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Hadamard Products of Schlicht Functions and the Pólya-Schoenberg Conjecture[1])

by ST. RUSCHEWEYH and T. SHEIL-SMALL

Introduction

A function $\varphi(z)$ is said to be *convex* if it is a schlicht conformal mapping of the unit disc $|z|<1\}$ onto a convex domain. $f(z)$ is said to be *starlike* if it is a schlicht conformal mapping of the disc onto a domain starlike with respect to the origin. Throughout the paper we shall assume that our convex and starlike functions vanish at $z=0$. The *Hadamard product* or *convolution* of two power series $f(z)=\sum_0^\infty a_n z^n$ and $g(z)=\sum_0^\infty b_n z^n$ is defined as the power series $(f*g)(z)=\sum_0^\infty a_n b_n z^n$. In 1958 G. Pólya and I. J. Schoenberg [2] made the following conjecture.

(0.1). *If $f(z)$ and $g(z)$ are convex, then so is $(f*g)(z)$.*

That the convolution is schlicht (and in fact close-to-convex) was first shown by Suffridge [11]. Pólya and Schoenberg themselves showed that the conjecture was true in several special cases, and after Suffridge other special cases of the following proposition were established.

(0.2). *If $\varphi(z)$ is convex and $f(z)$ close-to-convex, then $(\varphi*f)(z)$ is close-to-convex.*

In this paper we shall establish the truth of both (0.1) and (0.2). Our methods enable us to obtain a number of other similar results and to settle two other conjectures. One of these is an interesting subordination conjecture of Wilf [13] stronger than the Pólya-Schoenberg conjecture, yet nevertheless still true (see section 4). Our main proof of (0.1) and (0.2) appears in section 2, but the success of the method requires a careful study of certain geometric properties of convex and starlike functions and the expression of these in analytic terms. This work appears in section 1.

Finally mention should be made of the Mandelbrojt-Schiffer conjecture also appearing in Pólya and Schoenberg's paper, that if $\sum_1^\infty a_n z^n$ and $\sum_1^\infty b_n z^n$ are schlicht in $|z|<1$, then $\sum_1^\infty (a_n b_n/n)\, z^n$ also is schlicht in $|z|<1$. This has been disproved on many occasions and it is not even true that $\sum_1^\infty a_n z^n$ schlicht implies $\sum_1^\infty (a_n/n\, z^n)$ schlicht. In

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[1]) This work was done whilst the second author was on leave from the University of York, England, and was a guest of the University of Bonn, Germany, and was supported by the Sonderforschungsbereich 40 (Theoretische Mathematik), Bonn.

particular one cannot replace $f(z)$ in (0.2) by an arbitrary schlicht function. In section 5.3 the correct generalisation of (0.2) is briefly discussed.

1. Structural Inequalities for Starlike and Convex Functions

(1.1). It is well-known that a function $f(z)=\sum_1^\infty a_n z^n$ $(a_1 \neq 0)$ is starlike univalent in the unit disc if, and only if,

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0 \quad (|z|<1), \tag{1.1.1}$$

and that $\varphi(z)$ is convex if, and only if,

$$\mathrm{Re}\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right) > 0 \quad (|z|<1), \tag{1.1.2}$$

and $\varphi'(0)\neq 0$. It is then clear that $\varphi(z)$ is convex if, and only if, $z\varphi'(z)$ is starlike. Geometrically the conditions (1.1.1) and (1.1.2) are local in nature and are analytic formulations of the fact that for a starlike function f, $w=f(z)$ turns monotonically about the origin as z traverses the circle $|z|=r$ $(0<r<1)$, and that for a convex function φ the tangent vector increases monotonically. The global geometric structure of the image domains of the functions is therefore implicitly determined by these conditions. However it turns out that in the present problem it is essential to formulate analytic conditions of a more explicitly global nature.

(1.2). The first condition of this type which we shall need is known [10, 12] but for the sake of completeness we include the very short and simple proof of Suffridge [12].

(1.3). THEOREM. *If $\varphi(z)$ is convex in $|z|<1$, then for each z_0 $(|z_0|<1)$ the function*

$$z\left(\frac{\varphi(z)-\varphi(z_0)}{z-z_0}\right)^2 \tag{1.3.1}$$

is starlike.

Proof. The image D of $\varphi(z)$ being convex is in particular star-shaped relative to the point $\varphi(z_0)$, and hence if $|z|>|z_0|$

$$\mathrm{Re}\,\frac{z\varphi'(z)}{\varphi(z)-\varphi(z_0)} > 0. \tag{1.3.2}$$

Now if $h(z)$ denotes the function (1.3.1), we have

$$\frac{2z\varphi'(z)}{\varphi(z)-\varphi(z_0)}-\frac{z+z_0}{z-z_0}=\frac{zh'(z)}{h(z)}, \tag{1.3.3}$$

and this is analytic in z and z_0. By (1.3.2) it has non-negative real part for $|z|=|z_0|$. The maximum principle immediately implies that (1.3.3) has positive real part for every z and z_0 and the conclusion follows.

It can also be shown that the condition (1.3.1) is an analytic formulation of the geometrical fact that the chord from $\varphi(z_0)$ to $\varphi(z)(|z|=|z_0|)$ turns monotonically as the circle is traversed (see [10]). The condition of the theorem is also of course sufficient for convexity.

(1.4). A function $f(z)=\sum_1^\infty a_nz^n\,(a_1\neq 0)$ is said to be starlike of order $\frac{1}{2}$ if

$$\operatorname{Re}\frac{zf'(z)}{f(z)}>\tfrac{1}{2}\qquad(|z|<1). \tag{1.4.1}$$

This condition has no immediate geometrical interpretation, but is convenient to handle analytically and has a close relation to the general class of starlike functions. It is immediately verified that $g(z)$ is starlike if, and only if, $f(z)=z\sqrt{(g(z)/z)}$ is starlike of order $\frac{1}{2}$. Also

$$h(z)=h_1z+h_3z^3+h_5z^5+\cdots$$

is an odd starlike function if, and only if,

$$f(z)=h_1z+h_3z^2+h_5z^3+\cdots$$

is starlike of order $\frac{1}{2}$. If we put $z_0=0$ in theorem 1.3 we see that every convex function φ is starlike of order $\frac{1}{2}$, and in general for each $z_0\,(|z_0|<1)$

$$z\frac{\varphi(z)-\varphi(z_0)}{z-z_0}\qquad(|z|<1) \tag{1.4.2}$$

is starlike of order $\frac{1}{2}$. We shall require the following more general condition than (1.4.1).

(1.5). THEOREM. *$f(z)$ is starlike of order $\frac{1}{2}$ if, and only if,*

$$\operatorname{Re}\left(\frac{f(z_1)-f(z_2)}{z_1-z_2}\cdot\frac{z_2}{f(z_2)}\right)>\tfrac{1}{2} \tag{1.5.1}$$

for $|z_1|<1$, $|z_2|<1$.

Proof. The sufficiency is clear putting $z_1=z_2$. For the necessity, let $g(z)=(f(z))^2/z$, so that $g(z)$ is starlike. Consider the principal branch of $\arg(1-\sigma)$ for $|\sigma|<1$ and observe that it extends continuously to $|\sigma|=1$, $\sigma\neq 1$, to give

$$\arg(1-e^{i\varphi}) = -\frac{\pi}{2}+\frac{\varphi}{2} \qquad (0<\varphi<2\pi).$$

Writing $z=re^{i\theta}$ we have

$$\begin{aligned}
&\arg\left\{(1-e^{i\varphi})\frac{f(ze^{i\varphi})}{ze^{i\varphi}}\cdot\frac{z}{f(z)}\right\}\\
&=\tfrac{1}{2}\left\{-\pi+\varphi+\arg\frac{g(re^{i(\theta+\varphi)})}{re^{i(\theta+\varphi)}}-\arg\frac{g(re^{i\theta})}{re^{i\theta}}\right\}\\
&=\tfrac{1}{2}\{-\pi+\arg g(re^{i(\theta+\varphi)})-\arg g(re^{i\theta})\}
\end{aligned}$$

and since g is starlike this expression does not exceed $\pi/2$ in absolute value if $0<\varphi<2\pi$. Thus

$$\operatorname{Re}\left\{(1-\sigma)\frac{f(z\sigma)}{z\sigma}\cdot\frac{z}{f(z)}\right\}>0 \qquad (|z|<1) \tag{1.5.2}$$

for $|\sigma|=1$, $\sigma\neq 1$. Consider now $|z_1|=|z_2|<1$ and $z_1\neq z_2$, so that for some σ satisfying $|\sigma|=1$, $\sigma\neq 1$, we have $z_2=z_1\sigma$. We then obtain

$$\begin{aligned}
\operatorname{Re}\left(\frac{f(z_1)-f(z_2)}{z_1-z_2}\cdot\frac{z_2}{f(z_2)}\right) &= -\operatorname{Re}\frac{\sigma}{1-\sigma}+\operatorname{Re}\left(\frac{1}{1-\sigma}\,\frac{z_1\sigma}{f(z_1\sigma)}\cdot\frac{f(z_1)}{z_1}\right)\\
&> -\operatorname{Re}\frac{\sigma}{1-\sigma}=\tfrac{1}{2}
\end{aligned}$$

applying (1.5.2). The proof is now completed by application of the maximum principle.

(1.6). If $\varphi(z)$ is convex, the function (1.4.2) is starlike of order $\frac{1}{2}$ and so we can apply theorem 1.5 and obtain a three point condition for convexity which for our purposes is conveniently expressed in the following form. *If $\varphi(z)$ is convex, then for all complex numbers σ_1, σ_2 and σ_3 satisfying $|\sigma_k|\leqslant 1$ $(k=1, 2, 3)$ we have*

$$\operatorname{Re}\frac{\varphi(z)*z(1-\sigma_1 z)^{-1}(1-\sigma_2 z)^{-1}(1-\sigma_3 z)^{-1}}{\varphi(z)*z(1-\sigma_1 z)^{-1}(1-\sigma_2 z)^{-1}}>\tfrac{1}{2} \qquad (|z|<1). \tag{1.6.1}$$

The condition (1.5.1) can be similarly translated and we obtain: *if $f(z)$ is starlike of* *order* $\frac{1}{2}$, *then for any pair* σ_1, σ_2 *satisfying* $|\sigma_1| \leqslant 1$, $|\sigma_2| \leqslant 1$, *we have*

$$\operatorname{Re} \frac{f(z) * z(1-\sigma_1 z)^{-1}(1-\sigma_2 z)^{-1}}{f(z) * z(1-\sigma_1 z)^{-1}} > \tfrac{1}{2} \quad (|z|<1). \tag{1.6.2}$$

(1.7). We shall also need a relation of a different type for the starlike functions $f(z)$. This is based on the geometrical observation that a domain D starlike with respect to the origin is characterised by the property that every straight line through the origin intersects D in a single connected segment (finite or infinite). In the terminology of Robertson [5] the domain is "starlike in every direction". In order to give a sharp characterisation of this property in analytical terms a few boundary properties of starlike functions are required.

Let $f(z)$ be starlike and let

$$V(t) = \lim_{r \to 1} \arg f(re^{it}). \tag{1.7.1}$$

The limit $V(t)$ exists for all real t and has the following properties.

$$V(t) \text{ is monotonic increasing with } t. \tag{1.7.2}$$

$$V(t+2\pi) = V(t) + 2\pi \quad \text{for every } t. \tag{1.7.3}$$

$$V(t) = \tfrac{1}{2}(V(t+0) + V(t-0)) \quad \text{for every } t. \tag{1.7.4}$$

These results are established for example in [3]. We now set

$$\left.\begin{aligned} h(t) &= \inf\{s : V(s) \geqslant V(t) + \pi\} \\ k(t) &= \sup\{s : V(s) \leqslant V(t) + \pi\}. \end{aligned}\right\} \tag{1.7.5}$$

It is immediately verified that $h(t)$ and $k(t)$ are non-decreasing functions of t which satisfy

$$h(t+2\pi) = h(t) + 2\pi, \qquad k(t+2\pi) = k(t) + 2\pi \tag{1.7.6}$$

$$t \leqslant h(t) \leqslant k(t) \leqslant t + 2\pi \tag{1.7.7}$$

for every real t. We then have

(1.8). THEOREM. *For each real* t *let* t^* *denote any number satisfying* $h(t) \leqslant t^* \leqslant k(t)$. *Then*

$$\operatorname{Im}\left\{ e^{-iV(t)} e^{\frac{1}{2}i(t+t^*)} (1 - ze^{-it})(1 - ze^{-it^*}) \frac{f(z)}{z} \right\} \geqslant 0 \tag{1.8.1}$$

in $|z|<1$.

Proof. For fixed t we consider the function

$$W(\theta) = \lim_{r\to 1} \arg\left\{e^{-iV(t)}\, e^{\frac{1}{2}i(t+t^*)} \frac{f(re^{i\theta})}{re^{i\theta}} (1 - re^{i(\theta-t)})(1 - re^{i(\theta-t^*)})\right\}. \tag{1.8.2}$$

$W(\theta)$ is periodic in θ with period 2π and we aim to show that

$$0 \leqslant W(\theta) \leqslant \pi. \tag{1.8.3}$$

To do this it is sufficient to consider the range $t \leqslant \theta \leqslant t + 2\pi$. We consider four cases:

(i) $\theta = t$. Then

$$\lim_{r\to 1} \arg\{(1 - re^{i(\theta-t)})(1 - re^{i(\theta-t^*)})\} = \begin{cases} \frac{1}{2}(t - t^*) + \frac{\pi}{2} & \text{if} \quad t < t^* < t + 2\pi \\ 0 \quad \text{if} \quad t^* = t & \text{or} \quad t^* = t + 2\pi \end{cases}$$

Thus

$$W(\theta) = \begin{cases} \frac{\pi}{2} & \text{if} \quad t < t^* < t + 2\pi \\ 0 & \text{if} \quad t^* = t \\ \pi & \text{if} \quad t^* = t + 2\pi. \end{cases}$$

(ii) $t < \theta < t^*$. Then

$$\lim_{r\to 1} \arg\{(1 - re^{i(\theta-t)})(1 - re^{i(\theta-t^*)})\} = \theta - \tfrac{1}{2}(t + t^*)$$

and hence $W(\theta) = V(\theta) - V(t)$. Since $V(\theta)$ is increasing, $W(\theta) \geqslant 0$. Since $\theta < t^* \leqslant k(t)$, $V(\theta) \leqslant V(t) + \pi$. Thus (1.8.3) follows in this case.

(iii) $\theta = t^*$. If $t^* = t$ or $t + 2\pi$ then (1.8.3) follows from case (i). Thus we may assume that $t < t^* < t + 2\pi$. Then

$$\lim_{r\to 1} \arg\{(1 - re^{i(\theta-t)})(1 - re^{i(\theta-t^*)})\} = \tfrac{1}{2}(t^* - t) - \frac{\pi}{2},$$

which gives $W(t^*) = -V(t) + V(t^*) - \pi/2$. Now if $h(t) < t^* < k(t)$, then $V(t^*) = V(t) + \pi$ and so $W(t^*) = \pi/2$. If $h(t) = t^* < k(t)$, then $V(t^*) \leqslant V(t) + \pi$ and so $W(t^*) \leqslant \pi/2$. Also $V(t^* + 0) \geqslant V(t) + \pi$ and $V(t^* - 0) \geqslant V(t)$ and therefore applying (1.7.4), $V(t^*) \geqslant V(t) + \pi/2$, which gives $W(t^*) \geqslant 0$. If $h(t) < t^* = k(t)$, we obtain in a similar manner $\pi/2 \leqslant W(t^*) \leqslant \pi$. Lastly, if $h(t) = k(t) = t^*$, then

$$V(t) + \pi \leqslant V(t^* + 0) \leqslant V(t) + 2\pi$$
$$V(t) \leqslant V(t^* - 0) \leqslant V(t) + \pi$$

and applying (1.7.4) we obtain $0 \leqslant W(t^*) \leqslant \pi$.

(iv) $t^* < \theta < t+2\pi$. Then

$$\lim_{r\to 1} \arg\{(1 - re^{i(\theta-t)})(1 - re^{i(\theta-t^*)})\} = \theta - \tfrac{1}{2}(t + t^*) - \pi$$

and so $W(\theta) = V(\theta) - V(t) - \pi$. Since $\theta > h(t)$, $V(\theta) \geqslant V(t) + \pi$ and so $W(\theta) \geqslant 0$. Also $V(\theta) \leqslant V(t+2\pi) = V(t) + 2\pi$ so $W(\theta) \leqslant \pi$.

We now have (1.8.3) in all cases. Let

$$v(z) = \frac{1}{2\pi} \int_0^{2\pi} \left(\mathrm{Re}\, \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}}\right) W(\theta)\, d\theta.$$

By Poisson's formula we have for $|z| < R < 1$

$$\arg\left\{e^{-iV(t)}\, e^{\frac{1}{2}i(t+t^*)}\,(1 - ze^{-it})(1 - ze^{-it^*})\frac{f(z)}{z}\right\}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{\mathrm{Re}^{i\theta} + z}{\mathrm{Re}^{i\theta} - z}\right) \arg\left\{e^{-iV(t)}\, e^{\frac{1}{2}i(t+t^*)}\,(1 - \mathrm{Re}^{i(\theta-t)})\right.$$

$$\left.\times (1 - \mathrm{Re}^{i(\theta-t^*)})\frac{f(\mathrm{Re}^{i\theta})}{\mathrm{Re}^{i\theta}}\right\} d\theta$$

$\to v(z)$ as $R \to 1$ by the Lebesgue bounded convergence theorem. From (1.8.3) we have $0 \leqslant v(z) \leqslant \pi$ and the theorem follows.

(1.9). This result leads to new coefficient inequalities for starlike functions, but these will not concern us here. If the result is applied to odd starlike functions ($t^* = t + \pi$) the condition (1.5.1) for functions starlike of order $\frac{1}{2}$ again emerges. The theorem is used at a crucial moment in the proof of Pólya and Schoenberg's conjecture, although not in quite the sharp form which we have obtained here.

2. The Conjecture of Pólya and Schoenberg

(2.1). THEOREM. *Let $\varphi(z)$ and $\psi(z)$ be convex univalent functions in the unit disc. Then $(\varphi * \psi)(z)$ is convex univalent in this disc.*

(2.2). THEOREM. *Let $\varphi(z)$ be convex and $f(z)$ close-to-convex in the unit disc. Then $(\varphi * f)(z)$ is close-to-convex.*

(2.3). The proofs of these results go together and will occupy this section. We recall that a function f is *close-to-convex* if there is a convex function ψ such that

$$\operatorname{Re}\frac{f'(z)}{\psi'(z)} > 0 \qquad (|z| < 1).$$

It is well-known that close-to-convex functions are univalent. The following lemma plays a central role in the discussion and is a modification of a lemma appearing in [6].

(2.4). LEMMA. *Let $\varphi(z)$ and $g(z)$ be analytic in $|z|<1$ and satisfy $\varphi(0)=g(0)=0$, $\varphi'(0)\neq 0$, $g'(0)\neq 0$. Suppose that for each σ ($|\sigma|=1$) and α ($|\alpha|=1$) we have*

$$\varphi(z) * \frac{1+\alpha\sigma z}{1-\sigma z} g(z) \neq 0 \qquad (0<|z|<1). \tag{2.4.1}$$

Then for each function $F(z)$ analytic in $|z|<1$ and satisfying

$$\operatorname{Re} F(z) > 0 \qquad (|z|<1), \tag{2.4.2}$$

we have

$$\operatorname{Re}\frac{(\varphi * Fg)(z)}{(\varphi * g)(z)} > 0 \qquad (|z|<1). \tag{2.4.3}$$

Proof. We first show that

$$\operatorname{Re}\frac{\varphi(z) * \dfrac{1+\sigma z}{1-\sigma z} g(z)}{\varphi(z) * g(z)} > 0 \qquad (|z|<1). \tag{2.4.4}$$

The hypothesis (2.4.1) when $\alpha=-1$ implies that $\varphi * g \neq 0$ for $0<|z|<1$. Assume that $|\alpha|=1$ and $\alpha\neq -1$. Then we have

$$\varphi(z) * \frac{1+\alpha\sigma z}{1-\sigma z} g(z) = \tfrac{1}{2}(1+\alpha)\,\varphi(z) * \frac{1+\sigma z}{1-\sigma z} g(z) + \tfrac{1}{2}(1-\alpha)\,\varphi(z) * g(z)$$

and hence by (2.4.1)

$$\frac{\varphi(z) * \dfrac{1+\sigma z}{1-\sigma z} g(z)}{\varphi(z) * g(z)} \neq -\frac{1-\alpha}{1+\alpha} \qquad (|z|<1).$$

The left-hand member of this relation therefore takes no value on the imaginary axis, but clearly has the value 1 at $z=0$. Therefore (2.4.4) follows.

Consider then $F(z)$ satisfying (2.4.2). We may assume that $|F(0)|=1$, and we then have by the Herglotz formula

$$e^{-i\gamma}F(z) = \int_T \frac{1+\beta\sigma z}{1-\sigma z}\,\mathrm{d}\mu(\sigma) \tag{2.4.5}$$

where μ is a probability mass on the unit circle T and β and $e^{-i\gamma}$ are uniquely determined constants such that $|\beta|=1$, $\beta \neq -1$, and $\cos\gamma>0$ ($w=e^{i\gamma}(1+\beta z)/(1-z)$ maps $|z|<1$ onto $\mathrm{Re}\,w>0$). Then

$$\begin{aligned} e^{-i\gamma}(\varphi * Fg)(z) &= \int_T \varphi(z)*g(z)\frac{1+\beta\sigma z}{1-\sigma z}\,\mathrm{d}\mu(\sigma) \\ &= \tfrac{1}{2}(1+\beta)\int_T \varphi(z)*g(z)\frac{1+\sigma z}{1-\sigma z}\,d\mu(\sigma) + \tfrac{1}{2}(1-\beta)(\varphi*g)(z) \\ &= (\varphi(z)*g(z))\left\{\tfrac{1}{2}(1+\beta)\int_T H_\sigma(z)\,d\mu(\sigma) + \tfrac{1}{2}(1-\beta)\right\} \end{aligned}$$

where $H_\sigma(0)=1$ and by (2.4.4), $\mathrm{Re}\,H_\sigma(z)>0$. Thus

$$e^{-i\gamma}\frac{(\varphi*Fg)(z)}{(\varphi*g)(z)} = \tfrac{1}{2}(1+\beta)K(z) + \tfrac{1}{2}(1-\beta)$$

where $K(0)=1$ and $\mathrm{Re}\,K(z)>0$, and the condition (2.4.3) follows immediately.

(2.5). *Remark.* It is easily seen from this lemma that the condition (2.4.1) implies that $(\varphi*Fg)/(\varphi*g)$ takes only values in the convex hull of the range of F for every analytic F.

We require next what is essentially a special case of theorem 2.2 (see [6]).

(2.6). LEMMA. *Let $h(z)$ be analytic in $|z|<1$ with $h(0)=0$ and suppose that there exist constants α and β with $|\alpha|=|\beta|=1$ such that*

$$\mathrm{Re}\left\{(1-\alpha z)(1-\beta z)\frac{h(z)}{z}\right\}>0 \quad (|z|<1). \tag{2.6.1}$$

Then for every convex function $\varphi(z)$

$$\varphi(z)*h(z)\neq 0 \quad (0<|z|<1). \tag{2.6.2}$$

Proof. By the Herglotz formula there exists γ ($|\gamma|=1$) such that

$$h(z) = h'(0)\int_T \frac{z(1+\gamma\sigma z)}{(1-\alpha z)(1-\beta z)(1-\sigma z)}\,d\mu(\sigma) \tag{2.6.3}$$

for $|z|<1$, where μ is a probability mass on the unit circle T. Thus

$$\begin{aligned}
\frac{1}{h'(0)}(\varphi(z)*h(z)) &= \int_T \varphi(z)*\frac{z(1+\gamma\sigma z)}{(1-\alpha z)(1-\beta z)(1-\sigma z)}\,d\mu(\sigma) \\
&= \int_T \varphi(z)*[(1+\gamma)\,z(1-\sigma z)^{-1}(1-\alpha z)^{-1}(1-\beta z)^{-1} \\
&\qquad\qquad - \gamma z(1-\alpha z)^{-1}(1-\beta z)^{-1}]\,d\mu(\sigma) \\
&= \{\varphi(z)*z(1-\alpha z)^{-1}(1-\beta z)^{-1}\} \\
&\qquad \times\left\{(1+\gamma)\int_T \frac{\varphi(z)*z(1-\sigma z)^{-1}(1-\alpha z)^{-1}(1-\beta z)^{-1}}{\varphi(z)*z(1-\alpha z)^{-1}(1-\beta z)^{-1}}\,d\mu-\gamma\right\}
\end{aligned}$$

The expression in the first parentheses cannot vanish for $0<|z|<1$ by (1.6.2) and the expression in the second parentheses cannot vanish by (1.6.1). This proves the lemma.

(2.7). LEMMA. *Let $\varphi(z)$ be convex and $g(z)$ starlike in $|z|<1$. Then for each function $F(z)$ analytic in $|z|<1$ and satisfying*

$$\operatorname{Re} F(z) > 0 \quad (|z|<1), \tag{2.7.1}$$

we have

$$\operatorname{Re}\frac{(\varphi * Fg)(z)}{(\varphi * g)(z)} > 0 \quad (|z|<1). \tag{2.7.2}$$

Proof. By lemma 2.4 it will be sufficient to show that

$$\varphi(z)*\frac{1+\alpha\sigma z}{1-\sigma z}\,g(z) \neq 0 \quad (0<|z|<1) \tag{2.7.3}$$

for every α and σ satisfying $|\alpha|=|\sigma|=1$. According to lemma 2.6 a sufficient condition for this is that for each such α and σ we find constants a, b and c with $|a|=|b|=|c|=1$ such that

$$\operatorname{Re}\left\{a(1-bz)(1-cz)\frac{1+\alpha\sigma z}{1-\sigma z}\frac{g(z)}{z}\right\} > 0 \tag{2.7.4}$$

in $|z|<1$. However it is clear from theorem 1.8 that such a relation holds, for if we set $b=\sigma$, then c can be chosen $c=e^{-it^*}$ where $-\alpha\sigma=e^{-it}$. By the maximum principle the possibility of equality occurring in the theorem can be ignored by appropriate choice of a.

(2.8). We can now establish theorems 2.1 and 2.2. If f is close-to-convex, then for a suitable starlike g, we have

$$zf'(z) = g(z)\,F(z)$$

where $\operatorname{Re} F(z) > 0$. From (2.7.2) we obtain

$$\operatorname{Re} \frac{z(\varphi * f)'}{\varphi * g} = \operatorname{Re} \frac{\varphi * Fg}{\varphi * g} > 0 \tag{2.8.1}$$

and hence theorem 2.2 follows if we can show that $\varphi * g$ is starlike, which is equivalent to theorem 2.1. To do this we must establish the inequality

$$\operatorname{Re} \frac{z(\varphi * g)'}{\varphi * g} > 0 \quad (|z| < 1). \tag{2.8.2}$$

Since g is starlike, $\operatorname{Re}[zg'(z)/g(z)] > 0$, and hence (2.8.2) follows from (2.7.2) on putting $F(z) = zg'(z)/g(z)$. The two theorems are thus established.

(2.9). *Remark.* If we say "f is close to ψ" when

$$\operatorname{Re} \frac{f'(z)}{\psi'(z)} > 0 \quad (|z| < 1),$$

then we have shown that $\varphi * f$ is close to $\varphi * \psi$ for every φ and ψ convex with f close to ψ.

3. Convolution of Functions Starlike of Order $\frac{1}{2}$

Somewhat surprisingly the structure of functions starlike of order $\frac{1}{2}$ is also preserved under convolutions.

(3.1). THEOREM. *If φ and ψ are starlike of order $\frac{1}{2}$, then so is $\varphi * \psi$.*

(3.2). COROLLARY. *The Hadamard product of two odd starlike functions is starlike.*

(3.3). THEOREM. *Let φ and ψ be starlike of order $\frac{1}{2}$, and suppose that f satisfies*

$$\operatorname{Re} \frac{zf'(z)}{\psi(z)} > 0 \quad (|z| < 1). \tag{3.3.1}$$

Then

$$\operatorname{Re} \frac{\varphi * zf'(z)}{\varphi * \psi} > 0 \quad (|z| < 1) \tag{3.3.2}$$

and in particular $\varphi * f$ is close-to-convex.

(3.1) and (3.2) are of course equivalent and (3.2) may be preferred, since the odd starlike functions have an immediate geometrical interpretation. Theorem 3.3 can also be expressed in terms of odd functions for natural geometrical classes.

(3.4). LEMMA. *Let $h(z)$ be analytic in $|z|<1$ with $h(0)=0$, and suppose that there exists β satisfying $|\beta|=1$ such that*

$$\operatorname{Re}\left\{(1-\beta z)\frac{h(z)}{z}\right\}>0 \quad (|z|<1). \tag{3.4.1}$$

Then if $\varphi(z)$ is starlike of order $\frac{1}{2}$

$$\varphi(z)*h(z)\neq 0 \quad (0<|z|<1). \tag{3.4.2}$$

Proof. By the Herglotz formula we can write

$$\frac{h(z)}{h'(0)}=\int_T \frac{z(1+\gamma\sigma z)}{(1-\sigma z)(1-\beta z)}\,d\mu(\sigma)$$

where $|\gamma|=1$, $\gamma\neq -1$. Thus

$$\frac{1}{h'(0)}\varphi(z)*h(z)=\frac{\varphi(\beta z)}{\beta}\int_T\left\{(\gamma+1)\frac{\varphi(z)*z(1-\beta z)^{-1}(1-\sigma z)^{-1}}{\varphi(z)*z(1-\beta z)^{-1}}-\gamma\right\}d\mu.$$

The integrand lies in a half-plane not containing 0 by (1.6.2) and $\varphi(\beta z)\neq 0\,(0<|z|<1)$ so the lemma follows.

(3.5). LEMMA. *Let φ and ψ be starlike of order $\frac{1}{2}$. Then for each function $F(z)$ satisfying*

$$\operatorname{Re}F(z)>0 \quad (|z|<1) \tag{3.5.1}$$

we have

$$\operatorname{Re}\frac{\varphi(z)*F(z)\psi(z)}{\varphi(z)*\psi(z)}>0 \quad (|z|<1). \tag{3.5.2}$$

Proof. By lemma 2.4 it will be sufficient to show that for every α and σ satisfying $|\alpha|=|\sigma|=1$, we have

$$\varphi(z)*\frac{1+\alpha\sigma z}{1-\sigma z}\psi(z)\neq 0 \quad (0<|z|<1).$$

By 3.4 this relation will be proved if for each such α and σ we can find β and a ($|a| = |\beta| = 1$) such that

$$\operatorname{Re}\left\{a(1-\beta z)\frac{1+\alpha\sigma z}{1-\sigma z}\frac{\psi(z)}{z}\right\} > 0 \quad (|z|<1). \tag{3.5.3}$$

Since $\psi(z)$ is starlike of order $\frac{1}{2}$, it is immediately verified that for each σ ($|\sigma|=1$) the function

$$\frac{\psi(z)}{1-\sigma z}$$

is starlike, and hence if we apply theorem 1.8 to this function we obtain the required relation (3.5.3).

(3.6). Theorem 3.3 will now follow from the previous lemma as soon as we have established theorem 3.1. If φ and ψ are starlike of order $\frac{1}{2}$ we have to show that

$$\operatorname{Re}\frac{\varphi * z\psi'}{\varphi * \psi} > \tfrac{1}{2}.$$

This follows immediately from lemma 3.5 on putting $F(z) = z\psi'(z)/\psi(z)$ and applying remark 2.5.

4.A Subordination Theorem

The following subordination result was first conjectured by Wilf [13].

(4.1). THEOREM. *Let φ and ψ be convex in $|z|<1$ and suppose that f is subordinate to ψ. Then $\varphi * f$ is subordinate to $\varphi * \psi$.*

(4.2). LEMMA. *If $k(z)$ is convex and $h(z)$ analytic, then $h(z)$ is properly subordinate to $k(z)$ if, and only if, for each σ ($|\sigma|=1$)*

$$\operatorname{Re}\frac{zk'(z)}{k(z)-h(\sigma z)} > 0 \quad (|z|<1). \tag{4.2.1}$$

Here $h(z)$ properly subordinate to $k(z)$ means that $h(z) = k(\omega(z))$ where $|\omega(z)| < |z|$.

Proof. Suppose that $h(z) = k(\omega(z))$ where $|\omega(z)| < |z|$. If $|z_0| < |z| < 1$, then

$$\operatorname{Re}\frac{zk'(z)}{k(z)-k(z_0)} > 0,$$

and hence (4.2.1) is immediate from the maximum principle. Conversely, if the condition (4.2.1) holds, then it holds for every σ such that $|\sigma|\leqslant 1$. Suppose that for some w, $k(z)\neq w$, where $w=h(z_1)$ and $|z_1|<1$. Then if $|z_1|<|z|<1$

$$\operatorname{Re}\frac{zk'(z)}{k(z)-w}=\operatorname{Re}\frac{zk'(z)}{k(z)-h(\sigma z)}>0$$

where $\sigma=z_1/z$, so that $|\sigma|<1$. Since $k(z)\neq w$, the maximum principle gives

$$\operatorname{Re}\frac{zk'(z)}{k(z)-w}>0\qquad(|z|<1),$$

which is clearly false for $z=0$. Since also (4.2.1) implies that $h(0)=k(0)$, we deduce that h is subordinate to k.

(4.3). *Proof of Theorem* 4.1. We may assume that f is properly subordinate to ψ. We then have

$$\operatorname{Re}\frac{z\psi'(z)}{\psi(z)-f(\sigma z)}>0\qquad(|z|<1,|\sigma|=1).\tag{4.3.1}$$

To prove the theorem we must show that

$$\operatorname{Re}\frac{\varphi*z\psi'(z)}{(\varphi*\psi)(z)-(\varphi*f)(\sigma z)}>0\qquad(|z|<1,|\sigma|=1).\tag{4.3.2}$$

This is equivalent to

$$\operatorname{Re}\frac{\varphi*\dfrac{\psi(z)-f(\sigma z)}{z\psi'(z)}z\psi'(z)}{\varphi*z\psi'(z)}>0$$

which follows from (4.3.1) and lemma 2.7.

5. Applications

(5.1). *The de la Vallée Poussin means*

The de la Vallée Poussin means of an analytic function $f(z)=\sum_1^\infty a_n z^n$ are given by

$$V_n(z,f)=\frac{1}{\binom{2n}{n}}\sum_{k=1}^{n}\binom{2n}{n+k}a_k z^k\tag{5.1.1}$$

for $n=1,2,\ldots$. Pólya and Schoenberg [2] showed that these means are convex (starlike) whenever f is convex (starlike) (and conversely). In particular the polynomials

$$V_n(z) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^{n} \binom{2n}{n+k} z^k \tag{5.1.2}$$

are convex on putting $f(z) = z(1-z)^{-1}$. This apparently weaker statement now implies the full result, and furthermore the de la Vallée Poussin means of a close-to-convex function are close-to-convex. In addition Pólya and Schoenberg showed that $V_n(z,f)$ is subordinate to f for every convex f, and conjectured that in this case the stronger subordination condition

$$V_n(z, f) \prec V_{n+1}(z, f) \quad (n = 1, 2, \ldots)$$

held. This follows from theorem 4.1 as soon as it is shown for the polynomials (5.1.2). Pólya and Schoenberg verified this case, although the proof was not given.

(5.2). *Univalence of Partial Sums*

Let $P_n(z,f)$ denote the nth partial sum of the analytic function f. If f is convex, then $P_n(z,f)$ is convex in $|z|<r_1$, where r_1 is the radius of convexity of the polynomial $P_n(z) = z + z^2 + \cdots + z^n$, and if f is close-to-convex, then $P_n(z,f)$ is close-to-convex for $|z|<r_1$. Again if r_2 is the radius of close-to-convexity of $P_n(z)$, then $P_n(z,f)$ is close-to-convex in $|z|<r_2$ for every convex f. These results, which are sharp, have been obtained elsewhere and the values of r_1 and r_2 have been computed [7, 8].

It is also easy to show from our results that if $f(z)$ is starlike of order $\frac{1}{2}$, then

$$\operatorname{Re} \frac{f(z)}{P_n(z, f)} > \tfrac{1}{2} \quad (|z| < 1) \tag{5.2.1}$$

and that, writing $zg'(z) = f(z)$, the polynomials $P_n(z, g)$ are close-to-convex in $|z|<1$.

(5.3). *The class M*

M is defined as the class of functions $f(z) = z + a_2 z^2 + \cdots$ whose convolution with an arbitrary convex function is schlicht. Clearly the members of M are schlicht and M contains the class C of normalised close-to-convex functions. In [8] the following result has been established for M.

THEOREM. *Let Λ_1 and Λ_2 be continuous linear functionals on the space $\mathscr{A}$ of functions analytic in the disc, and suppose that $\Lambda_2 \neq 0$ on M. Let C_0 denote the class of functions $h(z)$ which have the form*

$$h(z) = \frac{z - \frac{1}{2}(x + y) z^2}{(1 - xz)^2} \quad (|z| < 1) \tag{5.3.1}$$

where either (a) $x \neq y$ *and* $|x|=|y|=1$, *or* (b) $x=y$ *and* $|x|\leqslant 1$. *Then for each* $f\in M$ *there exists* $h\in C_0$ *such that*

$$\frac{\Lambda_1 f}{\Lambda_2 f}=\frac{\Lambda_1 h}{\Lambda_2 h}. \tag{5.3.2}$$

In case (a) $h(z)$ maps the disc onto the plane cut along a half-line, and in case (b) $h(z)$ has the form $z(1-xz)^{-1}$ where $|x|\leqslant 1$. C_0 is therefore a subclass of the functions convex in one direction. The theorem in particular holds for C and provides considerable information on the structure of these functions. The theorem also implies that M lies in the closed convex hull of C_0 in the locally convex linear topological space $\mathscr{A}$. This follows immediately from general separation theorems (see e.g. [1] page 119). On the other hand there are members of M not in C [9]. It would be very interesting to completely characterise M either geometrically or by means of a general representation formula. The above theorem would appear to provide a basic method of attack on this problem and strongly suggests that M lies "very close" to C.

(5.4). It is to be hoped that the results and methods of this paper have other deeper applications than the simple ones which we have mentioned here. Also it is surely the case that there are many classes closed with respect to convolution, and when these have a specifically geometrical characterisation interesting information may result. We conclude by pointing out the following simple extension of theorems 2.1 and 2.2.

THEOREM. *Let* ψ *be convex and suppose that for a function* f, f'/ψ' *takes all its values in a convex domain* D. *Then for every convex* φ,

$$\frac{\varphi * zf'}{\varphi * z\psi'}$$

takes all its values in D.

This follows immediately from lemma 2.7 and remark 2.5. In particular if K_α denotes the class of close-to-convex functions f for which there exists a convex ψ such that

$$\left|\arg\frac{f'(z)}{\psi'(z)}\right|\leqslant\frac{\alpha\pi}{2}\qquad(|z|<1),$$

where $0\leqslant\alpha\leqslant 1$, then $f\in K_\alpha \Rightarrow f*\varphi\in K_\alpha$ for every convex φ. The class K_α has a simple geometrical characterisation (see [4]).

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