Integrability in Codimension 1

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Integrability in Codimension 1

by JOHN N. MATHER[1])

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Introduction

In this paper, we prove the main result announced in [4]. More precisely, we show that for certain groupoids Γ of homeomorphisms of $\mathbb{R}$, there is a discrete group G, and a mapping

$$BG \to \Omega B\Gamma$$

which induces isomorphism in integer homology.

Haefliger has shown [2, 3] that the cohomology of $B\Gamma$ measures the obstruction to finding a foliation on an open manifold with a given normal bundle. Our proof that $H_i(BG)=H_i(\Omega B\Gamma)$ arose out of an attempt to calculate the homology of this space. Unfortunately, in most cases $H_i(BG)$ is completely unknown. Nonetheless our main result does have applications. In dimensions 2 and 3, it is possible to homotope all Haefliger structures to certain special Haefliger structures. For example, on S^2 every Γ-structure is homotopic to one with two nodes and an arc, transverse to the Γ-structure, connecting the nodes. In §3, we show how this follows from our main results. In fact, it was this result, which we originally proved by an entirely different method, which suggested the main result of this paper.

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In our announcement [4], we stated that there are two spectral sequences connecting the homology of BG and the homology of $B\Gamma$, which resemble the Serre spectral sequence and the Eilenberg-Moore spectral sequence connecting the homology of a space and its loop space. This suggested that there should be a mapping $BG \to \Omega B\Gamma$ inducing isomorphism in homology. Quillen showed me how to construct such a mapping and how to use a classical spectral sequence comparison theorem to prove isomorphism in homology. I am indebted to him for this.

Quillen has told me that he has found another method of proving the main result of this paper, which may be simpler. His proof has not been published yet, however.

Moreover, Thurston has found a marvelous generalization of our main result to higher codimensions. His result has not been published either.

Now we outline the proof that there is a mapping $BG \to \Omega B\Gamma$ which induces isomorphism in homology. In §5, we construct (following Quillen) a space BBG which is a delooping of BG in the sense that there is a mapping $BG \to \Omega BBG$ which is an isomorphism in homology. In §6, we construct a Γ-structure on BBG, using technical results concerning cocycles defined over closed covers, which are proved in §1. From the universal property of $B\Gamma$ this gives rise to a mapping $BBG \to B\Gamma$, and most of the paper is devoted to constructing a homotopy inverse of this mapping.

If ω is a Γ-foliation on a space X, we construct in §10 a subcomplex $S(\omega)$ of the product of singular complexes $S(X) \times S(\mathbb{R})$, consisting of simplices on which the Γ-structure ω has certain properties. Then the geometric realization $|S(\omega)|_\Delta$ is a good model for X, in the sense that the projection $|S(\omega)|_\Delta \to X$ is a homotopy equivalence. On the other hand, to every semi-simplicial set Y without degeneracies, we associate a bi-semi-simplicial set AY in §4, and $|AY|$ has the same homotopy type as $|Y|_\Delta$. There is a natural mapping $|AS(\omega)|_\Delta \to BBG$ which pulls-back the Γ-structure on BBG to the pull-back of ω on $|AS(\omega)|_\Delta$ (§11). Thus, for any foliated space X, we have found a model for X and a mapping of the model into BBG. More generally, this applies to any space with a Γ-structure on it by Haefliger's normal bundle construction [3]. In particular we get a mapping $B\Gamma \to BBG$. In §12, we show that this mapping is a homotopy inverse of the mapping $BBG \to B\Gamma$ already constructed.

Part of this proof was given in a preprint [5]. This is absorbed into §§8, 9, and 10.

§1. Generalities on Haefliger structures.

Let $\mathscr{I}$ be a pseudogroup of homeomorphisms of a topological space Z. Thus, by definition, $\mathscr{I}$ consists of triples (U, h, V), where U and V are open subsets of Z, h is a homeomorphism of U onto V, and $\mathscr{I}$ satisfies the following conditions.

a) If $(U, h, V) \in \mathscr{I}$ and U' is an open subset of U, $V' = h(U')$, and $h' = h \mid U'$, then $(U', h', V') \in \mathscr{I}$.

b) If U and V are open subsets of Z, $h: U \to V$ is a homeomorphism, and $\{U_\alpha\}$ is a cover of U by open subsets such that $(U_\alpha, h \mid U_\alpha, h(U_\alpha)) \in \mathscr{S}$ for each α, then $(U, h, V) \in \mathscr{S}$.

c) If $(U, h, V) \in \mathscr{S}$, then $(V, h^{-1}, U) \in \mathscr{S}$.

d) $(Z, id, Z) \in \mathscr{S}$.

e) If (U, h, V) and (V, h', W) are members of $\mathscr{S}$, then so is $(U, h'h, W)$.

To a pseudogroup $\mathscr{S}$, we can associate a topological groupoid Γ. As a set Γ consists of germs of elements of $\mathscr{S}$. Given $\gamma \in \Gamma$, we let $s(\gamma)$ denote the source of γ and $t(\gamma)$ the target. We provide Γ with the sheaf topology. If $(U, h, V) \in \mathscr{S}$ the set of germs $\{h_x : x \in U\}$ is an open subset of Γ, and the collection of all such sets is a basis for the sheaf topology of Γ.

The product $\gamma'\gamma$ of two elements of Γ is their composition. This is defined if and only if $s(\gamma') = t(\gamma)$. We will identify any $z \in Z$ with the germ at z of id. Thus, Z is the set of units of Γ.

The four structure mappings, i.e. multiplication $\Gamma \times_Z \Gamma \to \Gamma$, inverse $\Gamma \to \Gamma$, and the source and target mappings $\Gamma \to Z$ are continuous, so Γ is a topological groupoid.

If $\{A_i\}_{i \in I}$ is a family of subsets of a topological space X, a *1-cocycle over* $\{A_i\}$ with values in Γ is a rule which assigns to each $i, j \in I$ a continuous mapping γ_{ij}: $A_i \cap A_j \to \Gamma$ such that if $i, j, k \in I$ and $x \in A_i \cap A_j \cap A_k$, then

$$\gamma_{ij}(x)\,\gamma_{jk}(x) = \gamma_{ik}(x). \tag{1.1}$$

(In particular, the left side is defined.) We let $\gamma_i = \gamma_{ii}$. It follows from (1.1) that for each $x \in X$, we have $\gamma_i(x)$ is a unit, so γ_i is a continuous mapping into Z.

Let $\{A_i\}_{i \in J}$ be a second family of subsets of X, and let γ' be a 1-cocycle over $\{A_i\}_{i \in J}$ with values in Γ. We will say γ and γ' are *compatible* if there is a 1-cocycle γ'' over $\{A_i\}_{i \in I \cup J}$ whose restriction to $\{A_i\}_I$ is γ and whose restriction to $\{A_i\}_J$ is γ'.

By definition a *Haefliger Γ-structure* on X is an equivalence class of 1-cocycles with values in Γ defined over *open covers*, where compatibility is the equivalence relation. If ω is a Haefliger Γ-structure on X and γ is a 1-cocycle over an open cover with values in Γ we will say γ *defines* ω provided $\gamma \in \omega$.

If $f: Y \to X$ is a continuous mapping and ω is a Γ-structure on X defined by a 1-cocycle $\{\gamma_{ij}\}_I$ over an open cover $\{U_i\}_I$, then we let $f^*\omega$ be the Γ-structure on Y defined by the 1-cocycle $(\gamma_{ij} f)$ over $\{f^{-1}U_i\}_I$.

If $Y \subset X$ and f is the inclusion mapping, then we say $f^*\omega$ is the *restriction* of ω to Y and write $\omega \mid Y = f^*\omega$. Two Γ-structures on $X \times I$ are said to be *homotopic* if there is a Γ-structure on $X \times I$ whose restrictions to $X \times 0$ and $X \times 1$ are the given Γ-structures. This is an equivalence relation. The "pull-back" f^* respects this relation.

Let γ be a 1-cocycle with values in Γ over an arbitrary family of subsets of X, and let ω be a Haefliger Γ-structure on X. We will say γ *defines* ω if for any 1-cocycle γ' with values in Γ over any *open cover* we have that $\gamma' \in \omega$ if and only if γ' is compatible with γ.

Note that compatibility is not an equivalence relation. However compatibility of cocycles defined over open covers is an equivalence relation. Moreover, if γ is a cocycle defined over an arbitrary family of subsets, γ' and γ'' are cocycles defined over open covers, then we have the following situation. If γ is compatible with γ' and γ' is compatible with γ'', then γ is compatible with γ''. However, if γ is compatible with both γ' and γ'', then γ' and γ'' need not be compatible.

LEMMA 1.1. *If X is a* CW *complex, then a* 1-*cocycle over the closed cells of X defines a Γ-structure.*

Proof. Let γ be a cocycle over the closed cells of X. We must show, first, that if γ' and γ'' are cocycles defined over open covers of X, and each is compatible with γ, then they are compatible with each other, and, second, that there is a cocycle over an open cover which is compatible with γ.

For the first assertion, we consider γ' defined over $\{U_i\}_I$ and γ'' defined over $\{U_i\}_J$. To define a cocycle γ''' over $\{U_i\}_{I \cup J}$ whose restrictions to the given covers are γ' and γ'', it is enough to define γ'''_{ij} for $i \in I$ and $j \in J$. Let $x \in U_i \cap U_j$. Using the compatibility of γ' and γ'' with γ, we see what $\gamma'''_{ij}(x)$ must be. For if $\gamma^{(iv)}$ denotes an extension of $\gamma \cup \gamma'$ and $\gamma^{(v)}$ an extension of $\gamma \cup \gamma''$, we let

$$\gamma'''_{ij}(x) = \gamma^{(iv)}_{ie}(x)\, \gamma^{(v)}_{ej}(x)$$

where e is a closed cell which contains x. This is independent of e. Since the topology on a CW complex is the weak topology, it follows that γ'''_{ij} is continuous. It is easily seen that γ''' is a cocycle.

For the second assertion, we consider the following sublemma.

SUBLEMMA. *Let K be a closed subset of X and let γ' be a cocycle over $\mathscr{E} \cup \{K\}$, where $\mathscr{E}$ is the collection of closed cells of X, such that the restriction of γ' to $\mathscr{E}$ is γ. Then there is a neighborhood N of K in X and an extension γ'' of γ' to $\mathscr{E} \cup \{K\} \cup \{N\}$, such that $\gamma_{KN}(x) \in Z$ for $x \in K$.*

Assuming the sublemma, it is now easy to complete the proof of the second assertion. We apply the sublemma, where K is a closed cell e. The lemma shows that γ can be extended to a cocycle γ^e over $\mathscr{E} \cup \{N_e\}$, where N_e is an open neighborhood of e in X. The argument used to prove the first assertion then shows that there is a unique cocycle γ' over $\mathscr{E} \cup \{N_e\}_{e \in \mathscr{E}}$ whose restriction $\mathscr{E} \cup \{N_e\}$ is γ^e, for each $e \in \mathscr{E}$. Then the restriction of γ' to $\{N_e\}_{e \in \mathscr{E}}$ is a cocycle over an open cover of X, and it is compatible with γ. Q.E.D.

Proof of the Sublemma. We first consider the case when X is a closed cell. Since γ' is a cocycle, the following diagram commutes.

$$\begin{array}{ccc} K & \xrightarrow{\gamma'_{KX}} & \Gamma \\ \cap & & \downarrow t \\ X & \xrightarrow{\gamma'_X} & Z \end{array}$$

Since t is a local homeomorphism, there is a neighborhood N of K in X and a mapping γ''_{NX} which extends γ'_{KX} such that the following diagram commutes.

$$\begin{array}{ccc} N & \xrightarrow{\gamma''_{NX}} & \Gamma \\ \cap & & \downarrow t \\ X & \xrightarrow{\gamma'_X} & Z \end{array}$$

Define $\gamma''_{LL'}=\gamma'_{LL'}$ if L, $L'\in\mathscr{E}\cup\{K\}$, $\gamma''_{NL}(x)=\gamma''_{NX}(x)(\gamma''_{LX}(x))^{-1}$ if $L\in\mathscr{E}\cup\{K\}$ and $x\in N\cap L$, $\gamma''_{LN}=(\gamma''_{NL})^{-1}$, and $\gamma''_N=t(\gamma''_{NX})$. Then γ'' is a cocycle over $\mathscr{E}\cup\{K\}\cup\{N\}$ having the required property.

Now the sublemma follows easily from the special case we have just considered, by an inductive argument.

Let X_k denote the k-skeleton of X. We will prove the following assertion by the induction on k.

INDUCTIVE ASSERTION. *There exists a neighborhood N_k of $K\cap X_k$ in X_k and a cocycle γ^k over $\mathscr{E}\cup\{K\}\cup\{N_k\}$ which extends γ' such that $\gamma^k_{KN_k}(x)\in Z$ for $x\in K$. Moreover, if $k\geqslant 1$ we can choose N_k so $N_{k-1}\supseteq N_k\cap X_{k-1}$ and $\gamma^{k-1}_{LN_{k-1}}(x)=\gamma^k_{LN_k}(x)$ for all $L\in\mathscr{E}\cup\{K\}$ and all $x\in L\cap N_{k-1}$.*

Proof. If $k=0$, the inductive assertion is trivial. Suppose $k\geqslant 1$, and that N_{k-1} and γ^{k-1} have been constructed. If e is a k-cell in X, we can apply the special case of the sublemma we have just proved, where e plays the role of X, $(K\cup N_{k-1})\cap e$ that of K, and γ^{k-1} that of γ. There results a 1-cocycle γ^e defined over a family of subsets of e, namely the cells of e, $N_{k-1}\cap e$, and a neighborhood N_e of $(K\cup N_{k-1})\cap e$ in e. Let $N_k=\bigcup_e N_e$, where the union is taken over all k-cells of X. Then the collection of cocycles we construct on k-cells yields a cocycle γ^k over $\mathscr{E}\cup\{N_{k-1}\}\cup\{N_k\}\cup\{K\}$ by the rule that $\gamma^k_{LN_k}(x)=\gamma^e_{L\cap e,\,N_k\cap e}(x)$ if $x\in e$. The restriction of this to $\mathscr{E}\cup\{K\}\cup\{N_k\}$ is the required cocycle. This proves the inductive assertion. Q.E.D.

Now let $N=\bigcup_k N_k$. For any $x\in N$ choose a k such that $x\in N_k$. If $L\in\mathscr{E}\cup\{K\}$, let $\gamma''_{LN}(x)=\gamma^k_{LN_k}(x)$. This defines the required cocycle γ'', and proves the sublemma. Q.E.D.

§2. Statement of the main Result

Throughout the rest of this paper, $\mathscr{I}$ will denote a pseudogroup of homeomorphisms of the real numbers, which satisfies the following conditions.

α) Each member of $\mathscr{I}$ preserves orientation.

β) If τ is a translation of $\mathbb{R}$, then $(\mathbb{R}, \tau, \mathbb{R}) \in \mathscr{I}$.

γ) Let $a, b, c, d, a', b', c', d'$ be real numbers such that $a<b<c<d$ and $a'<b'<c'<$ $<d'$. Let $((a, b), f, (a', b'))$ and $((c, d), g, (c', d'))$ be members of $\mathscr{I}$. Then there exists h such that $((a, d), h, (a', d'))$ is a member of $\mathscr{I}$ and $\varepsilon>0$ such that $h \mid (a, a+\varepsilon) =$ $\mid (a, a+\varepsilon)$, $h \mid (d, d-\varepsilon) = g \mid (d, d-\varepsilon)$.

Throughout the rest of this paper, we will let Γ denote the topological groupoid associated to $\mathscr{I}$.

The pseudogroup of orientation preserving C^r diffeomorphisms of open subsets of $\mathbb{R}$ satisfies the conditions α, β, and γ. On the other hand, the pseudogroup of orientation preserving analytic diffeomorphisms of open subsets of $\mathbb{R}$ does not satisfy γ.

A homeomorphism $h: U \to V$ of one open subset of $\mathbb{R}$ onto another will be said to be a *$\mathscr{I}$-homeomorphism* if $(U, h, V) \in \mathscr{I}$. If $h: Z \to Z$ is a self-mapping, the *support* $\operatorname{supp} h$ of h is defined to be the closure of $\{z \in Z : h(z) \neq z\}$.

We let G denote the group of compactly supported $\mathscr{I}$-homeomorphisms of $\mathbb{R}$ onto itself. We let BG denote the classifying space of G, where the latter is regarded as a discrete group.

Now we construct a Γ-structure on the suspension $S(BG)$ of BG. Let $\widetilde{BG}$ denote the universal covering of space BG. Let I denote the unit interval $[0, 1]$. Consider real numbers a, b, such that $0<a<b<1$.

It follows easily from conditions (β) and (γ) on $\mathscr{I}$ that there exists a $\mathscr{I}$-homeomorphism $h: (a, b) \to \mathbb{R}$. For, let ζ be a subset of (a, b), order-isomorphic to the set $\mathbb{Z}$ of integers. From (β) and the definition of pseudogroup, it follows that there is a $\mathscr{I}$-homeomorphism h_0 of an open neighborhood of ζ in (a, b) onto an open neighborhood of $\mathbb{Z}$ in $\mathbb{R}$ which carries ζ order isomorphically onto $\mathbb{Z}$. From (γ) it follows that for two successive members s, t of ζ, there is a $\mathscr{I}$-homeomorphisms h_s of a neighborhood of the closed interval $[s, t]$ onto a neighborhood of $[h_0(s), h_0(t)]$ whose germ at s and t equals the germ of h_0. Let $h: (a, b) \to \mathbb{R}$ be the mapping defined by letting $h \mid [s, t]$ $= h_s$ for any two successive members of γ. From the assumption that h_0 maps ζ in an order preserving way onto $\mathbb{Z}$, and the assumption that each h_s is a homeomorphism, it follows that h is a homeomorphism of (a, b) onto $\mathbb{R}$. From our constructions, it follows that each point in (a, b) has a neighborhood such that the restriction of h to that neighborhood is a $\mathscr{I}$-homeomorphism. Hence by the definition of pseudogroup, h is a $\mathscr{I}$-homeomorphism.

We construct an action of G on $\widetilde{BG} \times I$ by letting G act on each factor separately, as follows. We let G act on $\widetilde{BG}$ as the group of covering transformations. We let G act on I by the formula

$$\begin{aligned} g\cdot t &= h^{-1}gh(t), && t\in(a,b) \\ &= t && \text{otherwise}. \end{aligned}$$

We let $\tilde{\omega}$ be the Γ structure on $\widetilde{BG}\times I$ defined by the projection on the second factor:

$$BG\times I\to I\subset\mathbb{R}.$$

Clearly this Γ structure is G invariant.

We let $\sim$ denote the equivalence relation on $BG\times I$ defined by $(x_0, t_0)\sim(x_1, t_1)$ if

$$t_0=t_1=0,\qquad t_0=t_1=1,$$

or there exists

$$g\in G \quad\text{such that}\quad g\cdot(x_0, t_0)=(x_1, t_1)$$

There is a canonical homotopy equivalence

$$SBG\simeq\widetilde{BG}\times I/\sim$$

Furthermore, there is a unique Γ structure ω_h on $BG\times I/\sim$ such that $\pi^*\omega_h=\tilde{\omega}$, where π denotes the projection

$$\pi:\widetilde{BG}\times I\to\widetilde{BG}\times I/\sim.$$

By Haefliger's theorem ω_h defines a homotopy class of mappings

$$\widetilde{BG}\times I/\sim\to B\Gamma.$$

Composing with the canonical homotopy equivalence above, we get a homotopy class δ_h of mappings of SBG into $B\Gamma$. We let

$$\gamma_h:BG\to\Omega B\Gamma$$

denote the adjoint of δ_h. The main result of this paper is the following:

THEOREM. *The induced mappings*

$$\gamma_{h*}:H_i(BG,\mathbb{Z})\to H_i(\Omega B\Gamma,\mathbb{Z}),\qquad i\geqslant 0$$

are isomorphisms.

We do not know whether the homotopy class of γ_h depends on h. However, a certain amount in this direction can be said. We recall that if A is an automorphism of G, then A induces a mapping $BA: BG \to BG$, and BA is homotopic to the identity (where base points are allowed to move) if and only if A is an inner automorphism.

Let a', b' be a second pair of real numbers such that $0 < a' < b' < 1$. Let $h' : (a', b') \to \mathbb{R}$ be a $\mathscr{I}$-homeomorphism. We wish to compare γ_h and $\gamma_{h'}$. It is easily seen that there is a $\mathscr{I}$-homeomorphism g of I onto itself which is the identity in a neighborhood of the endpoints and maps (a, b) onto (a', b'). Let

$$f = h'gh^{-1} : \mathbb{R} \to \mathbb{R}.$$

Clearly f is a $\mathscr{I}$-homeomorphism of $\mathbb{R}$. We let A denote the automorphism of G defined by

$$A(k) = f^{-1}kf, \quad k \in G.$$

LEMMA 2.1. *The following diagram is homotopy commutative*:

$$\begin{tikzcd} BG \arrow[rr, "BA"] \arrow[dr, "\gamma_h"'] & & BG \arrow[dl, "\gamma_{h'}"] \\ & \Omega B\Gamma & \end{tikzcd}$$

Proof. It will be convenient to write $\mathscr{R}_h$ for the equivalence relation $\sim$ on $BG \times I$ defined above and ϱ_h for the action of G on $BG \times I$ also defined above. Note that $\mathscr{R}_h$ and ϱ_h depend on the choice of h.

Let x_0 be the base point of BG, so that $BA(x_0) = x_0$. Consider the unique lifting $\widetilde{BA} : \widetilde{BG} \to \widetilde{BG}$ of BA such that $\widetilde{BA}(e_0) = e_0$, where e_0 is the base point of $\widetilde{BG}$. Let

$$F : \widetilde{BG} \times I \to \widetilde{BG} \times I$$

be the homeomorphism defined by

$$F(x, t) = (\widetilde{BA}(x), g^{-1}(t)).$$

It is easily verified that

$$\varrho_h(k) \circ F(x, t) = F \circ \varrho_{h'}(A^{-1}(k))(x, t) \tag{2.1}$$

for all $k \in G$, and $(x, t) \in \widetilde{BG} \times I$. For, it is enough to verify the validity of this equation separately on each factor. On the factor $\widetilde{BG}$, this equation comes down to

$$k \cdot \widetilde{BA}(x) = \widetilde{BA}(A^{-1}(k) \cdot x)$$

which follows immediately from the group theoretic formula.

$$L_k A = A L_{A^{-1}(k)},$$

where L_k means multiplication on the left by k. On the factor I, the left side of the equation is

$$h^{-1}khg^{-1}(t) \quad \text{if} \quad t \in g(a, b)$$

and t otherwise, and the right side is

$$g^{-1}h'^{-1}A^{-1}(k)\,h'(t) \quad \text{if} \quad t \in (a, b')$$

and t otherwise. But $(a', b') = g(a, b)$ and

$$\begin{aligned} g^{-1}h'^{-1}A^{-1}(k)\,h'(t) &= g^{-1}h'^{-1}fkf^{-1}h'(t) \\ &= g^{-1}h'^{-1}h'gh^{-1}khg^{-1}h'^{-1}h'(t) \\ &= h^{-1}khg^{-1}(t) \end{aligned}$$

which verifies (2.1).

From (2.1) it follows that $\mathscr{R}'_h$-equivalent points go to $\mathscr{R}_h$-equivalent points. Thus F induces $\bar{F}$ in the diagram below.

$$\begin{array}{ccc} BA \times I/\mathscr{R}_{h'} & \xrightarrow{\bar{F}} & BA \times I/\mathscr{R}_h \\ \int\Big\uparrow & & \int\Big\downarrow \\ S(BG) & \xrightarrow{S(BA)^{-1}} & S(BG) \end{array}$$

Clearly this diagram is homotopy commutative and $\bar{F}^*\omega_h = \omega_{h'}$. It follows immediately that the diagram below is homotopy commutative.

$$\begin{array}{ccc} SBG & \xrightarrow{SBA} & SBG \\ & {\scriptstyle \delta_h}\searrow \quad \swarrow{\scriptstyle \delta_{h'}} & \\ & B\Gamma & \end{array}$$

The lemma follows immediately. Q.E.D.

COROLLARY 2.2. *The induced homomorphism γ_{h*} in homology is independent of h.*

Proof. It is enough to show that BA induces the identity in integer homology, in view of Lemma 2.1. To see this, we observe, first, that for any compact subset of $\mathbb{R}$ there exists $f' \in G$ such that the restrictions of f' and f to that compact subset are the same. It follows that for any finite subset J of G there exists $f' \in G$ such that $A \mid J = I(f') \mid J$, where $I(f')$ is the inner automorphism of G determined by f'. It follows that for any finite subcomplex K of BG there exists $f' \in G$ such that $BA|K = BI(f')|K$. Since $I(f')$ is an inner automorphism, $BI(f')$ is homotopic to the identity. Hence $BA|K$ is homotopic to the inclusion for any finite subcomplex K of BG. The corollary follows immediately. Q.E.D.

§3. Some Corollaries

In this section, we point out some consequences of the main result. Several of these we already pointed out in [4].

COROLLARY 3.1. $\pi_1(B\Gamma)=0$.

Proof. Since BG is connected, we have $H_0(\Omega B\Gamma)=H_0(BG)=\mathbb{Z}$. Hence $\Omega B\Gamma$ is connected. Q.E.D.

Recall that we assumed that $\mathcal{S}$ consists of only orientation preserving homeomorphisms (condition α in §1). If instead, we assume $\mathcal{S}$ satisfies (β) and (γ), but not (α), then together with Γ, we can consider the subgroupoid Γ_0 of Γ consisting o orientation preserving members of Γ. It is easily seen that Γ_0 satisfies (α), (β), and (γ), and that $B\Gamma_0$ is a 2-fold covering space of $B\Gamma$. In this case $\pi_1(B\Gamma)=\mathbb{Z}_2$, since $\pi_1(B\Gamma)=0$ by the above corollary. In the interesting special cases (the pseudogroups of C^r diffeomorphisms) this was already proved by Haefliger.

COROLLARY 3.2. *$B\Gamma$ is n-connected if and only if the reduced integer homology of G vanishes in dimensions $\leqslant n-1$.*

In particular $B\Gamma$ is contractible if and only if G is acyclic. In [6], we proved that G is acyclic in the case $\mathcal{S}$ is the pseudogroup of all orientation preserving homeomorphisms of open subsets of $\mathbb{R}$ onto open subsets of $\mathbb{R}$. Hence:

COROLLARY 3.3. *$B\Gamma$ is contractible in the case $\mathcal{S}$ is the pseudogroup of all orientation preserving homeomorphisms of open subsets of $\mathbb{R}$ onto open subsets of $\mathbb{R}$.*

Remark. Thurston [13], using results of Godbillon and Vey [1] has shown that if Γ is the groupoid of germs of orientation preserving C^r diffeomorphisms, and $r \geqslant 2$, then there is a surjective homomorphism

$$H_3(B\Gamma) \to \mathbb{R}.$$

Thus, the analogue of Corollary 3.3 is not true in this case.

COROLLARY 3.4. $\pi_2(B\Gamma) \simeq G/[G, G]$.

Proof. Since $\pi_1(B\Gamma)=0$, $\pi_2(B\Gamma) \simeq H_2(B\Gamma)$. From the Serre spectral sequence (or the Eilenberg-Moore spectral sequence), it follows that $H_2(B\Gamma) \simeq H_1(G) \simeq \simeq G/[G, G]$. Q.E.D.

COROLLARY 3.5. *There exists an exact sequence*

$$\begin{aligned}
&2(H_2(G)\otimes H_1(G))\oplus \operatorname{Ext}(H_1(G), H_1(G))\\
&\qquad\to H_3(G)\to H_4(B\Gamma)\to H_1(G)\otimes H_1(G)\\
&\qquad\to H_2(G)\to H_3(B\Gamma)\to 0.
\end{aligned}$$

Proof. By the Eilenberg-Moore spectral sequence relating the homology of $\Omega B\Gamma$ and the homology of G, and the main theorem of this paper.

Corollary 3.4 may be applied to give a collection of representatives of homotopy classes (in Haefliger's sense) of Γ-structures on S^2. Consider the composition

$$G \xrightarrow{proj} G/[G, G] \xrightarrow{\sim} \pi_2(B\Gamma).$$

The explicit construction in §1 of the mapping $\delta_h: SBG\to B\Gamma$ yields an explicit construction of the mapping $G\to\pi_2(B\Gamma)$.

Given $g\in G$, we construct ω_g on S^2 as follows. On $\mathbb{R}\times I$ consider the Γ structure $\tilde{\omega}$ defined by the projection on the second factor. Let the group $\mathbb{Z}$ act on $\mathbb{R}\times I$ as the product action of the following two actions: the action of $\mathbb{Z}$ on $\mathbb{R}$ given by $n\cdot t=t+n$ and the action of $\mathbb{Z}$ on I given by

$$\begin{aligned}
n\cdot t &= h^{-1}g^n h(t) \quad \text{if} \quad t\in(a, b)\\
&= t \qquad\qquad\qquad \text{otherwise},
\end{aligned}$$

where the interval $(a, b)\subset(0, 1)$ and the $\mathscr{I}$ homeomorphism $h:(a, b)\to\mathbb{R}$ are as in §1.

Let $\sim$ denote the equivalence relation on $\mathbb{R}\times I$ defined by $(x_0, t_0)\sim(x_1, t_1)$ if

$$t_0=t_1=0, \qquad t_0=t_1=1,$$

or there exists

$$n\in\mathbb{Z} \quad\text{such that}\quad n\cdot(x_0, t_0)=(x_1, t_1).$$

The quotient space $\mathbb{R}\times I/\sim$ is homeomorphic to S^2. It is easily seen that there is a unique Γ-structure ω_g on S^2 such that $\pi^*\omega_g=\tilde{\omega}$, where $\pi:\mathbb{R}\times I\to S^2$ denotes the projection.

Let $\iota_g:\mathbb{Z}\to G$ be the homomorphism defined by $\iota_g(1)=g$. Consider the induced mappings

$$\begin{aligned}
B\iota_g&:B\mathbb{Z}=S^1\to BG\\
SB\iota_g&:S^2=S(S^1)\to S(BG).
\end{aligned}$$

It is easily seen that

$$(SB\iota_g)^*\,\omega_h=\omega_g.$$

It follows that ω_g represents the image of g under the homomorphism $G \to \pi_2(B\Gamma)$, described above.

From the fact that the homomorphism $G/[G, G] \to \pi_2(B\Gamma)$ is an isomorphism, two results follow. First, every Γ-structure on S^2 is homotopic (in the sense of Haefliger) to ω_g for some $g \in G$. Second, ω_g is homotopic to a trivial Γ-structure if and only if g is a product of commutators.

A geometric description of ω_g may be given as follows. There are two nodes of ω_g (the points $\pi(\mathbb{R} \times 0)$ and $\pi(\mathbb{R} \times 1)$) and an arc J on S^2 connecting the two nodes and transversal to ω_g. If we consider any $t \in J$, and follow the leaf of ω_g which contains t around until we come back to J, we get a new point $\tilde{g}(t)$ on J, and the resulting mapping $\tilde{g}: J \to J$ has compact support in the interior of J and is conjugate to g.

It is possible to give a direct geometric proof that every Γ-structure on S^2 is homotopic to one having this form.

In a similar way, for any $c \in H_3(B\Gamma)$, we may give a geometric construction of a Γ-structure on S^3 which represents c. For, by Corollary 2.5, the homomorphism

$$H_2(G) \to H_3(B\Gamma)$$

is onto, so we may find $c' \in H_2(G)$ which maps onto c. Such an element is represented by a homomorphism $\alpha: \pi_1(M) \to G$, where M is a compact oriented surface, i.e. c' is the image of the fundamental class of M under the homomorphism $H_2(M) \to H_2(BG)$ induced by the homotopy class of mappings $M \to BG$ which corresponds to α.

In the same way as we constructed a Γ-structure on SBG we can construct a Γ-structure on SM, corresponding to α. Here we think of SM as two cones over M identified along their base, so SM has two vertices. There is a degree 1 mapping of S^3 onto SM such that the inverse image of each vertex is a bouquet of circles. Furthermore this mapping maps the complement of the two bouquets onto the complement of the vertices. If we pull back the Γ-structure on SM to S^3 by means of this mapping, we get a Γ-structure which has two bouquets of nodes, and otherwise is non-singular. This Γ-structure represents the given element of $H_3(B\Gamma)$.

The next application concerns $\mathscr{S}$-homeomorphisms of the circle. Let $S^1 = \mathbb{R}/\mathbb{Z}$ denote the circle and let $\pi: \mathbb{R} \to S^1$ denote the natural projection. Let U and V be open intervals in S^1 and $f: U \to V$ be a homeomorphism. Choose open intervals $\tilde{U}$ and $\tilde{V}$ in $\mathbb{R}$ such that π maps $\tilde{U}$ homeomorphically onto U and maps $\tilde{V}$ homeomorphically onto V. Let $\tilde{f}: \tilde{U} \to \tilde{V}$ lift f (so that $\pi\tilde{f} = \tilde{f}\pi$). Then we say $\tilde{f}$ is a $\mathscr{S}$-homeomorphism if $\tilde{f}$ is. Since $\mathscr{S}$ contains all triples $(\mathbb{R}, \tau, \mathbb{R})$, where τ is a translation, it follows that if one lifting $\tilde{f}$ of f is a $\mathscr{S}$-homeomorphism, then so is every lifting.

The collection of $\mathscr{S}$-homeomorphisms of open intervals onto open intervals extends uniquely to a pseudogroup $\mathscr{S}(S^1)$ of homeomorphisms of open subsets of the circle onto open subsets of the circle. If $(U, f, V) \in \mathscr{S}(S^1)$, we say $f: U \to V$ is a $\mathscr{S}$-homeomorphism.

The collection of $\mathscr{I}$-homeomorphisms of the circle onto itself forms a group, which we denote by $G(S^1)$.

Let U be an open interval in S^1. It is easily seen that there is a $\mathscr{I}$-homeomorphism $h:\mathbb{R}\to U$. If $g\in G$, we let $h_*g\in G(S^1)$ denote the homeomorphism defined by

$$\begin{aligned} h_*g(t) &= hgh^{-1}(t), && t\in U \\ &= t && \text{otherwise}. \end{aligned}$$

Then $h_*:G\to G(S^1)$ and it induces a mapping $h_*:H_1(G)\to H_1(G(S^1))$.

COROLLARY 3.6. *The mapping*

$$h_*:H_1(G)\to H_1(G(S^1))$$

is an isomorphism.

Proof. Surjectivity. Each element of $G(S^1)$ can be written as a product of $\mathscr{I}$-homeomorphisms of S^1 with support in an interval. Any such $\mathscr{I}$-homeomorphism is conjugate in $G(S^1)$ to a $\mathscr{I}$-homeomorphism which has support in U. But such a $\mathscr{I}$-homeomorphism is in the image of $h_*:G\to G(S^1)$.

Injectivity. Let $\tilde{\omega}$ denote the Γ structure on the cylinder $I\times S^1$ defined by the projection on the second factor. For any $g\in G(S^1)$, identify the ends of the cylinder by means of g, and let ω_g be the Γ structure on the quotient space which pulls back to $\tilde{\omega}$. Since the quotient space is a torus T^2, the obstruction to homotopy-triviality of ω_g is an element of $H_2(T^2,\pi_2(B\Gamma))\simeq H_1(G)$. Thus for each $g\in G(S^1)$, we get $\beta(g)\in H_2(T^2,\pi_2(B\Gamma))=H_1(G)$. It is easily seen that $\beta:G(S^1)\to H_1(G)$ is a homomorphism. Consider the induced homomorphism

$$\bar{\beta}:H_1(G(S^1))=G(S^1)/[G(S^1),G(S^1)]\to H_1(G).$$

It is easily seen that the composition

$$H_1(G)\xrightarrow{h*}H_1(G(S^1))\xrightarrow{\bar{\beta}}H_1(G)$$

is the identity. It follows immediately that h_* is injective.

Now let G^r denote the group of C^r diffeomrophisms of $\mathbb{R}$ with compact support and let $G^r(S^1)$ denote the group of orientation preserving diffeomorphisms of S^1.

COROLLARY 3.7. $H_1(G^\infty)=0$.

COROLLARY 3.8. *If* $4\leqslant r<\infty$, *then the mapping* $H_1(G^r)\to H_1(G^{r-4})$ *induced by the inclusion mapping is* 0.

Proofs. By Corollary 3.6, it is enough to prove these results for $G^r(S^1)$ in place of G^r. For this case, these follow from results of Moser [9].

§4. A quick Review of semi-simplicial Methods

In this section we give a quick review of semi-semplical methods in algebraic topology and some new results. These will be needed in the sequel. For more details, the reader is referred to [7], [11], and [12]. Our notation generally follows [11].

Let $\mathbf{n}$ denote the ordered set $\{0,\ldots, n\}$. Let CSS denote the category whose objects are $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$, $\cdots$ and whose morphisms are weakly order preserving mappings. Let Δ denote the subcategory of CSS having the same objects, but whose morphisms are strictly order preserving mappings. If $\mathscr{A}$ and $\mathscr{B}$ are categories, let $\mathscr{A}\times\mathscr{B}$ denote the category whose objects are given by

$$\mathrm{Ob}(\mathscr{A}\times\mathscr{B})=\mathrm{Ob}(\mathscr{A})\times\mathrm{Ob}(\mathscr{B})$$

and whose morphisms are given by

$$\mathrm{Morph}(\mathscr{A}\times\mathscr{B})=\mathrm{Morph}(\mathscr{A})\times\mathrm{Morph}(\mathscr{B}).$$

Let $\mathscr{A}^2=\mathscr{A}\times\mathscr{A}$, etc.

If $\mathscr{A}$ is a category, then an $\mathscr{A}$-set will mean a contravariant functor from $\mathscr{A}$ to the category of sets. Similarly, an *$\mathscr{A}$-group* (*space*) will mean a contravariant functor from $\mathscr{A}$ to the category of groups and homomorphisms (topological spaces and continuous mappings). The main cases we consider in the sequel are $\mathscr{A}=CSS$, Δ, CSS^2, Δ^2, so we have the notions of CSS-set, Δ-set, CSS-space, etc.

A morphism of $\mathscr{A}$-sets is a natural transformation of functors. Similarly for $\mathscr{A}$-groups and $\mathscr{A}$-spaces. Thus, the $\mathscr{A}$-sets form a category. There are several functors relating Δ-sets, CSS-sets, etc., which will be important in the sequel. For example, since $\Delta\subset CSS$, we may define a *forgetful functor* F from the category of CSS-sets to the category of Δ-sets. Thus, if X is a CSS-set, FX is the composition

$$\Delta\subset CSS\xrightarrow{X}\{\text{Sets}\}.$$

Similarly the diagonal embedding $CSS\to CSS^2$ induces the *diagonal functor* δ from CSS^2-sets (spaces, or groups) to CSS-sets (spaces, or groups). Also, the transpose $T: CSS^2\to CSS^2$, defined by $T(\mathbf{n},\mathbf{p})=(\mathbf{p},\mathbf{n})$, $T(f,g)=(g,f)$ induces the *transpose functor* t from CSS^2-sets (spaces, or groups) to CSS^2-sets (spaces, or groups).

Since the CSS-sets form a category, we have a notion of CSS-(CSS-sets). There is a natural equivalence E from the category of CSS^2-sets to the category of CSS-CSS-sets defined by

$$E(X)(\mathbf{n})(\mathbf{p})=X(\mathbf{n},\mathbf{p})$$
$$E(X)(f)(g)=X(f,g),$$

where f and g are morphisms in the category CSS.

There is another functor A from the category of Δ-sets to the category of CSS^2-sets which we will need to consider later. Let X be a Δ-set. For non-negative integers n and p, we let $\tilde{A}X(\mathbf{n}, \mathbf{p})$ be the set of triples (u, φ, F), where $u \in X(\mathbf{m})$ for some m, $\varphi: \mathbf{n} \to \mathbf{m}$ is a weakly order preserving mapping, and $F = (F_0, \cdots, F_p)$ is a weakly increasing sequence of subsets of $\mathbf{m}$ such that $\varphi(\mathbf{n}) \subset F_0$ and F_p lies in the interval $[\varphi(0), \varphi(n)]$. We let $AX(\mathbf{n}, \mathbf{p})$ be the set of equivalence classes of members of $\tilde{A}X(\mathbf{n}, \mathbf{p})$, where two members (u, φ, F) and (u', φ', F') of $\tilde{A}X(\mathbf{n}, \mathbf{p})$ are said to be equivalent if there exist morphisms j, j' in Δ such that $j^*u = j'^*u'$, $F_p \subset (\text{image } j) \cap \cap (\text{image } j')$, $j^{-1}\varphi = j'^{-1}\varphi$, and $j^{-1}F_i = j'^{-1}F_i$, for $i = 0, \ldots, p$. Note that under the hypothesis that $F_p \subset \text{image } j$, we have $(j^*u, j^{-\prime}\varphi, j^{-1}F)$ is equivalent to (u, φ, F).

If $f: \mathbf{n}' \to \mathbf{n}$ and $g: \mathbf{p}' \to \mathbf{p}$ are morphisms in the category CSS, then

$$(f, g)^*: AX(\mathbf{n}, \mathbf{p}) \to AX(\mathbf{n}', \mathbf{p}')$$

is defined by

$$(f, g)^*(u, \varphi, F) = (u, \varphi f, G),$$

where $G = (G_0, \ldots, G_{p'})$ and

$$G_i = F_{g(i)} \cap [\varphi f(0), \varphi f(n')].$$

It is easily verified that this defines a CSS^2-set. If $\Phi: X \to Y$ is a morphism of Δ-sets, then $\Phi_*: AX \to AY$, defined by

$$\Phi_*(u, \varphi, F) = (\Phi u, \varphi, F)$$

is a morphism of CSS^2-sets. Hence A is a covariant functor from the category of Δ-sets to the category of CSS^2-sets.

Next, we recall the definition of the geometric realization functors. We will think of $\{0, \ldots, n\}$ as the vertices of the affine n-simplex Δ^n, so any mapping $f: \mathbf{n} \to \mathbf{p}$ has a unique affine extension $\Delta^n \to \Delta^p$. We will usually denote the affine extension of f by the same symbol.

Let X be a CSS-space. The geometric realization $|X|$ of X is formed from the disjoint union

$$\bigcup_n X(\mathbf{n}) \times \Delta^n$$

by identifying $(u, f(t))$ and $(f^*(u), t)$ if $u \in X(\mathbf{n})$, $t \in \Delta^p$, and $f: \mathbf{p} \to \mathbf{n}$. The geometric realization $|X|_\Delta$ of a Δ-space is defined similarly. A CSS-set or Δ-set may be regarded as a CSS-space or Δ-space, where each $X(\mathbf{n})$ is provided with the discrete topology. If X is a CSS-set, then $|X|$ is a CW complex with one cell for each non-degenerate simplex of X, i.e., each member of $X(\mathbf{n})$ which cannot be represented in the form f^*u, where f is a surjective CSS-morphism other than the identity. If X is a Δ-set, then $|X|_\Delta$ is a CW-complex with one cell for each simplex of X.

Similarly, one can define the geometric realization $|X|$ of a CSS^2-space and the geometric realization $|X|_\Delta$ of a Δ^2-space. For instance if X is a CSS^2-space, $|X|$ is formed from

$$\bigcup_{n,p} X(\mathbf{n}, \mathbf{p}) \times \Delta^n \times \Delta^p$$

by identifying $(u, f(s), g(t))$ with $((f, g)^*(u), s, t)$ if $u \in X(\mathbf{n}, \mathbf{p})$, $s \in \Delta^{n'}$, $t \in \Delta^{p'}$, and $f: \mathbf{n}' \to \mathbf{n}$ and $g: \mathbf{p}' \to \mathbf{p}$ are CSS-morphisms. One may generalize this construction to CSS^n-spaces, Δ^n-spaces, etc.

Let X be a CSS-space and FX the Δ-space obtained by applying the forgetful functor. Then $|X|$ and $|FX|_\Delta$ are constructed by means of identifications from the same space, only there are more identifications in the construction of $|X|$. It follows that there is a projection $|FX|_\Delta \to |X|$. Likewise if X is a CSS^2-space and FX is the Δ^2-space obtained by applying the forgetful functor, then there is a projection $|FX|_\Delta \to |X|$.

PROPOSITION 4.1. *If X is a CSS-set or CSS^2-set then the projection $|FX|_\Delta \to |X|$ is a homotopy equivalence.*

Proof. This follows by a standard argument from Whitehead's theorem.

Let X be a CSS^2-space and δX the CSS-space obtained by applying the diagonal functor. Since $(\delta X)(n) = X(\mathbf{n}, \mathbf{n})$ we have an inclusion

$$\bigcup_{n} \delta X(\mathbf{n}) \times \Delta^n \subset \bigcup_{n,p} X(\mathbf{n}, \mathbf{p}) \times \Delta^n \times \Delta^p$$

defined by identifying $(u, t) \in \delta X(\mathbf{n}) \times \Delta^n$ with $(u, t, t) \in X(\mathbf{n}, \mathbf{n}) \times \Delta^n \times \Delta^n$. This respects identifications and defines a mapping $|\delta X| \to |X|$ which we call the *canonical mapping*. The following result generalizes the main result of [8] and is proved in the same way, as was observed in [12].

PROPOSITION 4.2. *If X is a CSS^2-set then the canonical mapping $|\delta X| \to |X|$ is a homeomorphism.*

If F is a functor from a category $\mathscr{A}$ to a category $\mathscr{B}$, then there is an induced functor $CSS(F)$ from the category of CSS-objects with values in $\mathscr{A}$ to the category of CSS-objects with values in $\mathscr{B}$. In particular the geometric realization functor $|?|$ from the category of CSS-spaces to the category of spaces induces a functor $CSS|?|$ from the category of CSS-(CSS-spaces) to the category of CSS-spaces. The functor from CSS^2-spaces to CSS-spaces obtained by composing with the natural equivalence

$E: CSS^2$-spaces $\to CSS$-$(CSS$-spaces), defined above, will be denoted $|?|''$. In other words, $|?|''$ is the composition

$$\{CSS^2\text{-spaces}\} \xrightarrow{E} \{CSS\text{-}CSS\text{-spaces}\} \xrightarrow{CSS\,|?|} \{CSS\text{-spaces}\}.$$

If X is a CSS^2-space, $|X|''$ will be called the geometric realization in the second factor. Similarly, the *geometric realization functor in the first factor* is defined by composing on the left with the transpose functor $t: \{CSS^2\text{-spaces}\} \to \{CSS^2\text{-spaces}\}$. If X is a CSS^2-set, its geometric realization in the first factor is denoted $|X|'$.

If X is a topological space, we will denote its singular complex by $S\,(X)$. This is a CSS-set, and there is an evaluation mapping $|S(X)| \to X$ which is a homotopy equivalence if X is a CW-complex. If X is a pointed space (i.e., comes equipped with a base point x_0), then we let $S_0(X)$ denote the subcomplex of $S\,(X)$ whose n-simplices consist of mappings of Δ^n into X which map all vertices of Δ^n to x_0. We will also need to consider a CSS-space $\Omega_* X$ defined as follows. The underlying CSS-set of $\Omega_* X$ is $S_0 X$. The topology on $\Omega_* X\,(\mathbf{n})$ is the compact open-topology on mappings $\Delta^n \to X$.

Clearly the identity is a mapping $S_0(X) \to \Omega_* X$ of CSS-spaces. Then we have natural mappings

$$|S_0 X| \xrightarrow{i} |\Omega_* X| \xrightarrow{ev} X,$$

where i denotes the inclusion mapping and ev the evaluation mapping.

If X is a connected CW complex, each of these mappings is a homotopy equivalence. For, $ev \circ i$ is a homotopy equivalence by an elementary argument based on Whitehead's Lemma, and the fact that ev is a homotopy equivalence is proved in [10].

Let X be a Δ-set. We define a mapping

$$\bigcup_{n,p} AX(\mathbf{n}, \mathbf{p}) \times \Delta^n \times \Delta^p \to \bigcup_n X(\mathbf{n}) \times \Delta^n,$$

as follows. If $(u, \varphi, F) \in AX\,(\mathbf{n}, \mathbf{p})$ and $(s, t) \in \Delta^n \times \Delta^p$, we let the image of $((u, \varphi, F), s, t)$ be $(u, \bar{\varphi}(s))$, where $\bar{\varphi}: \Delta^n \to \Delta^m$ denotes the affine extension of φ. This mapping respects identifications and defines a continuous mapping

$$|AX| \to |X|_\Delta,$$

which we will call the canonical mapping.

PROPOSITION 4.3. *The canonical mapping*, $|AX| \to |X|_\Delta$, *is a homotopy equivalence.*

Proof. Let $\mathbf{\Delta}^n(\mathbf{p})$ denote the set of strictly order preserving mappings of $\mathbf{p}$ into $\mathbf{n}$. The geometric realization of $\mathbf{\Delta}^n$ is, of course, the n-simplex Δ^n. Let $E^n = |A\mathbf{\Delta}^n|$. We first show that E^n is contractible. (For $n = 1, 2, 3$ it can be seen that E^n is homeomorphic to Δ^n, and it seems likely that this is true in general, though we have been unable to show this.)

First, E^n is connected. For, the vertices of E^n correspond to the vertices of Δ^n, and if p, q are two vertices of Δ^n connected by a 1-simplex u, then the corresponding vertices of E^n are connected by the (1, 0)-simplex (u, φ, F), where φ is the identity mapping of $\mathbf{1}$ and $F=(F_0)=(\mathbf{1})$.

Second, $\pi_1(E^n)=0$. The 1-cells in E^n are non-degenerate (1, 0)-simplices. (All (0, 1)-simplices are degenerate.) Every non-degenerate (1, 0)-simplex of E^n is the equivalence class of a (u, φ, F), where u is the identity mapping of $\mathbf{n}$, $\varphi:\mathbf{1}\to\mathbf{n}$ is *strictly* order preserving and $F=(F_0)$ where F_0 is a subset of $[\varphi(0), \varphi(n)]$ which contains $\varphi(0)$ and $\varphi(n)$. But this (1, 0)-simplex is homotopic modulo its endpoints to (u, φ, G) where $G=(G_0)$ and $G_0=[\varphi(0), \varphi(n)]$, since these two (1, 0)-simplices are the sides of the (1, 1) simplex (u, φ, H), where $H=(F_0, G_0)$, and the top and bottom of this (1, 1)-simplex are degenerate.

To see that $\pi_1(E)=0$ it is enough to show that a loop composed of 1-cells in E is trivial. By what we have just shown, it is enough to consider (1, 0)-simplices (u, φ, G) where $G=[\varphi(0), \varphi(n)]$ and u is the identity map of $\mathbf{n}$. But the collection of all $(k, 0)$-simplices (u, ψ, G), where u is as before, ψ is a strictly order preserving mapping $\mathbf{k}\to\mathbf{n}$ and $G=[\psi(0), \psi(\mathrm{k})]$ form an n-simplex in E^n. Since we have already shown that any loop in E^n is homotopic to one in this set, it follows that $\pi_1(E^n)=0$.

Third, $H_i(E^n)=0$, if $i\geqslant 1$. In the case $X=\Delta^n$, the canonical mapping is a mapping $E^n\to\Delta^n$. We filter E^n by the inverse images of the skeleta of Δ^n. We filter the cellular chain complex of E^n-accordingly. The resulting spectral sequence has $E^1_{p,q}=0$ if $q>0$ and the complex $(E^1_{p,0}, \delta)$ is the simplicial chain complex of Δ^n. It follows immediately that $H_i(E^n)=0$, $i\geqslant 1$.

Let $\partial E^n=|A(\partial\Delta^n)|$. In the same way as $|X|_\Delta$ is built up by attaching copies of Δ^n along $\partial\Delta^n$, the space $|AX|$ is built up by attaching copies of E^n along ∂E^n. From the fact that E^n is contractible it then follows, first, that ∂E^n has the homotopy type of S^{n-1} (by induction on n), and, second, that the mapping $|AX|\to|X|_\Delta$ is a homotopy equivalence. Q.E.D.

§5. The Delooping of BG

We recall that BG can be taken as the geometric realization of the CSS-set NG whose n-simplices are n-tuples $(g_1,\dots, g_n)$ of elements of G and whose face and degeneracy operators are given by

$$\partial_0(g_1,\dots,g_n)=(g_2,\dots,g_n)$$
$$\partial_i(g_1,\dots,g_n)=(g_1,\dots,g_ig_{i+1},\dots,g_n),\qquad 0<i<n$$
$$\partial_n(g_1,\dots,g_n)=(g_1,\dots,g_{n-1}),$$

and

$$\eta_i(g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, e, g_i, \ldots, g_n),$$

where e is the identity element of G.

Now we give Quillen's construction of a delooping of BG. For any real numbers $a \leqslant b$, let

$$G_{a,b} = \{g \in G : \operatorname{supp} g \subset (a, b)\}.$$

Clearly $G_{a,b}$ is a subgroup of G.

LEMMA 5.1. *The inclusion mapping* $BG_{a,b} \subset BG$ *induces isomorphism in integer homology.*

This is proved in the same way as the Lemma in [6].

We consider a category whose objects are real numbers, and whose morphisms from a to b are the elements of $BG_{a,b}$. We give the set of objects the discrete topology, and topologize the morphisms as the disjoint union $\bigcup_{a \leqslant b} BG_{a,b}$. For $a \leqslant b \leqslant c$, we have a mapping

$$BG_{a,b} \times BG_{b,c} \to BG_{a,c} \tag{1}$$

which comes from the morphism of CSS-sets defined on n-cells by

$$(g_1, \ldots, g_n) \cdot (g_1', \ldots, g_n') \to (g_1 g_1', \ldots, g_n g_n').$$

Note that the fact that this is a morphism of CSS-sets depends on the fact that every element of $G_{a,b}$ commutes with every element of $G_{b,c}$. We take the mapping (1) as the composition law. This defines a topological category. We take $B(BG)$ to be the geometric realization of the nerve of this category in the sense of Segal [12].

We will denote the nerve by $N(BG)$. We recall its definition. It is the CSS-space whose n-cells are *composable* n-tuples $(x_1, \ldots, x_n)$ of morphisms, i.e., n-tuples such that $x_1 x_2 \ldots x_n$ is defined. Its 0-cells are the objects of the category, and the topology on the set of n-cells is that induced from the topology on the category. Then $B(BG) = |N(BG)|$. This is Quillen's construction of a delooping of BG.

Another way of describing the set of n-cells of $N(BG)$ is as the disjoint union

$$\bigcup_{a(0) \leqslant \cdots \leqslant a(n)} BG_{a(0)a(1)} \times \cdots \times BG_{a(n-1)a(n)}.$$

Each BG_{ab} is the realization of a CSS-set $NG_{a,b}$, and therefore the set of n-cells of $N(BG)$ is the realization of the disjoint union

$$\bigcup_{a(0) \leqslant \cdots \leqslant a(n)} NG_{a(0)a(1)} \times \cdots \times NG_{a(n-1)a(n)}.$$

The boundaries and degeneracies in $N(BG)$ arise from CSS-mappings, so $B(BG)$ can also be represented as the geometric realization of a CSS^2-set, which we will denote $N(NG)$. Thus, a (p, q)-cell of $N(NG)$ is a $2p+1$ tuple $(a_0, \ldots, a_p, c_1, \ldots, c_p)$ where $a_0 \leqslant \cdots \leqslant a_p$ are real numbers, and each c_i is a q-cell in $BG_{a(i-1)a(i)}$.

Let $\Delta(\mathbb{R})$ denote the functor from CSS to (sets) of weakly order preserving mappings of objects of CSS into $\mathbb{R}$. Clearly $|\Delta(\mathbb{R})|$ is contractible. Clearly $\Delta(\mathbb{R})$ is isomorphic to the CSS-set of $(p, 0)$-cells of $N(NG)$. Hence $|\Delta(\mathbb{R})| \subset |N(NG)|$. On the other hand, all (0, 1)-cells of $N(NG)$ are degenerate, so the 1-skeleton $|N(NG)|_1 \subset \subset \Delta(\mathbb{R})$. Hence

$$\pi_1(B(BG)) = 0.$$

Now we define another space $B'(BG)$, which is also the realization of a CSS^2-set $N'(NG)$. If $c = (g_1, \ldots, g_q)$ is a q-cell in NG, we let the *interval* $I(c)$ be the smallest interval which contains $\operatorname{supp} g_1 \cup \cdots \cup \operatorname{supp} g_q$. Clearly $I(c)$ is a closed bounded interval in $\mathbb{R}$. If I_1 and I_2 are two closed bounded intervals in $\mathbb{R}$, we say $I_1 < I_2$ if $s \in I_1$ and $t \in I_2$ imply $s < t$. We let the set of (p, q)-cells of $N'(NG)$ be the set of p-tuples $(c_1, \ldots, c_p)$, where each c_i is a q-cell in NG, and $I(c_1) < \cdots < I(c_p)$. We define the "back" face and degeneracy operators pointwise

$$\partial_i''(c_1, \ldots, c_p) = (\partial_i c_1, \ldots, \partial_i c_p)$$
$$\eta_i''(c_1, \ldots, c_p) = (\eta_i c_1, \ldots, \eta_i c_p).$$

We define the "front" face and degeneracy operators by

$$\partial_0'(c_1, \ldots, c_p) = (c_2, \ldots, c_p)$$
$$\partial_i'(c_1, \ldots, c_p) = (c_1, \ldots, c_i c_{i+1}, \ldots, c_p) \qquad 0 < i < p$$
$$\partial_p'(c_1, \ldots, c_p) = (c_1, \ldots, c_{p-1})$$
$$\eta_i'(c_1, \ldots, c_p) = (c_1, \ldots, c_i, e_q, c_{i+1}, \ldots, c_p),$$

where if $c_i = (g_1, \ldots, g_q)$ and $c_{i+1} = (g_1', \ldots, g_q')$ then $c_i c_{i+1} = (g_1 g_1', \ldots, g_q g_q')$, and $e_q = (e, \ldots, e)$.

We have a CSS^2-mapping $N(NG) \to N'(NG)$, defined on (p, q)-cells by

$$(a_0, \ldots, a_p, c_1, \ldots, c_p) \to (c_1, \ldots, c_p).$$

We claim that the induced mapping on geometric realizations

$$B(BG) \to B'(BG)$$

is a homotopy equivalence. We have already shown that $B(BG)$ is simply connected. Moreover all (1, 0) and (0, 1)-cells of $B'(BG)$ are degenerate, so it is also simply connected. On the other hand, it is easily seen that a fiber of $B(BG)$ above a point of $B'(BG)$ is acyclic, so that the induced mapping in homology is an isomorphism. Then the above mapping is a homotopy equivalence, by Whitehead's theorem.

We may also regard $B'(BG)$ as the geometric realization of a CSS-space $N'(BG)$ by realizing $N'(NG)$ in the second factor. The 1-cells of this space form a space naturally homeomorphic to BG, and the natural mapping $BG\times\Delta^1\to B'(BG)$ induces an embedding of the reduced suspension $S(BG)$ in $B'(BG)$.

We will show (following Quillen) that the adjoint mapping $BG\to\Omega B'(BG)$ induces isomorphisms in integer homology. This is the sense in which $B'(BG)$ (and therefore also $B(BG)$) is a delooping of BG.

Quillen's proof is based on Segal's construction Ω_*, defined in §4. There is a CSS-mapping

$$N'(BG)\to\Omega_*(B(BG))$$

defined by sending an n-cell of $N'(BG)$ into the corresponding mapping of Δ^n into $B(BG)$. This induces a mapping on spaces

$$B'(BG)\to|\Omega_*(B(BG))|,$$

and from the results of §4 it follows that this mapping is a homotopy equivalence.

On the other hand, using the spectral sequence associated to the filtration of a CSS-space by its skeleta, we get a homomorphism of spectral sequences, each of which converges to a graded group, associated to the homology of $B'(BG)$. Moreover the E^1 terms in these spectral sequences can be computed from the homology of BG and the homology of $\Omega B'BG$, respectively, using Künneth's formula. Thus, for the spectral sequence associated to the filtration of $B'(BG)$ we have

$$E'_{p*}\simeq\bigotimes_p H_*(BG),$$

and for the filtration of $|\Omega_*BBG|$ we have

$$E'_{p*}\simeq\bigotimes_p H_*(\Omega B'BG),$$

where we take a *field* of coefficients. Then a standard spectral sequence comparison argument shows

$$H_*(BG)\to H_*(\Omega B'BG)$$

is an isomorphism, if the coefficient ring is a field. But, then this is still an isomorphism when the coefficient ring is the integers, by the universal coefficient formula.

We will show in subsequent sections that BBG has the homotopy type of $B\Gamma$.

Here we show that this assertion implies the main result in our announcement [4]. For, $B'BG$ has the same homotopy type as BBG. If we give BG^k the product CW-complex structure, coming from the CSS-structure of BG, the space of k-cells of $N'(BG)$ is a subcomplex. The resulting CW-complex structure on the space of k-cells induces a CW-complex structure on $B'BG$. Then the chain complex $\beta\beta G$ of [4] is isomorphic to the chain complex associated to this CW-complex structure on $B'BG$. Theorem 1 of [4] then follows from the assertion that $BBG \sim B\Gamma$.

§6. Construction of a Γ-Structure on BBG

It is enough to define a 1-cocycle $\gamma_{\sigma\tau}$ over the cover of BBG by its closed cells, by §1. First, we define $\gamma_\sigma = \gamma_{\sigma\sigma}$ for a (p, q)-cell $\sigma = (a_0, \ldots, a_p, c_1, \ldots, c_p)$. Let $c_i = (h_{1i}, \ldots, h_{qi})$ where $h_{ji} \in BG_{a(i-1), a(i)}$. Hence

$$H_i = \prod_{j=1}^{p} h_{ij}.$$

This notation is unambiguous, because h_{ij} and h_{ik} commute for $j \neq k$, since they have disjoint support. Let $f : \Delta^p \to \mathbb{R}$ be the unique affine extension of the set map $\mathbf{p} \to \mathbb{R}$ given by $f(0) = a_0, \ldots, f(p) = a_p$.

For $0 \leqslant t \leqslant 1$, let H_i^t be the homeomorphism of $\mathbb{R}$ given by $H_i^t(u) = tH_i(u) + (1-t)u$. Let $x = (x_0, \ldots, x_p) \in \Delta^p$ and $y = (y_0, \ldots, y_q) \in \Delta^q$. Let

$$\gamma_0(x, y) = H_1^{t(1)} \cdots H_q^{t(q)} f(x)$$

where

$$t(q) = y_q, \; t(q-1) = \frac{y_{q-1}}{1 - y_q}, \ldots, t(1) = \frac{y_1}{1 - y_q - \cdots - y_2}.$$

Now we define $\gamma_{\sigma\tau}$ when τ is a face of σ. When τ is a front face $\delta_i'\sigma$, we let $\gamma_{\sigma\tau}(x, y)$ be the germ at $\gamma_\tau(x, y)$ of the identity mapping. When τ is a back face $\delta_i''\sigma$, we let $\gamma_{\sigma\tau}$ be the germ of id at $\gamma_\tau(x, y)$, provided $i \geqslant 1$. When $\tau = \delta_0''\sigma$, we let $\gamma_{\sigma\tau}(x, y)$ be the germ of H_1 at $\gamma_\tau(x, y)$. It is easy to see that this defines a 1-cocycle over the closed cells of BBG, and hence a Γ-structure on BBG, by §1.

§7. Subdivision of Δ-Sets

DEFINITION. If X is a Δ-set, a subdivision of X will mean a pair (X', s), where X' is a second Δ-set and s is a homeomorphism of $|X'|_\Delta$ onto $|X|_\Delta$ with the property that if σ is an open cell in $|X|_\Delta$, then $s^{-1}\sigma$ is a union of open cells in $|X'|_\Delta$.

Consequences of the Definition. We note some consequences of this Definition.

First, if σ is a closed cell of $|X'|_\Delta$ then there is a least closed cell τ of $|X|_\Delta$ which contains $s(\sigma)$. For, it follows from the Definition that if p and q are two interior points of σ, then the unique open cell τ^0 of $|X|_\Delta$ which contains $s(p)$ is the same as the unique open cell which contains $s(q)$, and clearly its closure τ is the least closed cell which contains $s(\sigma)$.

Second, if $f: Y \to X$ is a Δ-mapping of Δ-sets, then there exists a unique (up to canonical isomorphism) triple $(Y', \tilde{s}, f')$ such that $(Y', \tilde{s})$ is a subdivision of Y, $f': Y' \to X'$ is a Δ-mapping, and the following diagram commutes:

$$\begin{array}{ccc} |Y'|_\Delta & \xrightarrow{\tilde{s}} & |Y|_\Delta \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ |X'|_\Delta & \xrightarrow{s} & |X|_\Delta \end{array}$$

We will write $(Y', \tilde{s}) = f^*(X', s)$, and call $(Y', \tilde{s})$ the subdivision of Y *induced* via f from (X', s). We will call f' the *canonical mapping* of $|Y'|_\Delta$ into $|X'|_\Delta$.

Now we prove the existence of $(Y', \tilde{s}, f')$. We let Y'_p denote the set of pairs (σ, τ) where $\sigma \in X'_p$, $\tau \in Y_m$, $m \geqslant p$, and $f\tau$ is the least cell of X whose geometric realization contains $s(\sigma)$. We note that if $(\sigma, \tau) \in Y'_p$, then there is a unique mapping $\eta = \eta_{\sigma,\tau}$ of the standard p-simplex Δ^p into $|\tau|_\Delta$ such that the following diagram commutes.

$$\begin{array}{ccc} \Delta^p & \xrightarrow{\eta} & |\tau|_\Delta \\ \downarrow & & \downarrow{\scriptstyle f} \\ |\sigma|_\Delta & \xrightarrow{s} & |f\tau|_\Delta \end{array}$$

Here the left vertical arrow denotes the natural mapping of Δ^p onto the geometric realization of the p-simplex σ. It is clear how to define η on the interior of Δ^p, since s maps the interior $|\sigma|^0_\Delta$ of $|\sigma|_\Delta$ into $|f\tau|^0_\Delta$ and f maps $|\tau|^0_\Delta$ homeomorphically onto $|f\tau|^0_\Delta$. We must show η extends continuously to the boundary of Δ^p. Let $x \in \partial\Delta^p$, let $\bar{x}$ be its image in $|\sigma|_\Delta$ and let $y = s(\bar{x})$. Then $f^{-1}y$ is a finite set of points $y_1, \ldots, y_k$. We may choose a neighborhood U of y such that $f^{-1}U$ is a disjoint union of sets $U_1, \ldots, U_k$ where U_i is a neighborhood of y_i. Let V be a neighborhood of x in the inverse image of $s^{-1}U$ under $\Delta^p \to |\sigma|_\Delta$, and suppose the intersection V^0 of V with the interior of Δ^p is connected. Then $\eta(V^0)$ is in some one of the U_i's. Let $\eta(x)$ be the corresponding y_i. It is easily verified that η, so defined, is continuous.

Now for $0 \leqslant i \leqslant p$, we can define the boundary operator $\partial_i: Y'_p \to Y'_{p-1}$. Let $\partial_i(\sigma, \tau) = (\partial_i\sigma, \tau_i)$, where τ_i is the least cell of $|Y|_\Delta$ which contains $\eta_{\sigma,\tau}(\partial_i\Delta^p)$. It is easily verified that the collection of sets $\{Y'_p\}$ and the boundary operators ∂_i defines a Δ-set Y'. We let $f': Y' \to X'$ be given by $f(\sigma, \tau) = \sigma$. Finally, it is easily seen that there is a unique mapping $\tilde{s}$ such that for any $(\sigma, \tau) \in Y'_p$, the composition

$$\Delta^p \to |\sigma, \tau|_\Delta \xrightarrow{\tilde{s}} |Y|_\Delta$$

is $\eta_{\sigma,\tau}$, where the first arrow denotes the natural mapping associated to the p-cell (σ, τ).

The uniqueness of $(Y', \tilde{s}, f')$ is easily seen.

Special subdivisions. Let Δ^n denote the n-simplex with the standard triangulation. This may be regarded as a Δ-set, and its boundary $\partial\Delta^n$ as a sub Δ-set. If (X, s) is a subdivision of $\partial\Delta^n$, we define the *conical extension* (CX, Cs) of (X, s), by letting CX be the cone on X, and letting $Cs: |CX|_\Delta = C|X|_\Delta \to \Delta^n$ be the homeomorphism which sends the line segment vx joining the vertex v of the cone $C|X|_\Delta$ to a point x in $|X|_\Delta$ affinely to the line segment $bs(x)$, where b is the barycenter of Δ^n. Clearly this is a subdivision of Δ^n.

If X is a Δ-set, a collection U of cells of X will be said to be an *open subcomplex* if whenever $\sigma, \tau \in X$, $\sigma \in U$, and σ is a generalized face of τ, we have $\tau \in U$.

If we are given a triple $(X, U, \{h\})$, where X is a Δ-set, U is an open subcomplex of X and $\{h\}$ is a collection of homeomorphisms, one h_u for each $u \in U$, where if u is an n-cell then h_u is a face preserving homeomorphism of Δ^n, we may define a subdivision $(X', s) = (X'_U, s'_{U,\{h\}})$ of X as follows. We define inductively a subdivision (X'_n, s_n) of the n-skeleton X_n. We let $X'_0 = X_0$, $s_0 = \text{id}$. Supposing (X'_{n-1}, s_{n-1}) has been defined, we see easily that there is a unique (up to unique isomorphism) subdivision (X'_n, s_n) of X_n such that

a) if u is an n-cell of X, not in U, and $f: \Delta^n \to |X|_\Delta$ is the associated mapping, then $f^*(X'_n, s_n)$ is the trivial subdivision (Δ_n, id) of Δ_n, and

b) if $u \in U$ is an n-cell, and $f: \Delta^n \to |X|_\Delta$ is the associated mapping, then $fh_u^*(X'_n, s_n)$ is the conical extension of $(fh_u \mid \partial\Delta^n)^* (X'_{n-1}, s_{n-1})$.

Finally, the subdivision of X is the limit of the subdivisions of the skeletons we have defined in this way.

In the special case when $U = X$ and each h_u is the identity, the subdivision we have just defined is barycentric subdivision. More generally, if each h_u is the identity, it is a generalization of barycentric subdivision, where each cell of U is affinely subdivided, but the cells of $X - U$ are left unchanged. In the general situation, each cell of $X - U$ is left unchanged, but the cells of U are subdivided.

When $U = X$, we will drop U from the above notation. When each h_u is the identity, we will drop $\{h\}$.

Product of a Δ-set with I. If K is a Δ-set, then it is possible to define another Δ-set which we denote by $K*I$, such that $|K*I|_\Delta$ is homeomorphic to $|K|_\Delta \times I$. We let $(K*I)_m$ consist of all pairs (u, g) where $u \in K_n$ for some n and $g: \mathbf{m} \to \mathbf{n} \times \mathbf{1}$ is a strictly order preserving mapping (with respect to the product order on $\mathbf{n} \times \mathbf{1}$) and the composition

$$\mathbf{m} \xrightarrow{g} \mathbf{n} \times \mathbf{1} \xrightarrow{\pi_1} \mathbf{n}$$

is onto, where π_1 denotes the projection on the first factor. If $f: \mathbf{p} \to \mathbf{m}$ is a strictly order preserving mapping, we let $F\colon \mathbf{q}\to\mathbf{n}$ be the unique strictly order preserving mapping whose image is the image of the composition

$$\mathbf{p} \xrightarrow{f} \mathbf{m} \xrightarrow{g} \mathbf{n}\times\mathbf{1} \xrightarrow{\pi_1} \mathbf{n}$$

We let $f^*(u, g)=((F^*u,(F\times id_1)^{-1}gf))$. This defines the Δ-set $K{*}I$.

Let $h_1\colon|K{*}I|_\Delta\to|K|_\Delta$ be defined as follows. If (u, g) is as above, h_1 maps the m-cell in $|K{*}I|_\Delta$ corresponding to (i, g) to the n-cell in $|K|_\Delta$ corresponding to u by linear extension of the mapping $\pi_1 g$. Let $h_2\colon|K{*}I|_\Delta\to I$ be defined so it takes the m-cell in $|K{*}I|_\Delta$ corresponding to (u, g) to I by linear extension of the mapping $\pi_2 g$. Then it is easily seen that

$$(h_1, h_2)\colon |K*I|_\Delta \to |K|_\Delta\times I$$

is a homeomorphism. We call this the canonical homeomorphism of $|K{*}I|_\Delta$ onto $|K|_\Delta\times I$.

Lifting mappings. Now let K be a Δ-set and X a topological space. It is easily seen that the Δ-mappings of K into the singular complex $S(X)$ correspond bijectively to the continuous mappings of $|K|_\Delta$ into X. Given a continuous mapping $f\colon|K|_\Delta\to X$, we let $\tilde f\colon|K|_\Delta\to|S(X)|_\Delta$ denote the geometric realization of the Δ-mapping $K\to S(X)$ which corresponds to f. We call $\tilde f$ the *canonical lifting* of f. This is a lifting in the sense that f is the composition

$$|K|_\Delta \xrightarrow{\tilde f} |S(X)|_\Delta \to X.$$

Lifting homotopies. Note that if $f_t\colon|K|_\Delta\to X$, $0\leqslant t\leqslant 1$, is a homotopy, then $\tilde f_t\colon|K|_\Delta\to|S(X)|_\Delta$ is *not* a homotopy, because it is discontinuous as a mapping of $|K|_\Delta\times I$ to $|S(X)|_\Delta$. For this reason, we introduce another homotopy, which we call the canonical lifting of the homotopy.

If $F\colon|K|_\Delta\times I\to X$ is a homotopy, we let $\tilde F\colon|K|_\Delta\times I\to S(X)$ denote $\widetilde{FH}H^{-1}$ where $H\colon|K{*}I|_\Delta\to|K|_\Delta\times I$ is the canonical homeomorphism. We call $\tilde F$ the canonical lifting of F. Note that if F is the constant homotopy $\tilde F$ is not generally the constant homotopy.

A homotopy associated to a special subdivision. There is one more explicit homotopy that we will need. Let $(K, U, \{h\})$ be a triple, where K is a Δ-set, U is an open subcomplex of K, and $\{h\}$ is a family of homeomorphisms, one for each $u\in U$, where if u is an n-cell of U, then h_u is a face preserving homeomorphism of Δ^n. Let $(K', s)=(K'_U, s_{U,\{h\}})$ be the associated subdivision. We have two mappings $|K|_\Delta\to|S(|K|_\Delta)|_\Delta$; one is the canonical lifting $\widetilde{id}$ of the identity mapping of $|K|_\Delta$; the other is the composition $\tilde s s^{-1}$:

$$|K|_\Delta \xrightarrow{s^{-1}} |K'|_\Delta \xrightarrow{\tilde s} |S(|K|_\Delta)|_\Delta.$$

We will construct a homotopy between these two mappings.

We construct a complex $M=(K*I)_U$ and a homeomorphism

$$S=S_{U,\{h\}}:|(K*I)_U|_\Delta \to |K|_\Delta \times I$$

as follows. For each Δ-subset L of K, we will construct a Δ-set M_L and a homeomorphism

$$S_L:|M_L|_\Delta \to |L|_\Delta \times I,$$

and let $M=M_K$, $S=S_K$. If L is the complement of U, we let $M_L=L*I$ and S_L be the canonical homeomorphism.

In the general situation, we define M_L and S_L inductively. Supposing u is a top dimensional cell of L and $L'=L-u$, we may also suppose $M_{L'}$ and $S_{L'}$ are defined and that $L'\subset M_{L'}$ and $S_{L'}(|L'|_\Delta)=|L'|_\Delta\times 0$. Let $n=\dim u$ and let $J=(\Delta^n\times 0)\cup(\partial\Delta^n\times I)$. It can be shown that there is a unique (up to canonical isomorphism) triple (J', S', η'), where J' is a Δ-set, $S'=|J'|_\Delta\to J$ is a subdivision of J, and $\eta':J'\to M_{L'}$ is a Δ-mapping such that the following diagram commutes.

$$\begin{array}{ccc} |J'|_\Delta & \xrightarrow{S'} & J \\ \downarrow{\scriptstyle\eta'} & & \downarrow{\scriptstyle\eta} \\ |M_{L'}\cup L|_\Delta & \to & (|L'|_\Delta\times I)\cup(|L|_\Delta\times 0), \end{array}$$

where η denotes the composition

$$J\subset \Delta^n\times I \xrightarrow{u\times id} |L|_\Delta\times I.$$

Let $S'':|J''|\to\Delta^n\times I$ be the unique subdivision of $\Delta^n\times I$ such that there is a Δ-isomorphism H of the cone CJ' onto J'' with the following property. Let v denote the vertex of $C|J'|_\Delta$ and x an arbitrary point of the base $|J'|_\Delta$ of this cone. Then the line segment vx is mapped affinely by $(h_u\times id)\circ S''\circ H$ onto the line segment $b'y$ in $\Delta^n\times I$, where $b'=b\times 1$, $b=$the barycenter of Δ^n and $y=(h_u\times id)\circ S'(x)$.

It is easily seen that there is a unique subdivision $S_L:|M_L|_\Delta\to|L|_\Delta\times I$ which pulls back to $(M_{L'}, S_{L'})$ on $L'\times I$ and to (J'', S'') on $\Delta^n\times I$.

Thus, we get a subdivision (M, S) of $|K|_\Delta\times I$. The restriction of this subdivision to $|K|_\Delta\times 0$ is the identity subdivision of the latter, and its restriction to $|K|_\Delta\times I$ is $(K'_U, S_{U,\{h\}})$.

Let $\pi_1:|K|_\Delta\times I\to|K|_\Delta$ denote the projection on the first factor. The composition

$$|K|_\Delta\times I\xrightarrow{S^{-1}}|(K*I)_U|\xrightarrow{\widetilde{\pi_1 S}}|S(|K|_\Delta)|_\Delta$$

is a homotopy connecting $\widetilde{id}$ and $\tilde{s}s^{-1}$. We will call this the homotopy *associated with* the data $(K, U, \{h\})$.

§8 Quasi-linear Mappings

We recall that a mapping f from a simplicial complex K to $\mathbb{R}$ is said to be *linear* if its restriction to each simplex of K is affine, in the usual sense. We will say a mapping $f:K\to\mathbb{R}$ is *quasi-linear* if there is a homeomorphism h of K onto itself which preserves each simplex of K and satisfies the condition: fh is linear.

LEMMA 8.1. *In order for $f:K\to\mathbb{R}$ to be quasi-linear, it is necessary and sufficient that its restriction to each simplex of K be quasi-linear.*

Proof. Necessity is obvious. To prove sufficiency, we let K_l denote the l-skeleton of K. We will construct a homeomorphism h_l of K_l onto itself, preserving all simplices, such that $f\circ h_l$ is linear, by induction on l. We let $h_0=id:K_0\to K_0$. Supposing h_{l-1} has been constructed, we construct h_l. It suffices to construct $h_l \mid \sigma$ such that $f\circ h_l \mid \sigma$ is linear and $h_l \mid \partial\sigma=h_{l-1} \mid \partial\sigma$, for each l-simplex separately. By hypothesis $f\circ h_{l-1} \mid \partial\sigma$ is linear and there exists a face preserving homeomorphism h_σ of σ such that $f\circ h_\sigma$ is linear. Then both $f\circ h_\sigma \mid \partial\sigma$ and $f\circ h_{l-1} \mid \partial\sigma$ are linear and they have the same values on the vertices of σ. Hence these two mappings are the same. Let H be the mapping of σ in to itself obtained by coning $h_\sigma^{-1}h_{l-1} \mid \partial\sigma$ with respect to a vertex v in the interior of σ. Then H extends $h_\sigma^{-1}h_{l-1} \mid \partial\sigma$ and it is a homeomorphism. Furthermore $f\circ h_\sigma\circ H=f\circ h_\sigma$, since this is true on $\partial\sigma$. Let $h_l \mid \sigma=h_\sigma H$. Clearly $h_l \mid \sigma$ extends $h_{l-1} \mid \partial\sigma$, and $f\circ h_l \mid \sigma=f\circ h_\sigma\circ H=f\circ h_\sigma$ is linear. Q.E.D.

We will use the above lemma only in the case K is of the form $\partial\sigma$, but it is just as easy to prove it in general.

LEMMA 8.2. *Suppose σ is a simplex and $f:\sigma\to\mathbb{R}$ is quasi-linear. Let $h':\mathbb{R}\to\mathbb{R}$ be a homeomorphism. Then $h'f$ is quasi-linear.*

Proof. We suppose h' is orientation preserving, which we may do, since the general result reduces immediately to this case. Let $h:\sigma\to\sigma$ be a face preserving homeomorphism such that fh is linear.

We prove the lemma by induction on $n=\dim\sigma$. For a 0-simplex, the lemma is obvious. Assume the result has been proved for an $(n-1)$-simplex. Then $h'fh$ is quasi-linear on each face of σ, so by Lemma 8.1, there exists a face-preserving homeomorphism $h_1:\partial\sigma\to\partial\sigma$ such that $h'fhh_1:\partial\sigma\to\mathbb{R}$ is linear. Let $[a, b]=f(\sigma)$, $\tau_1=f^{-1}(a)$, $\tau_2=f^{-1}(b)$, and let p_1 and p_2 be points in the interiors of the simplices τ_1 and τ_2 respectively. Now let $(h_2:\tau_i\to\tau_i)$=the cone construction on $h_1 \mid \partial\tau_i$ for $i=1, 2$. It is easily seen that there exists a face preserving homeomorphism $h_3:\partial\sigma\to\partial\sigma$ which preserves the level surfaces of the linear map $h'fhh_1$ and extends $h_1^{-1}h_2:\tau_i\to\tau_i$ for $i=1, 2$. Let $h_2=h_1h_3:\partial\sigma\to\partial\sigma$. This extends $h_2:\tau_i\to\tau_i$ and has the same properties as h_1: it is a face preserving homeomorphism of $\partial\sigma$ such that $h'fhh_2:\partial\sigma\to\mathbb{R}$ is linear.

We extend h_2 to a homeomorphism $\tilde{h}_2$ of σ onto itself, as follows. Let g denote the linear mapping of σ into $\mathbb{R}$ which extends $h'fhh_2$. Let l denote the line segment of σ joining p_1 and p_2. Since fh and g are linear, $(fh)^{-1}(t)\cap l$ is a single point $\gamma(t)$, for each $t\in[a, b]$, and $g^{-1}(t)$ is a single point $\mu(t)$ for each $t\in[h'(a), h'(b)]$. Clearly, h_2 maps $\partial\sigma\cap g^{-1}(t)$ homeomorphically onto $\partial\sigma\cap(fh)^{-1}(h')^{-1}(t)$. Let $\tilde{h}_2 \mid g^{-1}(t)$ be the mapping of $g^{-1}(t)$ onto $(fh)^{-1}(h')^{-1}(t)$ obtained by applying the cone construction to $h_2 \mid \partial\sigma\cap g^{-1}(t)$ with vertices $\mu(t)$ (in the source) and $\gamma h'^{-1}(t)$ (in the target). We may apply the cone construction here because $g^{-1}(t)$ and $(fh)^{-1}(h')^{-1}(t)$ are convex sets, with $\mu(t)$ and $\gamma h'^{-1}(t)$ in their interiors.

It is easily verified that $\tilde{h}_2$ is a homeomorphism of σ onto itself. Since $\tilde{h}_2$ extends h_2, it preserves faces. Then $g=h'fh\tilde{h}_2$ is linear, so $h'f$ is quasi-linear. Q.E.D.

§9. $\Sigma(\omega)$

If σ is a simplex, a Γ structure on σ will said to be *quasi-linear* if it is defined by a single (local) projection $f:\sigma\to\mathbb{R}$, and f is a quasi-linear mapping.

Let X be a topological space and let $S(X)$ denote the singular complex of X (regarded as a CSS-set). For any Γ structure ω on X, let $\Sigma(\omega)\subset S(X)$ be the CSS-set whose set of n-cells consists of all n-simplices $u:\Delta^n\to X$ such that $u^*\omega$ is quasi-linear.

Let U be an open set in X. We say U is a *product neighborhood for* ω if there is a function $f: U\to\mathbb{R}$ which defines ω in U, such that $f(U)$ is an open interval (a, b) and $f: U\to(a, b)$ is a trivial fibration. We will say ω is *locally trivial* at $x\in X$ if x is contained in a product neighborhood for ω. We will say ω is a *foliation* of X if it is locally trivial at every point in X.

PROPOSITION 9.1. *If X is normal and ω is a foliation of X, then the inclusion mapping $|\Sigma(\omega)|\to|S(X)|$ is a homotopy equivalence.*

Proof. We will show the equivalent statement that $|\Sigma(\omega)|_\Delta\to|S(X)|_\Delta$ is a homotopy equivalence. For this, it is enough to show the following. If K is a finite Δ-subset of $S(X)$ and $L=\Sigma(\omega)\cap K$, then there is a homotopy

$$h_t:(|K|_\Delta, |L|_\Delta) \to (|S(X)|_\Delta, |\Sigma(\omega)|_\Delta)$$

such that h_0 is the identity and h_1 maps $|K|_\Delta$ into $|\Sigma(\omega)|_\Delta$.

Let (K, L) be as above. For each n-cell u in L, let h_u be a face preserving homeomorphism of Δ^n and $f_u:\Delta^n\to\mathbb{R}$ a mapping which defines $u^*\omega$ such that $f_u h_u$ is linear. Since $u^*\omega$ is quasi-linear, we may find such f_u and h_u. If u is a cell of K and not a cell of L, let $h_u=id$.

Now consider the subdivision $(K', s_{\{h\}})$ of K. Clearly L' is a Δ-subset of K' and $(L', s_{\{h\}})$ is a subdivision of L. Using Lemma 8.2, we see easily that for each n-cell u

of L', $\varphi_u^* \omega$ is quasi-linear, where φ_u is the composition

$$\Delta^n \xrightarrow{u} |L'|_\Delta \xrightarrow{s\{h\}} |L|_\Delta \to |S(X)|_\Delta \to X,$$

and the last two arrows are the natural mappings. Note that this result would not be true if we used ordinary barycentric subdivision rather than this twisted form of barycentric subdivision.

Consider the composition

$$|K'|_\Delta \xrightarrow{s\{h\}} |K|_\Delta \to |S(X)|_\Delta \to X,$$

and its canonical lifting

$$|K'|_\Delta \xrightarrow{\psi} |S(X)|_\Delta$$

(cf. §7). The latter is the realization of a Δ-mapping $\psi: K' \to S(X)$, and by what we have just shown $\psi(L') \subset \Sigma(\omega)$. Moreover the composition

$$(|K|_\Delta, |L|_\Delta) \xrightarrow{s^{-1}\{h\}} (|K'|_\Delta, |L'|_\Delta) \xrightarrow{\psi} (|S(X)|_\Delta, |\Sigma(\omega)|_\Delta)$$

is homotopic to the inclusion

$$(|K|_\Delta, |L|_\Delta) \to (|S(X)|_\Delta, |\Sigma(\omega)|_\Delta),$$

by the homotopy constructed at the end of §7.

We have observed that if u is an n-cell of L', then $\psi(u)^*\omega$ is quasi-linear. Thus we may choose a projection $f_u': \Delta^n \to \mathbb{R}$ for $\psi(u)^*\omega$ and a face preserving homeomorphism h_u' of Δ^n such that $f_u' h_u'$ is linear. In fact, it can be shown that f_u' and h_u' can be chosen so that if $g: \Delta^n \to \mathbb{R}$ is the function which takes the value 0 on the front $(n-1)$-face and the value 1 on the last vertex, then $gh_u' = g$. Suppose h_u' has been so chosen for each cell $u \in L'$ and that $h_u' = id$ if u is a cell of K', not in L'. This family of homeomorphisms defines a subdivision $(K'', s_{\{h'\}})$ of K'. Continuing in this way we get a sequence of subdivisions, where $(K^{(n)}, s_{\{h^{(n-1)}\}})$ is a subdivision of $K^{(n-1)}$.

Let $\psi_n: (K^{(n)}, L^{(n)}) \to (S(X), \Sigma(\omega))$ denote the Δ-mapping corresponding to the composition

$$|K^{(n)}|_\Delta \to \cdots \xrightarrow{s\{h'\}} |K'|_\Delta \xrightarrow{s\{h\}} |K|_\Delta \to |S(X)|_\Delta \to X.$$

The composition

$$(|K|_\Delta, |L|_\Delta) \xrightarrow{s^{-1}\{h\}} (|K'|_\Delta, |L'|_\Delta) \xrightarrow{s^{-1}\{h'\}} \cdots \to (|K^n|_\Delta, |L^n|_\Delta) \xrightarrow{\psi_n} (|S(X)|_\Delta, |\Sigma(\omega)|_\Delta)$$

is homotopic to the inclusion mapping.

Moreover, it is easily seen that if $\mathscr{U}$ is a covering of X by open sets, then we may choose n so large that for any $u \in K^{(n)}$, the image of $\psi_n(u)$ lies in a member of $\mathscr{U}$.

Let C be a compact subset of X. Let $\mathscr{A}$ be the set of all pairs $(\mathscr{U}, \mathscr{V})$ where $U = \{U_1, \ldots, U_n\}$ and $\mathscr{V} = \{V_1, \ldots, V_n\}$ are collections of subsets of X, each indexed by the same finite set, each U_i is open, there is a projection $f_i: U_i \to \mathbb{R}$ for ω which is a trivial fibration, each V_i is closed, $V_i \subset U_i$, and $C \subset \operatorname{int} V_1 \cup \cdots \cup \operatorname{int} V_n$. If $(\mathscr{U}, \mathscr{V})$ and $(\mathscr{U}', \mathscr{V}')$ are two members of $\mathscr{A}$, we say the second is *subordinate* to the first if for each $U'_i \in \mathscr{U}'$, if U'_i meets some $V_j \in \mathscr{V}$ then U'_i lies in the corresponding $U_j \in \mathscr{U}$.

The hypotheses that X is normal and ω is a foliation imply that $\mathscr{A}$ is non-empty, and that for any member of $\mathscr{A}$, there is another member which is subordinate to it.

Let $d = \dim K$, and let C denote the image of the canonical mapping $|K|_\Delta \to X$. Let $(\mathscr{U}, \mathscr{V})$, $(\mathscr{U}', \mathscr{V}'), \ldots, (\mathscr{U}^{(d)}, \mathscr{V}^{(d)})$ denote a sequence of members of $\mathscr{A}$ each of which is subordinate to the preceding one. By the argument we gave above, we can choose l so large that for each cell u of $K^{(l)}$, image $(\psi_l(u): \Delta^n \to X)$ lies in a member of $\mathscr{V}^{(d)}$.

To complete the proof, we will construct for each cell u of $K^{(l)}$ a mapping $\psi'(u): \Delta^n \to X$ such that $\psi'(u) = \psi_l(u)$ if $u \in L^{(l)}$, such that $\psi': K^{(l)} \to S(X)$ is a Δ-mapping whose image lies in $\Sigma(\omega)$, such that the mappings

$$\psi', \psi_l: |K^{(l)}|_\Delta \to X$$

are homotopic rel. $|L^{(l)}|_\Delta$. By the construction in §7 for lifting homotopies this yields a homotopy between the two mappings of pairs

$$\tilde{\psi}', \tilde{\psi}_l: (|K^{(l)}|_\Delta, |L^{(l)}|_\Delta) \to (|S(X)|_\Delta, |\Sigma(\omega)|_\Delta).$$

The existence of this homotopy implies Theorem 9.1.

In order to construct ψ' and the homotopy, we choose for each n-cell u of $K^{(l)}$ a member V_u of $\mathscr{V}^{(d-n)}$ which contains image $\psi_l(u)$. We let U_u denote the corresponding member of $\mathscr{U}^{(d-n)}$.

If u is a 0-cell, we take $\psi'(u) = \psi_l(u)$ and connect $\psi'(u)$ and $\psi_l(u)$ by the constant homotopy. We suppose inductively that we have chosen $\psi'(u)$ and the homotopy, for any $(n-1)$-cell u, and that the images of $\psi'(u)$ and the homotopy lies in U_u. We show under this assumption that we may extend the construction to n-cells u with the same properties for the extend the construction. Let u be such an n-cell and let v be any boundary $(n-1)$-cell. Since $\psi_l(v) \subset V_u \cap U_v$ and $(\mathscr{U}^{d-n+1}, \mathscr{V}^{d-n+1})$ is subordinate to $(\mathscr{U}^{d-n}, \mathscr{V}^{d-n})$, $U_v \subset U_u$. It follows that the constructions we have already given yield a homotopy $\psi(u)_t \mid \partial \Delta^n$, $0 \leqslant t \leqslant 1$ and this homotopy stays in U_u. Furthermore by Lemma 6.1, $\psi'(u) \mid \partial \Delta^n$ is quasi-linear, so by the fact that there is a local projection $f: U_u \to \mathbb{R}$ for ω which is a trivial fibration, it follows that we can extend the homotopy of all to Δ^n with $\psi'(u)$ quasi-linear. Q.E.D.

§10. $S(\omega)$

Let X be a topological space and ω a Γ-structure on X. We let $S(\omega)\subset S(X)\times \times S(\mathbb{R})=S(X\times\mathbb{R})$ be the *CSS*-set whose set of n-cells consists of all pairs (u, v), where $u:\Delta^n\to X$ is in $\Sigma(\omega)$, $v:\Delta^n\to\mathbb{R}$ defines $u^*\omega$ in a neighbourhood of the 1-skeleton of Δ^n, and v is weakly order preserving on the vertices of Δ^n.

PROPOSITION 10.1. *If X is normal and ω is a foliation, then the projection $|S(\omega)|\to|S(X)|$ is a homotopy equivalence.*

Proof. Since $|S(X)\times S(\mathbb{R})|=|S(X)|\times|S(\mathbb{R})|$ and $|S(\mathbb{R})|$ is contractible, an equivalent assertion is that the inclusion mapping $|S(\omega)|\to|S(X)\times S(\mathbb{R})|$ is a homotopy equivalence. This, in turn, is equivalent to the assertion that the inclusion mapping $|S(\omega)|_\Delta\to|S(X)\times S(\mathbb{R})|_\Delta$ is a homotopy equivalence. To show this, it is enough to show that if K is a finite Δ-subset of $S(X)\times S(\mathbb{R})$ and $L=S(\omega)\cap K$, then there is a homotopy

$$h_t:(|K|_\Delta, |L|_\Delta)\to(|S(X)\times S(\mathbb{R})|_\Delta, |S(\omega)|_\Delta)$$

such that h_0 is the inclusion mapping and $h_1(|K|_\Delta)\subset|S(\omega)|_\Delta$.

Let $\pi_1:S(X)\times S(\mathbb{R})\to S(X)$ and $\pi_2:S(X)\times S(\mathbb{R})\to S(\mathbb{R})$ denote the projections on the first and second factors, respectively. The following lemma will be helpful.

LEMMA 10.2. *Suppose $f_i:(K, L)\to(S(X)\times S(\mathbb{R}), S(\omega))$, $i=0, 1$ are Δ-mappings, $f_0|L=f_1|L$, and $\pi_1 f_0=\pi_1 f_1$. Then there is a homotopy*

$$h_t:(|K|_\Delta, |L|_\Delta)\to(|S(X)\times S(\mathbb{R})|_\Delta, |S(\omega)|_\Delta)$$

such that $h_i=\bar{f}_i$, $i=0, 1$.

Proof. Let $r:S(\mathbb{R})\to\mathbb{R}$ be the natural mapping. Let h_t^0 be a homotopy rel. $|L|_\Delta$ connecting the mappings

$$r\pi_2\bar{f}_i:|K|_\Delta\to\mathbb{R}.$$

Let $H_2:K*I\to S(\mathbb{R})$ be the corresponding Δ-mapping. Let $H_1:K*I\to S(X)$ correspond to the constant homotopy $|K|_\Delta\to X$ which is $\pi_1\bar{f}_0$ at each stage. It is easily seen that the composition

$$|K|_\Delta\times I\to|K*I|_\Delta\xrightarrow{(H_1, H_2)}|S(X)\times S(\mathbb{R})|_\Delta$$

provides the required homotopy.

Now we return to the construction of the homotopy (1).

By the proof of Proposition 9.1., we may assume without loss of generality that for each n-cell u of K, there is an open set U_u in X and a locally trivial fibration f_u of U_u over an open interval which defines ω in U_u. Furthermore, we may assume π, $u(\Delta^n) \subset U_u$ and if u is a face of v, then $U_u \subset U_v$. For, in general, there is a relative homotopy h_t (as in (1)) such that h_0 is the inclusion, and h_1 is a Δ-mapping from a subdivision of K having the properties listed above.

Let $U = K - L$. We construct a Δ-mapping $h_1 : K'_U \to S(\omega)$ as follows. If $u \in L$, we let $h_1(u) = u$. Otherwise, we assume that u is an n-cell of U and that h_1 has been constructed on all the cells of K'_U which lie in ∂u. Furthermore if v is a face of u, and v' is a cell of K'_U which lies in v, we assume $h_1(v')(\Delta^n) \subset U_v \subset U_u$. Then $h_1(|\partial u|) \subset U_u$. We construct $\pi_1 h_1$ on u so that $f_u \pi_1 h_1(b) > f_u \pi_1 h_1(x)$ for any $x \in |\partial u|$ and the restriction of $f_u \pi_1 h_1$ to the line segment xb is quasi-linerr. Then it is easy to construct $\pi_2 h_1$ on the cells of K'_U in u such that the image of h_1 lies in $S(\omega)$. The construction of §7 and Lemma 10.2 show that $|h_1|_\Delta$ and the inclusion mapping are homotopic rel. L. Q.E.D.

§11. A Mapping $|AS(\omega)|_\Delta \to BBG$

Let X be a topological space and ω a Γ-structure on X. Let $S(\omega)$ by the Δ-set defined in §10, and let A be the functor from Δ-sets to CSS^2-sets defined in §4. In this section, we will define a mapping

$$R : |AS(\omega)|_\Delta \to BBG .$$

This mapping has the following property. Let Ω be the Γ-structure on BBG constructed in §6. Let π denote the composition of the natural mappings:

$$|AS(\omega)|_\Delta \to |S(\omega)|_\Delta \to X .$$

Then

$$\pi^* \omega = R^* \Omega . \tag{1}$$

The rest of this section is devoted to the construction of R.

Given a subset S of $\mathbf{m}$, we let $l(S)$ denote the path in Δ^m obtained by joining successive points of S by edges in Δ^m.

Let π_i $(i = 1, 2)$ denote the projections of $X \times \mathbb{R}$ on its first and second factors, respectively.

If $u \in S(\omega)_m$ and S and S' are two subsets of $\mathbf{m}$ with the same least and same greatest elements, then we define $H_{u,S,S'}$ as follows. Let g be a function $\Delta^m \to \mathbb{R}$ which defines $(\pi_1 u)^* \omega$. We have two mappings

$$l(S) \underset{\pi_2 u}{\overset{g}{\rightrightarrows}} \mathbb{R}$$

and from the fact that g and $\pi_2 u$ define the same Γ-structure in a neighborhood of the 1-skeleton, it follows that there is a unique Γ-homeomorphism F_S of a neighborhood U of $g(l(S))$ in $\mathbb{R}$ onto a neighborhood V of $\pi_2 u(l(S))$ in $\mathbb{R}$ such that the following diagram commutes

$$\begin{tikzcd} & U \arrow[d, "F_S"] \\ l(S) \arrow[ur, "g"] \arrow[r, "\pi_2 u"'] & V \end{tikzcd}$$

We may define a Γ-homeomorphism $F_{S'}$ in the same way, replacing S by S'. Note that $g(l(S))=g(l(S'))$ and $\pi_2 u(l(S))=\pi_2 u(l(S'))$, since S and S' have the same end-points. Hence we may assume that $F_{S'}$ has the same range and domain as F_S. Now we define

$$\begin{aligned} H_{u,S,S'}(t) &= F_{S'}F_S^{-1}(t) \quad \text{if} \quad t\in\pi_2 u(l(S)) \\ &= t \qquad\qquad\qquad \text{otherwise}. \end{aligned}$$

Note that the result is independent of g and is an element of G. Furthermore, $H_{u,S,S'}$ is the identity in a neighborhood of $\pi_2 u(S\cap S')$ and outside of $\pi_2 u(l(S))$. Clearly if S, S', and S'' are subsets of $\mathbf{m}$ with the same greatest and least points, then

$$H_{u,S,S''} = H_{u,S',S''}H_{u,S,S'}.$$

Consider $(u, \varphi, S)\in AS(\omega)_{p,q}$. Then $u\in S(\omega)_m$ for some m, and $S=(S_0, \ldots, S_q)$. Let $H_i=H_{u,S(i-1),S(i)}$ for $i=1, \ldots, q$. Let $a_i=\pi_2 u\varphi(i)$ for $i=1, \ldots, p$. Then H_i is the identity in a neighborhood of $\{a_0, \ldots, a_p\}$ and outside $[a_0, a_p]$. Hence

$$H_i = \prod_1^p h_{ij}$$

where $\operatorname{supp} h_{ij}\in(a_{j-1}, a_j)$. Hence $(a_0, \ldots, a_p, c_1, \ldots, c_p)$ is a (p, q)-cell of NNG, where $c_j=(h_{1j}, \ldots, h_{qj})$.

In other words, to each (p, q)-cell (u, φ, S) in $AS(\omega)$ we have associated a (p, q)-cell, which we will denote $r(u, \varphi, S)$, in BBG. It is easily verified that

$$r: AS(\omega)\to NNG$$

is a bisimplicial mapping.

Now we show that there exists a continuous mapping

$$R: |AS(\omega)|_\Delta \to BBG$$

such that $R(|u, \varphi, S|_\Delta)\subset r(u, \varphi, S)$ and $R^*\Omega=\pi^*\omega$.

For each (p, q) simplex σ of $AS(\omega)$, we will construct a mapping $R_\sigma : \Delta^p \times \Delta^q \to \to \Delta^p \times \Delta^q$. This will be done so as to respect identifications, i.e., so that there is a mapping $R: |AS(\omega)|_\Delta \to BBG$ such that the diagram below commutes, for any simplex σ of $AS(\omega)$:

$$\begin{tikzcd} \sigma \times \Delta^p \times \Delta^q \arrow[r, "R_\sigma"] \arrow[d] & r(\sigma) \times \Delta^p \times \Delta^q \arrow[d] \\ |AS(\omega)|_\Delta \arrow[r, "R"] & BBG \end{tikzcd}$$

Here σ is a (p, q)-simplex, and the vertical arrows are the canonical mappings.

The compatibility conditoin is equivalent to the following condition.

a) If $\alpha: \mathbf{p}' \times \mathbf{q} \to \mathbf{p} \times \mathbf{q}$ is a morphism in the category Δ^2 then the following diagram commutes:

$$\begin{tikzcd} \Delta^{p'} \times \Delta^{q'} \arrow[r, "R_{\alpha^*\sigma}"] \arrow[d, "\alpha"] & \Delta^{p'} \times \Delta^{q'} \arrow[d, "\alpha"] \\ \Delta^p \times \Delta^q \arrow[r, "R_\alpha"] & \Delta^p \times \Delta^q \end{tikzcd}$$

The condition that $R^*\Omega = \pi^*\omega$ is equivalent to:

b) Let π_σ denote the composition

$$\Delta^p \times \Delta^q = \sigma \times (\Delta^p \times \Delta^q) \to |AS(\omega)|_\Delta \to X \times \mathbb{R}.$$

Let $\gamma_{r(\sigma)}: \Delta^p \times \Delta^q \to \mathbb{R}$ be as in the definition of Ω in §6. Then $(\pi_1 \pi_\sigma)^*\omega$ is defined by $\gamma_{r(\sigma)} R_\sigma$.

In addition, we will construct the R_σ so that the following two conditions are satisfied.

c) The following diagram commutes:

$$\begin{tikzcd} \Delta^p \times \Delta^q \arrow[rr, "R_\sigma"] \arrow[dr] & & \Delta^p \times \Delta^q \arrow[dl] \\ & \Delta^q & \end{tikzcd},$$

where the slanted arrows are the projection on the second factor.

d) If σ is a $(1, 0)$-simplex (u, φ, S), where $S = (S_0)$, then the following diagram commutes:

$$\begin{tikzcd} \Delta^1 \arrow[r, "R_\sigma"] \arrow[d] & \Delta^1 \arrow[d] \\ {[a_0, a_1]} \arrow[r, "H_{u, T, S(0)}"] & {[a_0, a_1]} \end{tikzcd},$$

where each vertical arrow is the unique affine mapping which takes the ordered pair $(0, 1)$ to (a_0, a_1), $T = (\varphi(0), \varphi(1))$, and $a_0 = \pi_2 u\varphi(0)$, $a_1 = \pi_2 u\varphi(1)$.

We construct R_σ satisfying these conditions by induction. For a (0, 0)-simplex R_σ is the unique mapping $\Delta^0 \to \Delta^0$. To construct R_0 for a (p, q)-simplex σ, we assume inductively that R_τ has been constructed for all (p', q') simplices τ for which $p' \leqslant p$, $q' \leqslant q$, and $p' < p$ or $q' < q$, and that these R_τ satisfy the conditions (a)–(d), whenever these make sense.

If σ is a (1, 0) simplex, and $\pi_2 u\varphi(0) \neq \pi_2 u\varphi(1)$, then R_σ is uniquely determined by (d). Then (a) and (b) follow from (d), and (c) is vacuous in this case.

If σ is a (1, 0)-simplex and $\pi_2 u\varphi(0) = \pi_2 u\varphi(1)$, then we may choose R_σ to be any orientation preserving homeomorphism. Then (a)–(d) are easily verified.

Now let σ be a (1, 1)-simplex, (u, φ, S), and suppose

$$\pi_2 u(a_0) < \cdots < \pi_2 u(a_k) \tag{1}$$

where $a_0 < \cdots < a_k$ are the integers in the interval $[\varphi(0), \varphi(1)]$. Then there is a unique homeomorphism R_σ such that (a), (b), and (c) are satisfied. For, there is a unique homeomorphism

$$R_\sigma : \partial(\Delta^1 \times \Delta^1) \to \partial(\Delta^1 \times \Delta^1)$$

such that (a) is satisfied. Moreover $(\pi_1 \pi_\sigma)^* \omega$ is the horizontal foliation defined by $\pi_2 \pi_\sigma$. (Here we think of the first factor as "vertical", and the second factor as "horizontal", reversing the usual way of representing the two factors.) The foliation defined by $\gamma_{r(\sigma)}$ is transverse to the vertical lines. Hence to extend R_σ to the interior of $\Delta^1 \times \Delta^1$ satisfying (b) and (c), it is enough to verify that the following diagram commutes.

$$\begin{tikzcd} \Delta^1 \arrow[r, "R\partial'_1\sigma"] \arrow[d, "R\partial'_0\sigma"] & \Delta^1 \arrow[d, "\gamma_{r(\sigma)}"] \\ 0 \times \Delta^1 \arrow[r, "\gamma_{r(\sigma)}"] & \mathbb{R} \end{tikzcd}$$

But this is an easy consequence of the definitions.

In the case when σ is a (1, 1)-simplex, but (1) does not hold, the proof is similar, except that R_σ is no longer unique.

In the case when σ is a (2, 0)-simplex, the proof is similar to that which was just given.

In the case when σ is a $(p, 0)$ simplex with $p \geqslant 3$, we must show that if ω_1 and ω_2 are two quasi-linear linear foliations on Δ^p and $R_\sigma : \partial\Delta^p \to \partial\Delta^p$ is a face preserving mapping such that $R_\sigma^* \omega_2 = \omega_1$, then R_σ extends to a mapping $\Delta^p \to \Delta^p$ such that $R_\sigma^* \omega_2 = \omega_1$.

In the remaining cases, the assertion reduces to verifying a similar statement, which is again easily shown.

§ 12. End of the Proof

In this section, we finish the proof of the Main Theorem. We have shown (§5) that there is a mapping $BG \to \Omega(BBG)$ which induces isomorphism in integer homology, In §6, we have constructed a Γ-structure Ω on BBG. This gives rise to a mapping

$$U: BBG \to B\Gamma.$$

It is easily seen that the composition

$$BG \to \Omega(BBG) \xrightarrow{\Omega U} B\Gamma$$

induces the mapping in integer homology which appears in the statement of the Main Theorem. Hence it is enough to prove the following result.

THEOREM 12.1. *U is a homotopy equivalence.*

Proof. Let Ω' be the universal Γ-structure on $B\Gamma$. By replacing $B\Gamma$ with the graph of Ω', we may suppose Ω' is a foliation. Moreover $B\Gamma$ is a CW-complex and hence normal. Hence Proposition 10.1 implies

$$|S(\Omega')|_\Delta \to B\Gamma$$

is a homotopy equivalence. By Proposition 4.3,

$$|AS(\Omega')|_\Delta \to |S(\Omega')|_\Delta$$

is a homotopy equivalence. Hence the composition

$$\pi: |AS(\Omega')|_\Delta \to B\Gamma$$

of these two mappings is a homotopy equivalence. In §11, we have constructed a mapping

$$R: |AS(\Omega')|_\Delta \to BBG$$

such that $R^*\Omega = \pi^*\Omega'$, where Ω is the Γ-structure on BBG constructed in §6. Since $U^*\Omega' \sim \Omega$, it follows that $UR \sim \pi$ and U has a right inverse in the homotopy category.

The rest of this section is a proof that U has a left inverse in the homotopy category.

The space BBG is the realization of a CSS^2-set NNG (§5). We let δ denote the diagonal functor (§4). Since $|NNG| = BBG$, we may regard δNNG as a subset of the singular complex $S(BBG)$. In fact, $\delta NNG \subset \Sigma(\Omega)$.

For, let σ be a non-degenerate p-cell in δNNG. There is a unique non-degenerate cell τ in NNG such that σ, considered as a (p, p)-cell of NNG, is a degeneracy of τ. If τ is a (k, l) cell, there are uniquely determined weakly order preserving surjective mappings $f:\mathbf{p}\to\mathbf{k}$ and $g:\mathbf{p}\to\boldsymbol{l}$ such that $\sigma=(f, g)^*\tau$. We extend to $f:\Delta^p\to\Delta^k$ and $g:\Delta^p\to\Delta^l$. The Γ-structure Ω pulls back to a Γ-structure Ω_τ on $\Delta^k\times\Delta^l$ *via* the mapping of $\Delta^k\times\Delta^l$ into BBG which corresponds to τ. This pulls back *via* (f, g) to a Γ-structure Ω_σ on Δ^p, which we must show is defined by a quasi-linear function on Δ^p. By definition (§6), Ω_τ is defined by the function γ_τ. Hence Ω_σ is defined by $\gamma_\tau\circ(f, g)$. Now let $\Delta^{l-1}\subset\Delta^l$ be the simplex spanned by $\mathbf{l-1}$. Then $\Delta^k\times\Delta^l$ is the join of $\Delta^k\times\Delta^{l-1}$ and $\Delta^k\times l$. If $x\in\Delta^k\times\Delta^{l-1}$ and $y\in\Delta^k\times l$, then it follows easily from the definition of γ_τ that the restriction of γ_τ to the line segment joining x and y is either constant, or is a homeomorphism onto a closed interval in $\mathbb{R}$. Let $\mathbf{q}=g^{-1}(\mathbf{l-1})$ and $r=p-q-1$. Let $\Delta^q\subset\Delta^p$ be the simplex spanned by $\mathbf{q}$ and let $\Delta^r\subset\Delta^p$ be the simplex spanned by the remaining vertices. Again Δ^p is the join of Δ^q and Δ^r. It is easily seen that if $x\in\Delta^q$ and $y\in\Delta^r$ then $\gamma_\tau(x)\leqslant\gamma_\tau(y)$, and there is $\epsilon>0$ such that if $\gamma_\tau(y)<\gamma_\tau(x)+\epsilon$ then the restriction of γ_τ to the line segment joining x and y is linear. By induction we may assume $\gamma_\tau\mid\Delta^q$ is quasi-linear. It follows from construction that $\gamma_\tau\mid\Delta^r$ is linear. From these facts it is easy to see that $\gamma_\tau\mid\Delta^p$ is quasi-linear.

This completes the proof that $\delta NNG\subset\Sigma(\Omega)$.

Now we construct a real valued function v defined on a neighborhood of the 1-skeleton of $|\delta NNG|$ and defining Ω there. If σ is a non-degenerate 1-cell of δNNG, we let τ be the unique non-degenerate cell in NNG of which σ is a degeneracy (in NNG) Then we let $v=\gamma_\tau$ on $|\sigma|$. This defines v on the 1-skeleton, and there is a unique extension of v to a neighborhood of the 1-skeleton which defines Ω.

Finally, we extend v arbitrarily to all of BBG. Then v induces a mapping of CSS-sets $\delta NNG\to S(\mathbb{R})$, where the latter is the singular complex of $\mathbb{R}$. Since $\delta NNG\subset$ $\subset S(BBG)$, the graph of this mapping is a sub-CSS-set of $S(BBG)\times S(\mathbb{R})$. We will identify δNNG with the graph of this mapping. Then, by our construction, $\delta NNG\subset$ $\subset S(\Omega)$.

Now we apply the forgetful functor and regard δNNG and $S(\Omega)$ as Δ-sets. Then we may apply A tc each of these, so the above inclusion gives rise to an inclusion $A\delta NNG\subset AS(\Omega)$.

By replacing $B\Gamma$ with a homotopy equivalent space if necessary, we may suppose that $U^*\Omega'=\Omega$ and that Ω' is a foliation of $B\Gamma$. Then we have a commutative diagram.

$$\begin{array}{ccccc}
|A\delta NNG|_\Delta & \xrightarrow{\iota} & |AS(\Omega)|_\Delta & \xrightarrow{U_*} & |AS(\Omega')|_\Delta \\
 & & \downarrow{\scriptstyle\pi(\Omega)} & & \downarrow{\scriptstyle\pi(\Omega)'} \\
 & & BBG & \xrightarrow{U} & B\Gamma
\end{array}$$

Here ι is the geometric realization of the inclusion defined above. The functors

$\omega \to S(\omega)$ and $X \to AX$ and $|?|_\Delta$ compose to give a functor; U_* comes from U by this functor. $\pi(\Omega)$ and $\pi(\Omega)'$ are the mappings coming from the natural transformation.

Now $\pi(\Omega')$ and $\pi(\Omega) \circ \iota$ are homotopy equivalences. Consider $R: |AS(\Omega')|_\Delta \to \to BBG$. We will show $RU_*\iota$ induces an isomorphism in integer homology. Since BBG (and hence, also $|A\delta NNG|_\Delta$) is simply connected, this implies $RU_*\iota$ is a homotopy equivalence. Thus $U_*\iota$ has a left inverse. Since $\pi(\Omega')$ and $\pi(\Omega)\iota$ are homotopy equivalences, it then follows that U has a left inverse.

It remains only to show that $RU_*\iota$ induces an isomorphism in integer homology.

The proof that $RU_*\iota$ induces isomorphism in integer homology is based on an explicit calculation as follows.

Let C_0 denote the chain complex of the Δ^2-set $A\delta NNG$ (so we do not divide by degeneracies). Let C_2 denote the chain complex of the Δ^2-set NNG (so again we do not divide by degeneracies), and let C_2^N denote the normalized chain complex of the CSS^2-set NNG (so here we divide by degeneracies). From the definition of R, it follows that $RU_*\iota$ induces a chain mapping $\varrho: C_0 \to C_2^N$. We must show ϱ induces isomorphism in homology.

Let C_1 denote the chain complex of the Δ-set δNNG. The shuffle homomorphism $\theta: C_2 \to C_1$ induces an isomorphism in homology inverse to that given by the canonical mapping $|\delta NNG|_\Delta \to |NNG|_\Delta$. Moreover, if we consider the decomposition of $|A\delta NNG|$ into the sets E^n given in the proof of Proposition 4.3, this gives a "cell" structure on $|A\delta NNG|$ isomorphic to the standard cell structure on $|\delta NNG|_\Delta$. The cell structure on $|A\delta NNG|$, coming from its structure as a CSS^2-set, is a subdivision of this cell structure. Hence we get a subdivision mapping $\Psi: C_1 \to C_0^N$, where C_0^N denotes the normalized chain complex of the CSS^2-set $A\delta NNG$.

It is easy to verify that ϱ factors

$$C_0 \xrightarrow{proj.} C_0^N \xrightarrow{\bar{\varrho}} C_2^N .$$

Thus it suffices to show that $\bar{\varrho}$ induces isomorphisms in integer homology, or, equivalently that

$$\bar{\varrho}\Psi\theta: C_2 \to C_2^N$$

induces isomorphisms in integer homology.

Now let $\beta\beta G$ denote the complex defined in [4]. At the end of §5, we showed $\beta\beta G$ is the chains on $B'BG$ obtained by giving BG^k the product structure as a CW-complex instead of the product structures as a CSS-set. Thus C_2^N and $\beta\beta G$ have the same homology and there is an Alexander homomorphism

$$C_2^N \to \beta\beta G$$

which induces the canonical isomorphism in homology.

We filter $\beta\beta G$ as in [4], §2. We filter C_2 by letting $F_p C_2$ be generated by all (p', q) cells with $p' \leqslant p$. Then we have two chain homomorphisms of filtered complexes, arising from the compositions

$$C_2 \xrightarrow{\text{proj.}} C_2^N \xrightarrow{A} \beta\beta G$$

$$C_2 \xrightarrow{\bar{\varrho}\Psi\theta} C_2^N \xrightarrow{A} \beta\beta G.$$

A direct, albeit lengthy, calculation shows that the induced mappings on the associated graded objects are the same. Hence the two induced mappings on spectral sequences are the same on the E^1 terms and therefore likewise on the E^r $r \geqslant 1$. However the induced mapping on E^2 arising from the first composite above is an isomorphism by general nonsense. Hence, so is the induced mapping on E^2 arising from the second composite. Hence

$$A\bar{\varrho}\Psi\theta : C_2 \to \beta\beta G$$

induces isomorphisms in homology. Since A induces isomorphisms in homology, it follows that $\bar{\varrho}\Psi\theta$ does also. Q.E.D.

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