^{rr}=r^{-2\alpha}$, and for $f(r)\in C^2$,

$$\begin{aligned}\Delta f(r) &= -r^{-N+1-N\alpha}\varphi_0^{-\frac{1}{2}}\frac{\partial}{\partial r}(r^{N-1+(N-2)\alpha}\varphi_0^{\frac{1}{2}}f') \\ &= -r^{-2\alpha}\{f''+[N-1+(N-2)\alpha]\, r^{-1}f'\}.\end{aligned}$$

This vanishes if and only if

$$f(r) = a\int_1^r r^{-N+1-(N-2)\alpha}dr + b.$$

The rest of the proof is as for Lemma 1.

3. To show that there exist hyperbolic N-manifolds without H^2B-functions, it will be convenient to choose $\alpha=-\frac{3}{4}$ in No. 2. Then the equation $\Delta f(r)=0$ has a solution

$$\sigma(r)=\begin{cases}\log r & \text{for } N=2,\\ r^{-(N-2)/4} & \text{for } N\neq 2,\end{cases}$$

and the general solution is $a\sigma+$b. Let $S_{nm}(\theta)$ be the surface spherical harmonics, $n=1,2,\dots$, and $m=1,\dots,m_n$, where m_n is determined by

$$(1+x)(1-x)^{-N+1}=\sum_{n=0}^{\infty} m_n x^n.$$

We have $\Delta S_{nm}=n(n+N-2)r^{-\frac{1}{2}}S_{nm}$, and the equation $\Delta(f(r)S_{nm}(\theta))=0$ has the general solution $f(r)=ar^{p_n}+br^{q_n}$, where

$$p_n,\, q_n=\tfrac{1}{2}[-(N-2)/4\pm\sqrt{(N-2)^2/16+4n(n+N-2)}].$$

For a fixed r, any harmonic function $h(r,\theta)$ on $E_{-3/4}^{N}$ is C^{∞} on the (Euclidean) unit sphere, with an eigenfunction expansion

$$h(r, \theta) = f_0(r) + \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} f_{nm}(r)\, S_{nm}(\theta).$$

Given $0<r_1<r_2<\infty$, choose constants a_{nm}, b_{nm}, a, b, such that for $i=1, 2$,

$$a_{nm} r_i^{p_n} + b_{nm} r_i^{q_n} = f_{nm}(r_i),\ a\,\sigma(r_i) + b = f_0(r_i).$$

Then h has the expansion

$$h = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (a_{nm} r^{p_n} + b_{nm} r^{q_n})\, S_{nm} + a\sigma(r) + b$$

on $r=r_1$ and $r=r_2$, hence by the harmonicity on $r_1 \leqslant r \leqslant r_2$. The uniqueness is verified by choosing $0<r_1'<r_1<r_2<r_2'<\infty$, which gives on $r_1' \leqslant r \leqslant r_2'$ an expansion that on $r_1 \leqslant r \leqslant r_2$ must coincide with the above.

4. By a straightforward computation of Δ we find that the equation $\Delta f(r)=1$ has a solution $s(r)=-(8/N)r^{1/2}$, the general solution being $s+a\sigma+b$. Similarly, the equation $\Delta f(r)=\sigma(r)$ has a solution

$$\tau(r) = \begin{cases} s(r)(\log r - 4) & \text{for } N=2, \\ -2\log r & \text{for } N=4, \\ \dfrac{8}{N-4} r^{-(N-4)/4} & \text{for } N \neq 2, 4, \end{cases}$$

and the general solution is $\tau + a\sigma + b$. Set $P_n = N/8 + p_n$, $Q_n = N/8 + q_n$. The equation $\Delta u = r^{p_n} S_{nm}$ is satisfied by

$$u_{nm} = -\frac{1}{P_n} r^{p_n + \frac{1}{2}} S_{nm},$$

and the equation $\Delta v = r^{q_n} S_{nm}$ by

$$v_{nm} = -\frac{1}{Q_n} r^{q_n + \frac{1}{2}} S_{nm}.$$

Given a biharmonic function u on $E^N_{-3/4}$, let

$$\Delta u = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (a_{nm} r^{p_n} + b_{nm} r^{q_n})\, S_{nm} + a\sigma(r) + b,$$

and set

$$u_0 = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} a_{nm}u_{nm} + b_{nm}v_{nm} + a\tau(r) + bs(r).$$

Then $u=u_0+k$, where k is a harmonic function

$$k = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (c_{nm}r^{p_n} + d_{nm}r^{q_n})\, S_{nm} + c\sigma(r) + d.$$

In fact, the compact convergence of u_0 is entailed by that of Δu, and the statement follows by Nos. 3–4.

5. We are ready to state:

LEMMA 4. *$E^N_{-3/4} \in O^N_{H^2B}$ for every N.*

Proof. Let $u \in H^2B(E^N_{-3/4})$. Clearly $|(u, \varphi)| \leqslant \sup|u|(1, |\varphi|)$ for all $\varphi \in L^1$, in particular for the family of functions $\varphi_t = \varrho_t(r)S_{nm}$, where ϱ_1 is a fixed function $\in C$, $\varrho_1 \geqslant 0$, $\operatorname{supp}\varrho_1 \subset (1, 2)$, and $\varrho_t(r) = \varrho_1(r+1-t)$ for $t \geqslant 1$. By the orthogonality of $\{S_{nm}\}$,

$$(u, \varphi_t) = \int_t^{t+1} (C_1 a_{nm} r^{p_n+\frac{1}{2}} + C_2 b_{nm} r^{q_n+\frac{1}{2}} + C_3 c_{nm} r^{p_n} + C_4 d_{nm} r^{q_n})\, \varrho_t(r)\, r^{N/4-1} dr.$$

Here and later the C's are constants, not always the same. Clearly $\int_t^{t+1} \varrho_t(r)dr$ is constant as $t \to \infty$. If some $a_{nm} \neq 0$, then

$$(u, \varphi_t) \sim Ct^{p_n+N/4-\frac{1}{2}}, \quad \text{whereas} \quad (1, |\varphi_t|) = O(t^{N/4-1}).$$

A fortiori, we have a contradiction for n such that $p_n + N/4 - \frac{1}{2} > N/4 - 1$, that is, $p_n > -\frac{1}{2}$. Since $p_n > 0$ for all n, we obtain $a_{nm} = 0$ for all n, m.

If some $c_{nm} \neq 0$, we infer by $q_n + \frac{1}{2} < p_n$ for all n, N that

$$(u, \varphi_t) \sim Ct^{p_n+N/4-1}$$

as $t \to \infty$. Every n such that $p_n > 0$ is ruled out, and we conclude again that $c_{nm} = 0$ for all n, m.

Now choose $\varrho_t(r) = \varrho_1(r/t)$, with ϱ_1 as before, and $0 < t \leqslant 1$. Then $\operatorname{supp}\varrho_t \subset (t, 2t)$ and $\int_t^{2t} \varrho_t(r)dr = Ct$. If some $d_{nm} \neq 0$, then

$$(u, \varphi_t) \sim Ct^{q_n+N/4} \quad \text{and} \quad (1, |\varphi_t|) \sim O(t^{N/4})$$

as $t \to 0$. Inequality $q_n + N/4 < N/4$ gives a contradiction, and by $q_n < 0$ we deduce that for all n, m. In the same manner we see that $b_{nm}=0$ if $q_n+\frac{1}{2}<0$, that is, for all n, m.

Thus the function u reduces to $a\tau(r)+bs(r)+c\sigma(r)+d$. Since τ, s, σ are linearly independent and unbounded, we have $a=b=c=0$, and u is a constant.

We combine Lemmas 3 and 4 to conclude:

THEOREM 2. *For every* N, $\tilde{O}_G^N \cap O_{H^2B}^N \neq \emptyset$.

6. The existence of hyperbolic N-manifolds with H^2B-functions is given by the Euclidean N-ball. It remains to find a parabolic N-manifold without H^2B-functions.

LEMMA 5. $E_{-1}^N \in O_{H^2B}^N$ *for every* N.

Proof. The proof arrangement is the same as in Nos. 3–5, and we only point out the changes. We now have $\sigma(r)=\log r$ for every N, $p_n=-q_n=\sqrt{n(n+N-2)}$, and the expansion of a harmonic function h is as before. As to biharmonic functions, $s(r)=-\frac{1}{2}(\log r)^2$, $\tau(r)=-\frac{1}{6}(\log r)^3$, both for every N, and

$$u_{nm}=-\frac{1}{2p_n}r^{p_n}\log r\cdot S_{nm},\quad v_{nm}=\frac{1}{2p_n}r^{-p_n}\log r\cdot S_{nm}.$$

With this notation, there is again no change in the expansion of a biharmonic function u.

If some $a_{nm}\neq 0$, we have for $\varphi_t=\varrho_t(r)S_{nm}$, $\varrho_t(r)=\varrho_1(r+1-t)$,

$$(u,\varphi_t)\sim C\int_t^{t+1} r^{p_n}\log r\cdot r^{-1}\varrho_t(r)\,dr\sim Ct^{p_n-1}\log t,\quad (1,|\varphi_t|)=O(t^{-1})$$

as $t\to\infty$. Therefore $a_{nm}=0$ for $p_n-1>-1$, that is, for all n, m. That $c_{nm}=0$ for all n, m is concluded in the same manner.

Now choose $\varrho_t(r)=\varrho_1(r/t)$, $t\to 0$. If some $b_{nm}\neq 0$, then

$$(u,\varphi_t)\sim Ct^{-p_n}\log t,\quad\text{and}\quad (1,|\varphi_t|)=O(1).$$

Thus all n with $-p_n<0$ are ruled out, and we have $b_{nm}=0$ for all n, m. Similarly all $d_{nm}=0$.

The function u again reduces to the radial terms of its expansion, and as before we infer that u is a constant.

We have established:

THEOREM 3. *For every* N, $O_G^N \cap O_{H^2B}^N \neq \emptyset$.

7. We may combine our results in the following form:

THEOREM 4. *The totality $\{R^N\}$ of Riemannian N-manifolds decomposes, for every N, into the four disjoint nonempty classes*

$$\{R^N\} = O_G^N \cap O_{H^2B}^N + O_G^N \cap \tilde{O}_{H^2B}^N + \tilde{O}_G^N \cap O_{H^2B}^N + \tilde{O}_G^N \cap \tilde{O}_{H^2B}^N.$$

We append a bibliography of recent work in the field.

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