Categories with Ultraproducts

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Introduction

The aim of this paper is to characterize those categories $\mathbf{M}$ with underlying set functor $U:\mathbf{M}\to\mathbf{S}$ which are equivalent to the category $\mathbf{M}(T)$ (cf. I.5.) of models of a theory T and homomorphisms, for a certain large class of first order theories T. They are called special theories (cf. I.7.). They include for example equational algebraic theories with finitary operations, universal Horn theories (cf. IV.4.), the theory of (total) order, the theory of dense order without extreme elements (cf. I.9.), as well as many others.

In part I we introduce the notion of U-objects $\langle A, \vec{a}\rangle$ (cf. I.1.) of some category $\mathbf{M}$ with underlying set functor $U:\mathbf{M}\to\mathbf{S}$. They are objects A of $\mathbf{M}$ with a distinguished finitary tuple $\vec{a}$ of elements of UA. U-maps are classes of homomorphisms which coincide on the elements of $\vec{a}$ (cf. I.2.). the sets $U\text{-Hom}(\langle A, \vec{a}\rangle, M)$ of U-maps from a U-object $\langle A, \vec{a}\rangle$ to the various objects $M\in\mathbf{M}$ give rise to the so called U-representable functor $U\text{-Hom}(\langle A, \vec{a}\rangle, -):\mathbf{M}\to\mathbf{S}$.

Roughly speaking a theory T turns out to be special if and only if the functors $\mathbf{M}(T)\to\mathbf{S}$, induced by the predicate constants, are U-representable (cf. lemma I.8.).

In part II we define the notion of U-subcategories of $\mathbf{M}$, as small categories of U-objects of $\mathbf{M}$ and U-maps between such (cf. II.1. and II.2.). Then we introduce U-dense U-subcategories and U-adequate U-subcategories (cf. II.4. and 6.), which are formally analogous to the standard notions of dense and adequate subcategories respectively (cf. [U] and [I]), and they turn out to be equivalent, too (cf. lemma II.6. and also [U] lemma 1.7.).

In the case of a special theory T, the U-subcategory $\mathbf{N}(T)$ in $\mathbf{M}(T)$ having as objects the U-representing U-objects for the predicate constants is an example of a U-dense U-subcategory in $\mathbf{M}(T)$, (cf. II.9.).

Any U-dense U-subcategory $\mathbf{N}$ of $\mathbf{M}$ gives rise to a category $\mathbf{M}(L_{\mathbf{N}})$ of structures and homomorphisms as well as to a full imbedding $\mathbf{M}\subseteq\mathbf{M}(L_{\mathbf{N}})$ (cf. II.10., 11.), which in case of a special theory T reproduces the full imbedding of $\mathbf{M}(T)$ into the category $\mathbf{M}(L(T))$ of all $L(T)$-structures. (cf. II.12.).

In part III we fully imbed the category $\mathbf{M}$ into the category $U-\prod\mathbf{M}$ of so called U-direct products (cf. III.2.). $\mathbf{M}$ is assumed to have a U-dense U-subcategory $\mathbf{N}$ in which $U:\mathbf{M}\to\mathbf{S}$ is U-representable. $U-\prod\mathbf{M}$ consists of direct products of the so called U-corepresentable functors $U\text{-Hom}(-,\mathbf{M}):\mathbf{N}^{op}\to\mathbf{S}$, $M\in\mathbf{M}$. (cf. II.1.). In the case of a special theory T the category of U-direct products is equivalent to the category $\prod \mathbf{M}(T)$ of standard direct products of models of T (cf. III.6.).

We then introduce the U-ultraproducts (cf. III.9.) in U-$\prod \mathbf{M}$ in formal analogy with Ohkuma's definition (cf. [O]). In the case of a special theory T they turn out to be the standard ultraproducts (cf. III.14.). More generally, the full imbedding $\mathbf{M} \subseteq \mathbf{M}(L_{\mathbf{N}})$ commutes with ultraproducts, provided that U-representable functors do so (cf. III.12).

In part IV we introduce the notion of an ultra dense U-subcategory $\mathbf{N}$ of a category M which is closed with respect to U-ultraproducts. Such an $\mathbf{N}$ is a U-dense U-subcategory of $\mathbf{M}$ containing a U-representing U-object of $U:\mathbf{M} \to \mathbf{S}$. Furthermore U-representable functors commute with ultraproducts and any functor $G:\mathbf{N}^{\mathrm{op}} \to \mathbf{S}$ is corepresentable if and only if some ultrapower G^I/D of G is U-corepresentable. It follows that $\mathbf{M}$ has an ultradense U-subcategory $\mathbf{N}$ if and only if $\mathbf{M}$ is equivalent to the category $\mathbf{M}(T)$ of all models and their homomorphisms of a special theory T (cf. IV.5.). This is the functorial characterization of categories of models we aimed at. It concerns a larger class of theories T than F. W. Lawvere's and O. E. Kean's characterizations which deal with algebraic theories and universal Horn theories respectively (cf. [L] and [Kn]).

In part V we investigate those special theories which have only finitely many predicate constants and are F-axiomatized (cf. V.1.). They turn out to have a purely syntactical description: they are "locally atomistic" (cf. V.7. and 8.). The category $\mathbf{M}(T)$, for such a T, is characterized by the conditions that it be closed with respect to direct limits of directed systems and that it have a finite ultra dense U-subcategory (cf. theorem V.9.). The model theoretic methods we use in part V are developped in full generality in [F1] and [F2]. Here we will sometimes sketch them only in a form adapted to what we need.

I. U-Representable Functors

1. *U-Objects.* Let $\mathbf{S}$ be the category of sets and let $U:\mathbf{M} \to \mathbf{S}$ be an "underlying set functor" of some category $\mathbf{M}$. By a U-object $\langle A, \vec{a} \rangle$ of $\mathbf{M}$ we understand an ordered pair where A is an object of $\mathbf{M}$ and $\vec{a}$ is a finitary tuple $(a_1, \ldots, a_n)$ of elements from UA.

2. *U-maps* in $\mathbf{M}$ are equivalence classes $[f]:\langle A, \vec{a} \rangle \to \mathbf{M}$ of morphisms $f:A \to M$ in $\mathbf{M}$. Two morphisms $f, g:A \rightrightarrows$ M being in the same class $[f]=[g]:\langle A, \vec{a} \rangle \to M$ if $f(a_k)=g(a_k)$, $\vec{a}=(a_1, \ldots, a_k, \ldots a_n)$. The set of U-maps $\langle A, \vec{a} \rangle \to M$ will be denoted by U-Hom$(\langle A, \vec{a} \rangle, M)$. For any fixed U-object $\langle A, \vec{a} \rangle$ in $\mathbf{M}$ one can view U-Hom $\times$ $\times$ $(\langle A, \vec{a} \rangle, -):\mathbf{M} \to \mathbf{S}$ as a functor. The natural imbedding U-Hom$(\langle A, \vec{a} \rangle, -) \subseteq U^n$ (for $\vec{a}=(a_1, \ldots, a_n)$) $[f]:\langle A, \vec{a} \rangle \to M \mapsto f(\vec{a}) \in [U(M)]^n$ will be called the canonical imbedding.

3. *U-representable Functors.* A functor $G:\mathbf{M}\to\mathbf{S}$ which is naturally equivalent to U-Hom$(\langle A, \vec{a}\rangle, -)$ will be called U-representable by the U-object $\langle A, \vec{a}\rangle$. If a subfunctor G of U^n is the image of the canonical imbedding $U\text{-Hom}(\langle A, \vec{a}\rangle, -)\subseteq U^n$ it will be called canonically U-representable by $\langle A, \vec{a}\rangle$ (or "cU-representable").

4. *Dense Orders Without Extreme Elements.* Let M be the category of densely ordered sets without extreme elements, the morphisms being strictly order preserving maps (i.e. imbeddings). $U:\mathbf{M}\to\mathbf{S}$ is the usual underlying set functor. It follows that U is cU-representable for example by the U-object $\langle Q, 0\rangle$, where Q is the order of the rationals. Furthermore let $V:\mathbf{M}\to\mathbf{S}$ be the subfunctor of $U^2:\mathbf{M}\to\mathbf{S}$ where VM consists of all pairs $(a, b)\in(UM)^2$ which fulfill $M\vDash a<b$ (i.e. a precedes b in the order of M). It can easily be shown that V is cU-representable by the U-object $\langle Q, 0, 1\rangle$.

5. *Categories of Models.* Let T be a first order theory, having equality "$=$" as one of its predicate constants $p, q, \ldots$. By $\mathbf{M}(T)$ we understand the category whose objects are the models $A, B, \ldots$ of T and whose morphisms are those maps $f:A\to B$ which preserve the validity for the predicate constants, e.g. if $A\vDash p(a_1, a_2)$ then $B\vDash p(f(a_1), f(a_2))$. We will call them homomorphisms. By $U:\mathbf{M}(T)\to\mathbf{S}$ we denote the corresponding underlying set functor.

For n-ary predicate constant p of T we denote by $p(M)$, $M\in T$, the set of all n-tuples $(a_1,\ldots, a_n)$ of elements in M which fulfill $M\vDash p(a_1,\ldots, a_n)$. Thus $p:\mathbf{M}(T)\to\mathbf{S}$ becomes a functor, namely a subfunctor of $\mathrm{U}^n:\mathbf{M}(T)\to\mathbf{S}$. By abuse of language we say that the predicate constant p is U-representable if the corresponding functor $p:\mathbf{M}(T)\to\mathbf{S}$ is U-representable. E.g. in the case of dense orders without extreme elements we have stated that the predicate constant $x<y$ is U-representable. (cf. I.4.). If p is cU-representable by $\langle A, \vec{a}\rangle$ then it follows that $A\vDash p(\vec{a})$.

U itself is cU-representable by $\langle E, e\rangle$ if and only if the binary predicate $x=y$ (equality) is cU-representable by $\langle E, e, e\rangle$.

6. *Standard Direct Products.* Let T be a first order theory. By $L(T)$ we mean T without its nonlogical axioms. The models of $L(T)$ are sometimes also called $L(T)$-structures. By the standard direct product $\prod_I M(i)$ of some family $\{M(i)\}_{i\in I}$ of $L(T)$-structures one understands the $L(T)$-structure whose underlying set $U(\prod_I M(i))$ is the direct product $\prod_I UM(i)$, and $p(\prod_I M(i))=\prod_I pM(i)$. $\prod_I M(i)$ is also the direct product with respect to the category $\mathbf{M}(L(T))$. The category $\prod\mathbf{M}(T)$ is defined to be the full subcategory of $\mathbf{M}(L(T))$ whose objects are standard direct products $\prod_I M(i)$ of models in $\mathbf{M}(T)$. For any full subcategory $\mathbf{K}$ of $\mathbf{M}(L(T))$ we define $\prod\mathbf{K}$ to be the full subcategory of $\mathbf{M}(L(T))$ whose objects are the standard direct products of objects in $\mathbf{K}$.

7. *Special Theories.* T is called a special theory if for any standard direct product $\prod_I M(i)$ of models, any finitary tuple $\vec{m}$ of elements in $\prod_I M(i)$ and any predicate constant p of T with $\prod_I M(i) \models p(\vec{m})$ there exists a model $N \in \mathbf{M}(T)$ with a finitary tuple $\vec{n}$ of elements and a homomorphism $f: N \to \prod_I M(i)$ in $\prod \mathbf{M}(T)$, fulfilling $(Uf)(\vec{n}) = \vec{m}$.

8. LEMMA. *T is special if and only if all its predicate constants are canonically U-representable.*

Proof. Let the predicate constants of T be canonically U-representable and assume $\prod_I M(i) \models p(\vec{m})$. Since p is cU-representable, say by $\langle A, \vec{a} \rangle$ such that $A \models p(\vec{a})$, there is a U-map $[f(i)]: \langle A, \vec{a} \rangle \to M(i)$ with $f(i)(\vec{a}) = \vec{m}(i)$, for any $i \in I$. Hence $\{f(i)\}_{i \in I}: A \to \prod_I M(i)$ and $\vec{a} \mapsto \vec{m}$. Thus T is special. Conversely let T be special. Let $\|T\|$ be the cardinality of the set of all formulas of T. Let $\{\langle N(i)\, \vec{n}(i) \rangle\}_{i \in I}$ be the set of all possible (nonisomorphic) U-objects $\langle N(i), \vec{n}(i) \rangle$ with $\|N(i)\| \leqslant \|T\|$ such that $N(i) \models p\vec{n}(i))$. Since T is special there exists $N \in \mathbf{M}(T)$, and $\vec{n}$ such that $N \models p(\vec{n})$ and furthermore some $f: \{f(i)\}_{i \in I}: N \to \prod_I N(i)$ such that $f(i): \vec{n} \mapsto \vec{n}(i)$. We claim that the U-object $\langle N, \vec{n} \rangle$ is a cU-representing object for p. Let $M \models p(\vec{m})$ be given. According to the downwards Löwenheim-Skolem theorem there exists some elementary submodel $M' \prec M$ containing $\vec{m}$ with cardinality $\|M'\| \leqslant \|T\|$. $\langle M', \vec{m} \rangle$ is isomorphic to $\langle N(i), \vec{n}$ (i)$\rangle$for some $i \in I$. Thus there is the homomorphism $f(i): N \to N(i) = M' \prec M$ sending $\vec{n} \to \vec{m}$. Hence $U\text{-Hom}(\langle N, \vec{n} \rangle, M) \cong p(M)$ sending $[f] \to f(\vec{n})$. Q.E.D.

9. *The theory of dense order without extreme elements* is easily seen to be an example of a special theory (cf. also I.4.). Thus the predicate constants $x = y$ and $x < y$ are again proved to be cU-representable.

II. U-dense U-subcategories

1. *U-maps Between U-objects* $[f]: \langle A, \vec{a} \rangle \to \langle B, \vec{b} \rangle$ are those *U-maps* $[f]$: $\langle A, \vec{a} \rangle \to B$ such that $Uf: UA \to UB$ maps the a_k's onto some b_l's.

2. *U-subcategories.* A small category $\mathbf{N}$ consisting of U-objects of a category $\mathbf{M}$ with underlying set functor $U: \mathbf{M} \to \mathbf{S}$, and U-maps between those U-objects is called a U-subcategory of $\mathbf{M}$. For every $M \in \mathbf{M}$ one gets then the functor $U\text{-Hom}(-, M)$: $\times$ $\times\, \mathbf{N}^{\mathrm{op}} \to \mathbf{S}$ since $U\text{-Hom}(-,-): \mathbf{N}^{\mathrm{op}} \times \mathbf{M} \to \mathbf{S}$ is a functor (in both variables).

3. *U-direct Limits.* Let Δ be a diagram of U-objects and U-maps of $\mathbf{M}$. Such a

diagram will be called a U-diagram. An object $M \in \mathbf{M}$ is called a U-direct limit of Δ, $M = U\text{-}\varinjlim \Delta$, if the following holds:

(a) There is a U-map $\langle A, \vec{a}\rangle \to M$ from every $\langle A, \vec{a}\rangle \in \Delta$ such that for any U-map $\langle A, \vec{a}\rangle \to \langle B, \vec{b}\rangle$ in Δ the diagram

$$\begin{tikzcd} \langle A, \vec{a}\rangle \arrow[r] \arrow[dr] & \langle B, \vec{b}\rangle \arrow[d] \\ & M \end{tikzcd}$$

of U-maps commutes.

(Such a family $\{\langle A, a\rangle \to M\}_{\langle A, \vec{a}\rangle \in \Delta}$ of U-maps will be called a compatible family of U-maps from Δ to M.)

(b) For every compatible family $\{\langle A, \vec{a}\rangle \to N\}_{\langle A, \vec{a}\rangle \in \Delta}$ of U-maps into some object $\mathbf{N} \in \mathbf{M}$ there exists precisely one morphism $M \to N$ in $\mathbf{M}$ such that the diagrams

$$\begin{tikzcd} \langle A, \vec{a}\rangle \arrow[r] \arrow[dr] & M \arrow[d] \\ & N \end{tikzcd}$$

commute (as U-maps).

It follows immediately that any two U-direct limits of the same U-diagram are isomorphic.

The U-maps $\langle A, \vec{a}\rangle \to M$ are called the universal U-maps.

4. *U-dense U-subcategories.* Let N be a U-subcategory of $\mathbf{M}$. To every object $M \in \mathbf{M}$ we will associate the following U-diagram $\Delta(M)$ in $\mathbf{N}$: It contains a copy of $\langle A, \vec{a}\rangle$ for every possible U-map $\langle A, \vec{a}\rangle \to M$. Those U-maps will be called the canonical U-maps into M. A U-map $\langle A, \vec{a}\rangle \to \langle B, \vec{b}\rangle$ between such U-objects belongs to $\Delta(M)$ if for the corresponding canonical U-maps $\langle A, \vec{a}\rangle \to M \to \langle B, \vec{b}\rangle$ the diagram

$$\begin{tikzcd} \langle A, \vec{a}\rangle \arrow[r] \arrow[dr] & \langle B, \vec{b}\rangle \arrow[d] \\ & M \end{tikzcd}$$

commutes.

$\mathbf{N}$ will be called U-dense in $\mathbf{M}$ if for every $M \in \mathbf{M}$ M is a U-direct limit of the U-diagram $\Delta(M)$ such that the canonical U-maps and the universal U-maps coincide.

5. *Dense Orders Without Extreme Elements* (cf. I.4.). Let $\mathbf{N}$ be the U-category in

M consisting of the U-objects $\langle Q, 0\rangle$ and $\langle Q, 0, 1\rangle$ and all possible U-maps between them. It is easy to show that **N** is U-dense in **M**. (cf. also I.9. and II.9.)

6. *U-adequate U-subcategories.* Let **N** be a U-subcategory of **M**. We define $Y:\mathbf{M}\to\mathbf{S}^{\mathbf{N}^{op}}$ to be the functor which assigns to $M\in\mathbf{M}$ the functor $Y(M)=U\text{-Hom}(-, \mathbf{N}^{op}\to\mathbf{S}$ (cf. 2.).

The U-subcategory **N** of **M** is called U-adequate in **M** if the functor $Y:\mathbf{M}\to\mathbf{S}^{\mathbf{N}^{op}}$ is full and faithful, i.e. any natural transformation $U\text{-Hom}(-, M)\to U\text{-Hom}(-, N)$ is induced by some morphism $M\to N$, and two different morphisms $M \underset{g}{\overset{f}{\rightrightarrows}} N$ induce different natural transformations $U\text{-Hom}(-, f)\neq U\text{-Hom}(-, g)$.

7. LEMMA. *Some U-subcategory* **N** *of* **M** *is U-dense in* **M** *if and only if it is U-adequate in* **M**.

Proof. Let **N** be U-dense in **M**. Let the natural transformation $\alpha: U\text{-Hom}(-, M)\to\to U\text{-Hom}(-, N)$ be given, $M, N\in\mathbf{M}$. Apply α to all the canonical U-maps $\langle A, \vec{a}\rangle\to\to M$, $\langle A, \vec{a}\rangle\in\Delta(M)$. Thus one gets a compatible family of U-maps $\langle A, \vec{a}\rangle\to N$, which induces a morphism$M\to N$, since $M\cong U\text{-}\overrightarrow{\lim}\,\Delta(M)$. If two morphisms $M \underset{g}{\overset{f}{\rightrightarrows}} N$ are different then there must be some $\langle A, \vec{a}\rangle\in\Delta(M)$ such that the two compositions $\langle A, \vec{a}\rangle\to M \underset{g}{\overset{f}{\rightrightarrows}} N$ are different since M is the $U\text{-}\overrightarrow{\lim}\,\Delta(M)$ and $\langle A, \vec{a}\rangle\to M$ is universal. But this means that the induced natural transformation $U\text{-Hom}(-, f)$ and $U\text{-Hom}(-, g)$ act differently on $\langle A, \vec{a}\rangle\to M$. Conversely let **N** be U-adequate in **M**. In order to show that for any $M\in\mathbf{M}$ M is a U-direct limit of $\Delta(M)$ and the canonical U-maps $\{\langle A, \vec{a}\rangle\to M\}_{\langle A,\vec{a}\rangle\in\Delta(M)}$ are the universal ones, let $\{\langle A, \vec{a}\rangle\to N\}$ be any compatible family of U-maps from $\Delta(M)$ to some $N\in\mathbf{M}$. The correspondence between the family $\{\langle A, \vec{a}\rangle\to M\}_{\langle A,\vec{a}\rangle\in\Delta(M)}$ and the family $\{\langle A, \vec{a}\rangle\to N\}_{\langle A,\vec{a}\rangle\in\Delta(M)}$ is easily seen to be a natural transformation $U\text{-Hom}(-, M)\to U\text{-Hom}(-, N)$. The latter is induced by some morphism $f: M\to N$ in **M**, since **N** is U-adequate in **M**. Thus

$$\begin{array}{ccc} \langle A, \vec{a}\rangle & \to & M \\ & \searrow \quad \swarrow{\scriptstyle f} & \\ & N & \end{array}$$

is commutative (as U-maps) for every $\langle A, \vec{a}\rangle$ in $\Delta(M)$. It is left to be shown that there is no other morphism $g: M\to N$ which makes this triangle commutative. $f\neq g$ implies $U\text{-Hom}(-, f)\neq U\text{-Hom}(-, g)$, since **N** is U-adequate in **M**. Hence there is $\langle A, \vec{a}\rangle\to M$ on which $U\text{-Hom}(-, f)$ and $U\text{-Hom}(-, g)$ act differently. Thus $U\text{-Hom}(-, g)$ is not the original correspondence between $\{\langle A, \vec{a}\rangle\to M\}_{\langle A,\vec{a}\rangle\in\Delta(M)}$ and $\{\langle A, \vec{a}\rangle\to N\}_{\langle A,\vec{a}\rangle\in\Delta(M)}$. Q.E.D.

8. *Categories* $\mathbf{M}(T)$ *of Models* (cf. I.5.). Let $\mathbf{N}(T)$ be the U-subcategory of $\mathbf{M}(T)$ containing one (ore more) U-representing U-object of each canonically U-representable predicate constant of T, the morphisms being all possible U-maps between them.

9. LEMMA. $\mathbf{N}(T)$ *is U-dense (or U-adequate) in* $\mathbf{M}(T)$ *if T is special.*

Proof. Since T is special, all its predicate constants are U-representable (cf. lemma I.8.). Let any natural transformation $\alpha: U\text{-Hom}(-, M) \to U\text{-Hom}(-, N)$ be given. Since $p(M) \cong U\text{-Hom}(\langle A, \vec{a}\rangle, M)$ α represents a homomorphism $M \to N$. Two different morphisms $M \rightrightarrows N$ induce different maps $UM \rightrightarrows UN$. If $\langle E, e\rangle$ U-represents U (cf. I.5.) we have two different maps $U\text{-Hom}(\langle E, e\rangle, M) \rightrightarrows U\text{-Hom}(\langle E, e\rangle, N)$. Hence the induced natural transformations $U\text{-Hom}(-, M) \rightrightarrows U\text{-Hom}(-, N)$ cannot be equal. Thus we have shown that $\mathbf{N}(T)$ is U-adequate in $\mathbf{M}(T)$, thus it is U-dense.

Q.E.D.

10. $\mathbf{N}$-*structures.* Let $\mathbf{N}$ be a U-subcategory of $\mathbf{M}$. By $L_{\mathbf{N}}$ we denote the first order theory (without nonlogical axioms) which has one n-ary predicate constant $p_{\langle A, \vec{a}\rangle}$ for each U-object $\langle A, \vec{a}\rangle \in \mathbf{N}$ with $\vec{a} = (a_1, \dots, a_n)$. We are going to define a functor $s: \mathbf{M} \to \to \mathbf{M}(L_{\mathbf{N}})$ from $\mathbf{M}$ into the category $\mathbf{M}(L_{\mathbf{N}})$ of $L_{\mathbf{N}}$-structures and homomorphisms. For this let $U_{\mathbf{N}}: \mathbf{M}(L_{\mathbf{N}}) \to \mathbf{S}$ be the underlying set functor. For $M \in \mathbf{M}$, $s(M)$ in $\mathbf{M}(L_{\mathbf{N}})$ is defined by $U_{\mathbf{N}}(s(M)) = UM$ and $p_{\langle A, \vec{a}\rangle}(M) =$ image of $\{U\text{-Hom}(\langle A, \vec{a}\rangle, M) \subseteq \subseteq (U_{\mathbf{N}}^n(s(M))\}$ for $\langle A, \vec{a}\rangle \in \mathbf{N}$. For $f: M \to N$ in $\mathbf{M}$ we set $U_{\mathbf{N}}(s(f)) = U(f)$: $U_{\mathbf{N}}(s(M)) \to U_{\mathbf{N}}(s(N))$. $s(f)$ is a homomorphism because f induces a natural transformation $U\text{-Hom}(-, f): U\text{-Hom}(-, M) \to U\text{-Hom}(-, N)$, i.e.

$$\begin{array}{cccc} U\text{-Hom}(\langle A, \vec{a}\rangle, f): & U\text{-Hom}(\langle A, \vec{a}\rangle, M) & \to & U\text{-Hom}(\langle A, \vec{a}\rangle, N). \\ & \| & & \| \\ & P_{\langle A, \vec{a}\rangle(M)} & \to & P_{\langle A, \vec{a}\rangle(N)} \end{array}$$

Notice that $s(M) \in \mathbf{M}(L_{\mathbf{N}})$ can be represented by $U\text{-Hom}(-, M): \mathbf{N}^{\mathrm{op}} \to \mathbf{S}$, while the homomorphisms are the natural transformations $U\text{-Hom}(-, M) \to U\text{-Hom}(-, N)$, provided that U is U-representable in $\mathbf{N}$.

11. THEOREM. *Let* $\mathbf{N}$ *be a U-subcategory of* $\mathbf{M}$ *which contains some cU-representing U-object* $\langle E, e\rangle$ *for U. The functor* $s: \mathbf{M} \to \mathbf{M}(L_{\mathbf{N}})$ *is full and faithful if and only if* $\mathbf{N}$ *is U-dense (or U-adequate) in* $\mathbf{M}$.

Proof. Let $s: \mathbf{M} \to \mathbf{M}(L_{\mathbf{N}})$ be full and faithful. Any natural transformation U-Hom$(-, M) \to U\text{-Hom}(-, N)$, for $M, N \in \mathbf{M}$, corresponds to a map $UM = U$-Hom $(\langle E, e\rangle, M) \to UN = U\text{-Hom}(\langle E, e\rangle, N)$ which is a homomorphism $s(M) \to s(N)$ in $\mathbf{M}(L_{\mathbf{N}})$. The latter is induced by a uniquely determined morphism $M \to N$ in $\mathbf{M}$. Thus $\mathbf{N}$ is U-adequate in $\mathbf{M}$.

Conversely, let $\mathbf{N}$ be U-adequate in $\mathbf{M}$. Any homomorphism $s(M)\to s(N)$ in $\mathbf{M}(L_{\mathbf{N}})$ corresponds to a unique natural transformation $U\text{-Hom}(-, M)\to U\text{-Hom}(-, N)$, which in turn is induced by a unique morphism $M\to N$ in $\mathbf{M}$. Hence s is full and faithful. Q.E.D.

12. *Special Theories.* For a special theory T, the category $\mathbf{N}(T)$ is U-dense in $\mathbf{M}(T)$ (cf. lemma II.9.). Theorem 11 implies that $s:\mathbf{M}(T)\to\mathbf{M}(L_{\mathbf{N}(T)})$ is a full imbedding. If one chooses $\mathbf{N}(T)$ such that it contains precisely one cU-representing U-object for each predicate constant of T then it follows that $L_{\mathbf{N}(T)}=L(T)$ and $\mathbf{M}(L_{\mathbf{N}(T)})=\mathbf{M}(L(T))$. The full imbedding $s:\mathbf{M}(T)\to\mathbf{M}(L_{\mathbf{N}(T)})$ then coincides with the canonical imbedding $\mathbf{M}(T)\subseteq\mathbf{M}(L(T))$.

III. Ultraproducts

1. *U-corepresentable Functors.* Let $\mathbf{N}$ be a U-subcategory of $\mathbf{M}$. A contravariant functor G in $\mathbf{S}^{\mathbf{N}^{\mathrm{op}}}$ of the form $G\cong U\text{-Hom}(-, M)$, $M\in\mathbf{M}$, is called a U-corepresentable functor (with M as U-corepresenting object).

If $\mathbf{N}$ is U-dense (or U-adequate) in $\mathbf{M}$ it follows immediately that the U-corepresenting object M of some U-corepresentable functor is uniquely determined up to isomorphism.

For the sake of simplicity we will in part III assume that $\mathbf{N}$ is a U-dense U-subcategory in $\mathbf{M}$ containing a cU-representing object $\langle E, e\rangle$ of $U:\mathbf{M}\to\mathbf{S}$.

2. *U-direct Products.* We are going to define a subcategory $U-\prod\mathbf{M}$ in the category $\mathbf{S}^{\mathbf{N}^{\mathrm{op}}}$ of contravariant functors from $\mathbf{N}$ into the category of sets. The objects of $U-\prod\mathbf{M}$ are the direct products of U-corepresentable functors $\prod_{i\in I}U\text{-Hom}(-, M(i)):\mathbf{N}\to\mathbf{S}$, $M(i)\in\mathbf{M}$, $i\in I$, I non empty. The morphisms are the natural transformations between such functors. $U-\prod\mathbf{M}$ is called the category of U-direct products of $\mathbf{M}$. It follows readily that $U-\prod\mathbf{M}$ is closed with respect to direct products and the full imbedding $U-\prod\mathbf{M}\subseteq\mathbf{S}^{\mathbf{N}^{\mathrm{op}}}$ commutes with such. Furthermore the functor $\mathbf{M}\xrightarrow{Y}U-\prod\mathbf{M}$, $M\mapsto U\text{-Hom}(-, M)$ is a full imbedding, since $\mathbf{N}$ is U-adequate in $\mathbf{M}$ (cf. II.7.)

3. *Extended Functor $U:U-\prod\mathbf{M}\to\mathbf{S}$.* $U:\mathbf{M}\to\mathbf{S}$ can be extended to a functor $U-\prod\mathbf{M}\to\mathbf{S}$ which we will call again U. We set $U(\prod_{i\in I}U\text{-Hom}(-, M(i)))=$ $=\prod_{i\in I}U\text{-Hom}(\langle E, e\rangle, M(i))\cong\prod_{i\in I}U(M(i))$. For a natural transformation α: $\prod_{i\in I}\text{Hom}(-, M(i))\to\prod_{j\in J}\text{Hom}(-, N(j))$ we determine $U\alpha$ by $U\alpha=\alpha(\langle E, e\rangle)$: $\prod_{i\in I}U\text{-Hom}(\langle E, e\rangle, M(i))\to\prod_{j\in J}U\text{-Hom}(\langle E, e\rangle, N(j))$. Notice that U: $U-\prod\mathbf{M}\to\mathbf{S}$ thus is the functor "evaluation at $\langle E, e\rangle$".

4. *Extended Functor $U\text{-Hom}(-,-):\mathbf{N}^{\mathrm{op}}\times U-\prod\mathbf{M}\to\mathbf{S}$.* The notions of U-objects and U-maps (cf. I.1., I.2.) in $U-\prod\mathbf{M}$ are determined by the definition of $U:U-\prod\mathbf{M}\to\mathbf{S}$ in 3. Thus it makes sense to talk about U-subcategories and U-den-

sity as well as U-adequacy. Any U-object (U-map) of $\mathbf{M}$ can be regarded as a U-object (U-map) of $U-\prod \mathbf{M}$ in view of $\mathbf{M} \overset{Y}{\subseteq} U-\prod \mathbf{M}$. U-subcategories of $\mathbf{M}$ thus become U-subcategories of $U-\prod \mathbf{M}$. A straight forward computation shows that for a U-object $\langle A, \vec{a}\rangle$ of $\mathbf{M}$ $U\text{-Hom}(\langle A, \vec{a}\rangle, -): U-\prod \mathbf{M} \to \mathbf{S}$ is given by $U\text{-Hom}(\langle A, \vec{a}\rangle, \prod_I U\text{-Hom}(-, M(i))) = \prod_I U\text{-Hom}(\langle A, \vec{a}\rangle, M(i))$ i.e. by "evaluation at $\langle A, \vec{a}\rangle$". The functor $U\text{-Hom}(-,-): \mathbf{N}^{\mathrm{op}} \times U-\prod \mathbf{M} \to \mathbf{S}$ thus is just the evaluation functor.

5. LEMMA. *The U-dense U-subcategory* $\mathbf{N}$ *of* $\mathbf{M}$ *is also U-dense in the category* $U-\prod \mathbf{M}$ *of U-direct products of* $\mathbf{M}$.

Proof. $\mathbf{N}$ is U-adequate in $U-\prod \mathbf{M}$, since the morphisms $\prod_I U\text{-Hom}(-, M(i)) \to \prod_J U\text{-Hom}(-, N(j))$ in $U-\prod \mathbf{M}$ coincide with the natural transformations

$$\begin{array}{ccc} U\text{-Hom}(-, \prod\limits_I U\text{-Hom}(-, M(i))) & \to & U\text{-Hom}(-, \prod\limits_J U\text{-Hom}(-, N(j))) \\ \| & & \| \\ \prod\limits_I U\text{-Hom}(-, M(i)) & \to & \prod\limits_J U\text{-Hom}(-, N(j)) \end{array}$$

between the corresponding U-Hom functors (cf. III.4.). The rest follows from lemma II.7. Q.E.D.

6. LEMMA. *Let* $\mathbf{N}$ *be a U-dense subcategory of* $\mathbf{M}$ *containing a cU-representing U-object $\langle E, e\rangle$ of $U{:}\mathbf{M} \to \mathbf{S}$. Then there is an equivalence of categories* $U-\prod \mathbf{M} \cong \prod s(\mathbf{M})$.

Proof. Recall that $s(\mathbf{M})$ is the image of the full imbedding $s{:}\mathbf{M} \to \mathbf{M}(L_{\mathbf{N}})$ (cf. theorem II.11.), and $\prod s(\mathbf{M})$ is the full subcategory of $\mathbf{M}(L_{\mathbf{N}})$ whose objects are the standard direct products of structures in $s(\mathbf{M})$. It follows readily that the functors

$$\left\{\begin{array}{l} F{:}U - \prod \mathbf{M} \to \prod s(\mathbf{M}) \\ \prod\limits_I U\text{-Hom}(-, M(i)) \overset{F}{\mapsto} \prod\limits_I s(M(i)) \end{array}\right\} \text{ and } \left\{\begin{array}{l} G{:}\prod s(\mathbf{M}) \to U - \prod \mathbf{M} \\ \prod\limits_I s(M(i)) \overset{G}{\mapsto} \prod\limits_I U\text{-Hom}(-, M(i)) \end{array}\right\}$$

are inverse up to natural equivalence. Q.E.D.

7. *Special Theories. For a special theory T the category $U-\prod \mathbf{M}(T)$ of U-direct products of models of T and U-transformations is equivalent to the category $\prod \mathbf{M}(T)$ of standard direct products of models of T and homomorphisms.*

Proof. According to lemma 6 it is left to be shown that $\prod \mathbf{M}(T) \simeq \prod s(\mathbf{M}(T))$. We know from II.12. that $s{:}\mathbf{M}(T) \subseteq \mathbf{M}(L_{\mathbf{N}(T)})$ is a full imbedding. Thus we have a diagram of full imbeddings

$$\begin{array}{ccc} \mathbf{M}(L(T)) & \supseteq & \mathbf{M}(T) \\ & & {\scriptstyle s}\ \wr\!| \\ \mathbf{M}(L_{\mathbf{N}(T)}) & \supseteq & s(\mathbf{M}(T)) \end{array}$$

It is easily seen that the equivalence s can be extended to an equivalence $\prod \mathbf{M}(T) \simeq \simeq \prod s(\mathbf{M}(T))$. Q.E.D.

8. COROLLARY. $\mathbf{N}(T)$ *is U-dense in* $\prod \mathbf{M}(T)$, *provided that* T *is special.*
Proof. Apply lemma III.6.

9. *U-Ultraproducts.* Let $M(i)$, $i \in I$, be a collection of objects in the category $\mathbf{M}$. Assume D is an ultrafilter on I. Let Δ be the following diagram in $U-\prod \mathbf{M}$: Its objects are the U-direct products $\prod_J U\text{-Hom}(-, M(j))$, $J \in D$. There is a morphism $p_{JK}: \prod_J U\text{-Hom}(-, M(j)) \to \prod_K U\text{-Hom}(-, M(k))$ in Δ for every pair $J, K \in D$ with $K \subset J$, namely the "projection". If the direct limit of Δ in $U-\prod \mathbf{M}$ exists, it will be called the U-ultraproduct $U-\prod_I M(i)/D$ of $\{M(i)\}_{i \in I}$. $\mathbf{M}$ is said to be closed with respect to U-ultraproducts if $U-\prod_I M(i)/D$ exists and is in $\mathbf{M}$ for any set I and any ultrafilter D on I and any family $\{M(i)\}_{i \in I}$ of objects in $\mathbf{M}$. We say that a U-representable functor $U\text{-Hom}(\langle A, \vec{a}\rangle, -)$, $\langle A, \vec{a}\rangle \in \mathbf{N}$, commutes with some U-ultraproduct $U-\prod_I M(i)/D$ if there is an isomorphism $U\text{-Hom}(\langle A, \vec{a}\rangle, U-\prod_I M(i)/D) \cong \cong \prod_I U\text{-Hom}(\langle A, \vec{a}\rangle, M(i))/D$. We say that $U\text{-Hom}(\langle A, \vec{a}\rangle, -)$ commutes with U-ultraproducts if there is a natural equivalence $U\text{-Hom}(\langle A, \vec{a}\rangle, U-\prod M(i)/D) \cong \cong \prod U\text{-Hom}(\langle A, \vec{a}\rangle, M(i)/D)$ for every U-ultraproduct $U-\prod M(i)/D$ in $\mathbf{M}$.

10. THEOREM. *Let* $\mathbf{N}$ *be a U-dense U-subcategory of* $\mathbf{M}$ *containing a cU-representing U-object* $\langle E, e\rangle$ *of* $U: \mathbf{M} \to \mathbf{S}$. *It follows that for some collection* $\{M(i)\}_{i \in I}$ *of objects in* $\mathbf{M}$, D *an ultrafilter on* I, $U\text{-}\prod_I M(i)/D$ *exists in* $\mathbf{M}$ *and U-representable functors commute with this U-ultraproduct if and only if the functor* $\prod_I U\text{-Hom}(-, M(i))/D$ *is U-corepresentable in* $\mathbf{M}$.

Proof. If $U-\prod_I M(i)/D$ exists in $\mathbf{M}$ and U-representable functors commute with it we can write $U\text{-Hom}(-, U-\prod_I M(i)/D) \cong \prod_I U\text{-Hom}(-, M(i))/D$ which proves one direction.

Conversely let $U\text{-Hom}(-, M) \cong \prod_I U\text{-Hom}(-, M(i))/D: \mathbf{N}^{op} \to \mathbf{S}$ be a natural equivalence. This implies that $U\text{-Hom}(-, M)$ in $\mathbf{S}^{\mathbf{N}^{op}}$ is the "ultraproduct in the category $\mathbf{S}^{\mathbf{N}^{op}}$". A fortiori it is the ultraproduct in the category $U-\prod \mathbf{M}$. Since M is in $\mathbf{M}$, we conclude $M \cong U-\prod_I M(i)/D$. That U-representable functors commute with this ultraproduct follows from the given natural equivalence. Q.E.D.

The following is a straight forward

11. COROLLARY. *Let* $\mathbf{N}$ *be U-dense in* $\mathbf{M}$ *and containing a cU-representing object* $\langle E, e\rangle$ *of* $U: \mathbf{N} \to \mathbf{S}$. $\mathbf{M}$ *is closed with respect to U-ultraproducts and U-representing functors commute with them if and only if ultraproducts of U-corepresentable functors are U-corepresentable.*

12. LEMMA. *$s:\mathbf{M}\to\mathbf{M}(L_{\mathbf{N}})$ carries U-ultraproducts into standard ultraproducts.*
Proof.

$$\begin{array}{rl} s: U - \prod_I M(i)/D \mapsto & U\text{-Hom}(-, U - \prod_I M(i)/D) \\ & \wr\Vert \\ & \prod_I U\text{-Hom}(-, M(i))/D \\ & \wr\Vert \\ & \prod_I s(M(i))/D \end{array}$$

Q.E.D.

13. THEOREM. *Let $\mathbf{N}$ be a U-dense subcategory of $\mathbf{M}$ containing a cU-representing U-object $\langle E, e\rangle$ of $U:\mathbf{M}\to\mathbf{S}$. Ultraproducts of U-corepresentable functors $\mathbf{N}^{\mathrm{op}}\to\mathbf{S}$ are again U-corepresentable if and only if the full imbedding* (cf. II.11.) *$s:\mathbf{M}\mapsto\to\mathbf{M}(L_{\mathbf{N}})$ sends $\mathbf{M}$ onto a subcategory $s(\mathbf{M})$ of $\mathbf{M}(L_{\mathbf{N}})$ which is closed with respect to standard ultraproducts.*

Proof. Since $\mathbf{M}$ is closed with respect to U-ultraproducts (cf. corollary 11) $s(\mathbf{M})$ is closed with respect to standard ultraproducts (cf. lemma 12).

Conversely, $s(\mathbf{M})$ being closed with respect to standard ultraproducts means that $\prod_I U\text{-Hom}(-, M(i))/D \cong s(M) = U\text{-Hom}(-, M)$ for an appropriate $M\in\mathbf{M}$. Thus it follows that ultraproducts of U-corepresentable functors are again U-corepresentable.

Q.E.D.

14. *Special Theories. For a special theory T the U-ultraproducts in $\mathbf{M}(T)$ coincide with the standard ultraproducts.*

Proof. This follows from theorem III.13. and II.12. Q.E.D.

15. THEOREM. *Let L be any first order theory without nonlogical axioms. Let $\mathbf{K}\subseteq\mathbf{M}(L)$ be any full subcategory. By $\tilde{\mathbf{K}}$ we understand the full subcategory of $\mathbf{M}(L)$, whose objects are isomorphic to some object in $\mathbf{K}$. It follows that for $\mathbf{K}\subseteq\mathbf{M}(L)$ the closure $\tilde{\mathbf{K}}$ is equal to $\mathbf{M}(T)\subseteq\mathbf{M}(L)$ for an appropriate theory T if and only if $\mathbf{K}$ is closed with respect to standard ultraproducts and the full subcategory whose objects are in $\mathrm{Ob}(\mathbf{M}(L))-\mathrm{Ob}(\tilde{\mathbf{K}})$ is closed with respect to ultrapowers.*

This theorem can be found for example in [K] (theorem 2.8). It depends upon Keisler's ultrapower theorem saying that two structures are elementarily equivalent if and only if they have isomorphic ultrapowers (cf. [K], theorem 2.6.). The proof of the latter uses the generalized continuum hypothesis. But this is not necessary according to Shelah (cf. [Sh]).

16. *Special Theories. Any contravariant functor $G\in\mathbf{S}^{\mathbf{N}^{\mathrm{op}}}$ such that some ultrapower $G^I/D\in\mathbf{S}^{\mathbf{N}^{\mathrm{op}}}$ is U-corepresentable in $\mathbf{M}(T)$ is itself U-corepresentable in $\mathbf{M}(T)$, provided that T is a special theory.*

Proof. We know that $G^I/D \cong U\text{-Hom}(-, M): \mathbf{N}(T) \to \mathbf{S}$, and there is a natural imbedding

$$G \subseteq G^I/D.$$

Consider the $L(T)$-structure N whose underlying set UN consists of all elements in the image of $G(\langle E, e\rangle) \subseteq G(\langle E, e\rangle)^I/D \cong U\text{-Hom}(\langle E, e\rangle, M) = UM$.

For any U-object $\langle A, \vec{a}\rangle$ in $\mathbf{N}$ it follows by naturality that the composition $G\langle A, \vec{a}\rangle \subseteq G\langle A, \vec{a}\rangle^I/D \cong U\text{-Hom}(\langle A, \vec{a}\rangle, M) \cong p_{\langle A,\vec{a}\rangle} M \subseteq (UM)^n$ factors through $(UN)^n \subseteq (UM)^n$.

We set $p_{\langle A,\vec{a}\rangle}(N)$ = image of $G(\langle A, \vec{a}\rangle)$ in $(UN)^n$.

Thus we get an injective homomorphism of $N \to M$, making the following diagram commutative

$$\begin{array}{ccc} G(\langle A, \vec{a}\rangle) & \cong & P_{\langle A,\vec{a}\rangle}(N) \\ \cap\| & & \cap| \\ G(\langle A, \vec{a}\rangle)^I/D & \cong & P_{\langle A,\vec{a}\rangle}(M) \end{array}$$

Hence $N^I/D = M$. According to theorem III.15, N is in $\mathbf{M}(T)$. Thus we get an isomorphism $G(\langle A, \vec{a}\rangle) \cong p_{\langle A,\vec{a}\rangle}(N) \cong U\text{-Hom}(\langle A, \vec{a}\rangle, N)$ which is natural in $\langle A, \vec{a}\rangle$.

Q.E.D.

IV. Ultradense U-Subcategories

1. DEFINITION. Let $\mathbf{M}$ be a category with underlying set functor $U: \mathbf{M} \to \mathbf{S}$. A U-subcategory $\mathbf{N}$ of $\mathbf{M}$ is called ultradense in $\mathbf{M}$ if

(a) $\mathbf{N}$ is U-dense in $\mathbf{M}$;

(b) $U: \mathbf{M} \to \mathbf{S}$ is cU-representable by a U-object $\langle E, e\rangle$ in $\mathbf{N}$;

(c) ultraproducts of U-corepresentable functors in $\mathbf{S}^{\mathbf{N}^{\mathrm{op}}}$ are again U-corepresentable;

(d) Any functor G in $\mathbf{S}^{\mathbf{N}^{\mathrm{op}}}$ for which some ultrapower G^I/D is U-corepresentable is itself U-corepresentable.

Notice that (c) can be replaced by the condition that $\mathbf{M}$ is closed with respect to U-ultraproducts and U-representable functors carry them into ultraproducts (cf. III.11).

2. LEMMA. *The category* $\mathbf{M}(T)$ *of models of some special theory* T *has an ultradense subcategory, namely* $\mathbf{N}(T)$ (cf. II.12).

Proof. IV.1 (a) follows from II.9; (b) follows from I.5 and I.8; (c) follows from III.11, 14 and 15 and (d) follows from III.16. Q.E.D.

3. *Dense Orders Without Extreme Elements.* Let $\mathbf{M}$ be as in I.4. It follows from

lemma 2 that the U-subcategory $\mathbf{N}$ of $\mathbf{M}$ (cf. II.5) consisting of the U-objects $\langle Q, 0\rangle$, $\langle Q, 0,1\rangle$ and all possible U-maps between them is an ultradense subcategory of $\mathbf{M}$.

4. $\mathbf{M}(T)$ *Closed with Respect to Standard Direct Products.* If T is such that $\mathbf{M}(T)$ is closed with respect to standard direct products, there exists an ultradense U-subcategory $\mathbf{N}$ of $\mathbf{M}(T)$. E.g. if T is an equational algebraic theory in the sense of Birkhoff with finitary operations of if T is a universal Hom theory (cf. [Kn]). Notice that for the special theory T of dense order without extreme elements (cf. IV.3) $\mathbf{M}(T)$ is not closed with respect to standard direct products.

5. THEOREM. *Any category* $\mathbf{M}$ *with underlying set functor* $U: \mathbf{M}\to\mathbf{S}$ *is equivalent to the category* $\mathbf{M}(T)$ *(with a compatible underlying set functor* $U(T):\mathbf{M}(T)\to\mathbf{S}$), *for an appropriate special theory* T *if and only if* $\mathbf{M}$ *has an ultradense subcategory* $\mathbf{N}$.

Proof. One direction is proved by lemma IV.2.

Conversely, assume that $\mathbf{M}$ has an ultradense subcategory $\mathbf{N}$. Since $\mathbf{N}$ is U-dense in $\mathbf{M}$ (cf. IV.1 (a)) it follows that $s:\mathbf{M}\to\mathbf{M}(L_{\mathbf{N}})$ is a full imbedding (cf. II.11). In order to show that $\mathbf{M}$ is equivalent to $\mathbf{M}(T)$, where T is a theory with the same language as $L_{\mathbf{N}}$ is, it suffices to show that $s(\mathbf{M})$ in $\mathbf{M}(L_{\mathbf{N}})$ is closed with respect to standard ultraproducts and the full subcategory of $\mathbf{M}(L_{\mathbf{N}})$ whose objects are those $L_{\mathbf{N}}$-structures in $\mathbf{M}(L_{\mathbf{N}})$ which are not in $s(\mathbf{M})$ is closed with respect to standard ultrapowers (cf. theorem III.15). Property 1 (c) guarantees that $s(\mathbf{M})$ is closed with respect to standard ultraproducts, according to theorem III.13.

It remains to be shown that the full subcategory whose objects are in $\mathbf{M}(L_{\mathbf{N}})-\overline{s(\mathbf{M})}$ is closed with respect to standard ultrapowers. Let $N\in\mathbf{M}(L_{\mathbf{N}})$ be any $L_{\mathbf{N}}$-structure such that for some ultrafilter D on I the standard ultrapower N^I/D is in $\overline{s(\mathbf{M})}$. I.e. there is some object $M\in\mathbf{M}$ such that $s(M)\cong N^I/D$. Let G in $\mathbf{S}^{\mathbf{N}^{\mathrm{op}}}$ be the following subfunctor of the U-corepresentable functor $U\text{-Hom}(-, M):\mathbf{N}\to\mathbf{S}$. $G(\langle A, \vec{a}\rangle)\subseteq$ $\subseteq U\text{-Hom}\langle A, \vec{a}\rangle, M)$ consists of those U-maps $[f]:\langle A, \vec{a}\rangle\to M$ such that $f(\vec{a})$ is in the image of $p_{\langle A,\vec{a}\rangle}(N)\subseteq p_{\langle <A,\vec{a}\rangle}(N^I/D)\cong p_{\langle A,\vec{a}\rangle}(M)$. It follows that $G^I/D\cong$ $=U\text{-Hom}(-, M)$. Property 1 (d) guarantees that G itself is U-corepresentable by some object $N'\in\mathbf{M}$. Since $U\text{-Hom}(\langle A, \vec{a}\rangle, N')=G(\langle A, \vec{a}\rangle)\cong p_{\langle A,\vec{a}\rangle}(N)$ it follows that $s(N')=N$. Hence $N\in\overline{s(\mathbf{M})}$. Thus we know that $\mathbf{M}\simeq\mathbf{M}(T)\subseteq\mathbf{M}(L_{\mathbf{N}})$. The theory T is special because the predicate constants are canonically U-representable by the U-objects of the form $\langle sA, \vec{a}\rangle$, where $\langle A, \vec{a}\rangle\in\mathbf{N}$. (cf. also lemma I.8). Q.E.D.

V. Locally Atomistic Theories

1. *F-axiomatized Theories* (cf. [F2] III.22). Let T be a theory. The set of its predi-

cate constants will be called F. T will be called F-axiomatized if the axioms of T can be written in the following form

I. universal sentences;

II. equivalences of the form $\forall \vec{x}(P(\vec{x}) \Leftrightarrow Q(\vec{x}))$ where $P(\vec{x})$ is a conjunction of predicate constants and $Q(\vec{x})$ arises from predicate constants by applying conjunction, disjunction and existential quantification.

Furthermore, for all formulas $Q_i(\vec{x})$ in the inductive build up of $Q(\vec{x})$ there is an axiom (or sentence) of the form $\forall \vec{x}(P_i(\vec{x}) \Leftrightarrow Q_i(\vec{x}))$ in T, where $P_i(\vec{x})$ is a conjunction of predicate constants.

2. *$T(N, F)$-types* (cf. also [F2] II.10). Let N be any $L(T)$-structure. By $T(N, F)$ we understand the theory T together with one new individual constant a for each element a of N and the additional axioms $p(a_1, \ldots, a_n)$ for any predicate constant p and n-tuple $(a_1, \ldots, a_n)$ for which $N \models p(a_1, \ldots, a_n)$ holds.

By an m-ary $T(N, F)$-type $I(x_1, \ldots, x_m)$ we understand a set of formulas $p(a_1, \ldots, a_k, y_1, \ldots, y_l)$, where p is a predicate constant, $a_1, \ldots, a_k$ are individual constants of $T(N, F)$ and $y_1, \ldots, y_l$ are some of the variables $x_1, \ldots, x_m$. Furthermore, I has to be consistent with $T(N, F)$, otherwise we call I a virtual $T(N, F)$-type.

A principal n-ary $T(N, F)$-type or $T(N, F)$-character is a $T(N, F)$-type which is realized by some n-tuple $(b_1, \ldots, b_n)$ of elements in each model of $T(N, F)$. An n-ary $T(N, F)$-type I is called real if there is a model M of T and an n-tuple of elements $b_1, \ldots, b_n$ such that the $T(N, F)$-type of $(b_1, \ldots, b_n)$ in M is precisely I.

An n-ary $T(N, F)$-atom is an n-ary principal $T(N, F)$-type which is maximal (i.e. not properly contained in another principal $T(N, F)$-type) and which is real.

If N is the empty structure the $T(N, F)$-types are called T-types.

According to [F2] III.28 we have

3. THEOREM. *The following statements are equivalent*:

(a) *T is F-axiomatized*

(b) $\mathbf{M}(T)$ *is closed with respect to direct limits of directed systems of models and homomorphisms*

(c) *T can be axiomatized in the way that an $L(T)$-structure N is a model of T if and only if*:

(a) *The unary principal $T(N, F)$-types are realized in N.*

(b) *For any finite substructure $M \subseteq N$ and any element $m \in N$ the virtual $T(M, F)$-type of m is a real $T(M, F)$-type.*

4. *Atomistic theories.* A theory T is called atomistic if any principal T-type is contained in a T-atom.

5. *Prime models* (cf. also [F1] I.19). A prime model 0 of a theory T is a model 0 which permits a homomorphism $0 \to M$ for any model M of T.

6. THEOREM (cf. also [F1] IV.17). *An F-axiomatized theory T with only finitely many predicate constants is atomistic if and only if it has a denumerable (or finite) prime model* 0.

Proof. Assume first that 0 is a denumerable prime model of T. Consider all n-tuples $\vec{a}(k)$, $k \in K$ of elements in 0, which realize a certain given n-ary principal T-type I. Let I_k be their respective T-types. Since there are only finitely many predicate constants there are only finitely many distinct n-ary T-types I_k. They are partially ordered by inclusion. This order must have some maximal element, say J. It follows immediately that J is a T-atom containing I.

Conversely, we assume that T is atomistic in order to construct a prime model which is at most denumerable.

First we add a denumerable set $\{c_i\}_{i<\omega}$ of new individual constants to the language of T, thus yielding a new theory S.

Let $\vec{c}_1, \vec{c}_2, \ldots$ be a denumeration of all finite tuples of new individual constants in S.

Let $I_1(x), I_2(x), \ldots$ be a denumeration of all finite unary S-types. We are going to define inductively a sequence $S_0, S_1, \ldots, S_n, \ldots, n<\omega$ of theories such that S_{n+1} extends S_n. Set $S_0 = S$.

Assuming that S_n is already determined consider the finite unary S-types $I_i(x)$, $0<i\leqslant n$. If I_i is a principal S_n-type add the formulas $I_i(c_{k(i)})$ to S_n as new axioms, where $k(i)$ is the smallest index such that $c_{k(i)}$ does not occur in the nonlogical axioms of S_n or in the axioms $I_l(c_{k(l)})$, $0<l<i$.

Now let $I(\vec{x})$ be the set of predicate constants $p(\vec{x})$ such that $p(\vec{c})$ is a nonlogical axiom of S_n or $p(\vec{c})$ is in one of the $I_i(c_{k(i)})$'s, $0<i\leqslant n$. Thus $I(\vec{x})$ is a principal S-type. Let $(d_1, \ldots, d_m) = \vec{d}$ consist of those c_k's in $\vec{c}_{n+1}$ which do not already occur in $I(\vec{c})$. Let $J(\vec{x}, y_1, \ldots, y_m)$ be an atom which contains $I(\vec{x})$. S_{n+1} is defined then to be S_n together with all formulas of $J(\vec{c}, \vec{d})$.

It follows that S_n is a principal extension of T.

The union $S_\omega = \bigcup_{n<\omega} S_n$ is consistent; it is even a principal extension of T. The following structure 0 can be proved to be a prime model of T. Its underlying set is the set $\{c_i\}_{i<\omega}$ modulo the relation $c_i \sim c_j$ if $S_\omega \vdash c_i = c_j$. The predicate constants are determined on 0 by

$$0 \models p(\vec{c}) \Leftrightarrow S_\omega \vdash p(\vec{c}).$$

Notice that the finiteness condition on T guarantees that T-types are equivalent to finite conjunctions of predicate constants. This is useful for showing that S_ω is a principal extension of T, and that 0 is prime. For the proof that 0 is actually a model of T one has to use theorem 3 ((a) and (c)). Q.E.D.

7. *Locally Atomistic Theories.* Let T be a first order theory. For any predicate constant p of T, let $T_{p(\vec{c})}$ be the theory T together with a finitary tuple $\vec{c}$ of new individual constants and the new axiom $p(\vec{c})$.

T is called locally atomistic if $T_{p(\vec{c})}$ is atomistic for each predicate constant p of T.

According to theorem 6 we know that an F-axiomatized theory T with only finitely many predicate constants is locally atomistic if and only if the theories $T_{p(\vec{c})}$ have prime models. But this means that the predicate constants are canonically U-representable (cf. I.5).

From lemma I.8 we can thus conclude

8. LEMMA. *An F-axiomatized theory T with only finitely many predicate constants is special if and only if it is locally atomistic.*

9. THEOREM. *A category* $\mathbf{M}$ *with underlying set functor* $U\colon \mathbf{M} \to \mathbf{S}$ *is equivalent to the category* $\mathbf{M}(T)$ (compatible with $U(T)\colon \mathbf{M}(T) \to \mathbf{S}$) *for an F-axiomatized locally atomistic theory T with only finitely many predicate constants if and only if* $\mathbf{M}$ *is closed with respect to direct limits of directed systems and has a finite ultradense U-subcategory.*

Proof. Apply theorem IV.5, theorem V.3 and lemma V.8. Q.E.D.

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