Quadratic Spaces with Few Isometries (Quadratic Forms and Linear Topologies VI)

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Introduction

What sort of metric automorphisms do always exist on infinite dimensional quadratic spaces? Clearly, we always have the symmetries about (nondegenerate) hyperplanes, the identity **1** of the space, **−1**, and of course finite products of these isometries; they form an invariant subgroup $\mathfrak{J}$ in the full orthogonal group of the space. In the finite dimensional case $\mathfrak{J}$ is already the full orthogonal group. In the infinite case however, $\mathfrak{J}$ usually represents only a negligible part of the orthogonal group associated with the space. In this note we shall show that there are quadratic spaces of arbitrarily large dimension whose full orthogonal groups equal $\mathfrak{J}$. In § 1 we shall describe how to define such spaces over prescribed (non denumerable) base fields.

The spaces E which we shall investigate below share the following property on subspaces F,

$$F \subset E \ \& \dim F \geqslant \aleph_0 \rightarrow \dim F^{\perp} < \dim E. \tag{*}$$

In particular, if such a space E is decomposed orthogonally, $E = E_1 \oplus E_2$, then one of the summands E_i necessarily is of finite dimension. Spaces with such few orthogonal splittings are an extreme counterpart to quadratic spaces admitting orthogonal bases. For subspaces F of spaces wich admit orthogonal bases we invariably have $\dim E/F^{\perp} = \dim F$ which sharply contrasts (*). We see in particular that $\dim E \neq \aleph_0$ for all E satisfying (*). The construction given in § 1 yields spaces which actually satisfy the stronger property on subspaces F,

$$F \subset E \ \& \dim F \geqslant \aleph_0 \rightarrow \dim F^{\perp} \leqslant \aleph_0. \tag{**}$$

The notion which stands in the center of our discussion of spaces with small orthogonal group $\mathfrak{O}$ in the sense indicated above ($\mathfrak{O} = \mathfrak{J}$) is that of a locally algebraic isometry (§ 2). An isometry T on E is called locally algebraic if T admits for every $x \in E$ a polynomial $f_x(T)$ (with coefficients in the base field of E) that annihilates x, $f_x(T)x = 0$. If f_x does not depend on x we call T algebraic. Theorem 3 of § 2 says that the spaces constructed in § 1 admit locally algebraic isometries only; in other words, there are infinite dimensional (**)-spaces E with property (λ): 'Every isometry on E is locally algebraic'. By means of somewhat complicated examples one can however show that (**) does not, in general, imply (λ) (the converse implication is seen not to be true either by Theorem 3 of §2). Spaces with property (λ) and which, in addition, satisfy (*) absolutely (i.e. which preserve (*) under extensions of the base field) are seen to have trivial quotient $\mathfrak{O}/\mathfrak{J}$ (Corollary 1 of Theorem 3 in §2).

In [3] it is shown that certain spaces constructed in §1 satisfy Witt's cancellation theorem: If $E=E_1\oplus E_2=F_1\oplus F_2$ are orthogonal decompositions of E with E_1 and F_1 isometric, then E_2 and F_2 must be isometric; a rare thing indeed to happen in the infinite dimensional case.

Notations.

Generally speaking, forms $\Phi: E\times E\to k$ are additive in each argument and satisfy $\Phi(\lambda x, y)=\lambda\Phi(x, y)$, $\Phi(x, \lambda y)=\Phi(x, y)\lambda^{\alpha}$ with respect to some fixed involution α (=antiautomorphism of period 2) of the division ring k. We shall however always assume below that k is *commutative*. We shall furthermore assume Φ to be *ε-hermitean*, i.e. $\Phi(y, x)=\varepsilon\Phi(x, y)^{\alpha}$ with $\varepsilon=+1$ (hermitean) or $\varepsilon=-1$ (antihermitean). If α is the identity, then k is necessarily commutative and we speak of symmetric and antisymmetric forms respectively. In any case, '$x\perp y$', defined as usual to be '$\Phi(x, y)=0$', is a symmetric relation. $E^{\perp}$ is called the *radical* of E (rad E). If rad $E=(0)$ we call Φ nondegenerate and – in analogy with algebras – the space (E, Φ) *semisimple*. Φ is said to be *tracevalued* if for every $x\in E$ there is a $\xi\in k$ such that $\Phi(x, x)=\xi+\varepsilon\xi^{\alpha}$. We shall always assume Φ to be tracevalued, a non trivial requirement only when chark $k=2$ ([1] §4, No. 2). We shall make use of Witt's theorem in §2 below ([1] §4, No. 3): Let E be a space with a non degenerate form Φ which is hermitean or anti-hermitean, and tracevalued if it is hermitean. Then any isometry (=vectorspace isomorphism that preserves Φ) between finite-dimensional subspaces can be extended to a isometric automorphism of E.

Let (E, Φ) be an ε-hermitean k-vectorspace with respect to the involution α. Assume that the division ring k' contains k and admits an extension (involution) of α to k'. We know that the abelian group $E'=k'\otimes_k E$ may be regarded as a vectorspace over k and as a vectorspace over k'. The form $\Phi': E'\times E'\to k'$, defined by $\Phi'(\sum\lambda_i\otimes x_i, \sum\mu_j\otimes y_i)=\sum\lambda_i\Phi(x_i, y_j)\mu_j^{\alpha}$ for $\lambda_i, \mu_j\in k'$ is ε-hermitean. We say that Φ satisfies (*), or (**), *absolutely*, if the form Φ' possesses these properties for all extensions k' of k. (E', Φ') is called the *k'-ification* of (E, Φ) or the space obtained from (E, Φ) by extending the ring of scalars.

A space (E, Φ) is called anisotropic if it contains no isotropic elements, i. e. no vectors $x\neq 0$ with $\Phi(x, x)=0$.

Unless stated otherwise, (E, Φ) will be assumed to be of infinite dimension.

§1. The Existence of Spaces with Property **

In this short section we shall describe the construction of infinite dimensional spaces (E, Φ) where Φ is an ε-hermitean form satisfying (**) absolutely.

Let α be an involution of the commutative field k, $\operatorname{card} k > \aleph_0$. Let $X \subset k$ be a maximal subset of algebraically independent elements over the prime field k_0 so that k is an algebraic extension of $k_0(X)$. Let $\varepsilon = +1$ or -1. Since α is of period 2, there is a subset $Y \subset X$ with $\operatorname{card} Y = \operatorname{card} X$ ($=\operatorname{card} k$) and for every $\eta \in Y$ either $\varepsilon\eta^\alpha = \eta$ or $\varepsilon\eta^\alpha \notin Y$. Let then $(e_\iota)_{\iota \in I}$ be a basis of a k-vectorspace with $\operatorname{card} k \geqslant \operatorname{card} I > \aleph_0$. We define an ε-hermitean form Φ on $E \times E$ as follows: Pick an ordering on I. For all $\iota < \kappa$ in I set $\Phi(e_\iota, e_\kappa) = \varepsilon\Phi(e_\kappa, e_\iota)^\alpha = \eta_{\iota\kappa} \in Y$ such that all elements $\eta_{\iota\kappa}$ $(\iota < \kappa)$ are different. Furthermore $\Phi(e_\iota, e_\iota) = \varepsilon\Phi(e_\iota, e_\iota)^\alpha \in k$ such that no $\Phi(e_\iota, e_\iota)$ equals a $\Phi(e_\iota, e_\kappa)$ with $\iota \neq \kappa$. *We assert that Φ satisfies* (**).

Proof. Let U and V be subspaces of E with $\dim V > \dim U = \aleph_0$, $(u_i)_{i \in \mathbb{N}}$ and $(v_\iota)_{\iota \in J}$ bases of U and V respectively. $u_i = \sum \alpha_{i\kappa} e_\kappa$, $v_\iota = \sum \beta_{\iota\kappa} e_\kappa$ where the first sum extends over the finite set $M_i = \{\kappa \in I \mid \alpha_{i\kappa} \neq 0\}$, the second over the finite set $N_\iota = \{\kappa \in I \mid \beta_{\iota\kappa} \neq 0\}$. Set $M = \bigcup_{\mathbb{N}} M_i$, $N = \bigcup_J N_\iota$. Thus $\operatorname{card} N > \operatorname{card} M = \aleph_0$. Our assertion is proved if we can exhibit a pair $u, v \in U \times V$ with $\Phi(u, v) \neq 0$. Such a pair is found as follows.

(i) X contains a denumerable subset A such that $\{\alpha_{i\kappa} \mid i \in \mathbb{N}, \kappa \in M_i\}$ is contained in the algebraic closure in k of the subfield $k_0(A)$.

(ii) There is a $\varrho_0 \in N \setminus M$ such that

$$A \cap \{\Phi(e_\nu, e_{\varrho_0}), \Phi(e_{\varrho_0}, e_\nu) \mid \nu \in I \setminus \{\varrho_0\}\} = \emptyset .$$

Let $\varrho_0 \in N_{\nu_0}$.

(iii) X contains a finite subset B such that $\{\beta_{\nu_0\mu} \mid \mu \in N_{\nu_0}\}$ is contained in the algebraic closure in k of $k_0(B)$. Since M is infinite, there is a $\kappa_0 \in M$ such that $\Phi(e_{\kappa_0}, e_{\varrho_0})$, $\Phi(e_{\varrho_0}, e_{\kappa_0}) \notin B$. Let $\kappa_0 \in M_{\iota_0}$.

(iv) Notice that $\kappa_0 \neq \varrho_0$. If $\kappa_0 < \varrho_0$ we let

$$C = \{\Phi(e_\kappa, e_\varrho) \mid (\kappa, \varrho) \in M_{\iota_0} \times N_{\nu_0} \setminus \{(\kappa_0, \varrho_0)\}\};$$

if $\varrho_0 < \kappa_0$ we let

$$C = \{\Phi(e_\varrho, e_\kappa) \mid (\varrho, \kappa) \in N_{\nu_0} \times M_{\iota_0} \setminus \{(\varrho_0, \kappa_0)\}\}.$$

Thus, if $\kappa_0 < \varrho_0$ we see by (ii), (iii), (iv) that $\eta_{\kappa_0\varrho_0} = \Phi(e_{\kappa_0}, e_{\varrho_0}) \notin A \cup B \cup C$; similarly, if $\varrho_0 < \kappa_0$ we have $\eta_{\varrho_0\kappa_0} = \Phi(e_{\varrho_0}, e_{\kappa_0}) \notin A \cup B \cup C$. Thus, if k_1 is the algebraic closure in k of $k_0(A \cup B \cup C)$ we see that $\eta_{\kappa_0\varrho_0} \notin k_1$ if $\kappa_0 < \varrho_0$ and $\eta_{\varrho_0\kappa_0} \notin k_1$ when $\varrho_0 < \kappa_0$. In the first case we consider

$$\Phi(u_{\iota_0}, v_{\nu_0}) = \sum_{(\kappa,\rho)\neq(\kappa_0,\rho_0)} \alpha_{\iota_0\,\kappa}\beta_{\nu_0\varrho}\,\phi(e_\kappa, e_\varrho) + \alpha_{\iota_0\kappa_0}\,\beta_{\nu_0\varrho_0}\,\eta_{\kappa_0\varrho_0}\,.$$

If we had $\Phi(u_{\iota_0}, v_{\nu_0})=0$ then we had a nontrivial linear equation for $\eta_{\kappa_0\varrho_0}$ with coefficients in k_1, so $\eta_{\kappa_0\varrho_0}\in k_1$. If $\varrho_0<\kappa_0$ we conclude in the same manner that $\Phi(v_{\nu_0},u_{\iota_0})\neq 0$. Clearly our proof remains valid if we pass to the form Φ' on the k'-ification $E'=k'\otimes_k E$ of E with respect to some overfield k' of k (admitting an extension of α). This proves our assertions. We note our result as

THEOREM 1. *For $\varepsilon=+1$ and for $\varepsilon=-1$ there exist ε-hermitean forms Φ over any commutative field k with given involution and* card $k>\aleph_0$ *which satisfy* (**) *absolutely; we may choose the dimension of Φ to be* card k.

We had card $k\geqslant\dim E$ for the spaces E in the above construction. We do not know if this is necessarily so for spaces with property (**). It is easy to see that (**) does imply $(\text{card}\,k)^{\aleph_0}\geqslant\dim E$. Thus, at least in the special cases where card k is a beth (e.g. when $k=\mathbb{R}$ or $\mathbb{C}$), (**) does imply card $k\geqslant\dim E$.

THEOREM 2. *Let $k=k_0(X)$ be a purely transcendental extension of k_0 and* card $X>\aleph_0$. *If – in the notation of the preceding construction – Φ is chosen symmetric with $\Phi(e_\iota, e_\kappa)=\xi_{\iota\kappa}\in X(\iota, \kappa\in I$ and* card $I>\aleph_0$) *such that $\xi_{\iota\kappa}=\xi_{\nu\mu}$ if and only if $\{\iota, \kappa\}=\{\nu, \mu\}$, then $\Phi(x, x)$ is a square in k only when $x=0$.*

This result is proved in [3]; it guarantees the existence of *anisotropic* forms with property (**) over all fields of a certain type. In the special case where k_0 is assumed orderable the part of theorem 2 ruling out isotropic vectors follows directly from Jacobi's diagonalization formula (for finite spaces). It is clear that after extending the base field Φ may admit isotropic vectors. The fact that k is a purely transcendental extension of some k_0 is not however crucial for the existence of an anisotropic Φ over k satisfying (**). We give an example of such a form over $\mathbb{R}$ by specifying a subspace of an infinite separable Hilbertspace (H, Φ) over the reals: Note that the collection of all sets M of linearly independent vectors $x, y, \cdots$ with $\{\Phi(x, y) \mid x, y\in M\}$ algebraically independent over $\mathbb{Q}$ is inductively ordered by inclusion. Let M_0 be a maximal element by Zorn's lemma. If card $M_0>\aleph_0$, then the restriction of Φ to the span of M_0 satisfies (**) as we have demonstrated above. Assume by way of contradiction that card $M_0\leqslant\aleph_0$. Let $(x_i)_{i\in J}$ be the elements of M_0 in some ordering, and let $A=\{\Phi(x_i, x_j) \mid i, j\in J\}$. Introduce an orthonormal basis $(e_i)_{i\in J}$ in the span X of the x_i $(i\in J)$, $e_i=\sum \alpha_{ij}x_j$ with (α_{ij}) triangular. Then $(\alpha_{ij})^{-1}=(\beta_{ij})$ is triangular and α_{ij}, $\beta_{ij}\in\overline{\mathbb{Q}(A)}$ (real closure). Since card $A\leqslant\aleph_0$ we can pick a family $(t_i)_{i\in J}$, the t_i in $\mathbb{R}$ and algebraically independent over $\overline{\mathbb{Q}(A)}$ with $\sum_J t_i^2=t<\infty$. The closure $\bar{X}$ of X in H (in the normtopology of Φ) contains a vector x with $\Phi(x, e_i)=\lambda_i t_i$ for any choice of λ_i with, say, $0<\lambda_i<1$. We have $\Phi(x, x_i)=\sum \beta_{ij}\lambda_j t_j$. It follows that the set $\{\Phi(x, x_j) \mid j\in J\}$ is algebraically independent over $(\mathbb{Q}A)$ for λ_i rational. If we can arrange for $\Phi(x, x)=\sum (\lambda_i t_i)^2$ to be outside $\overline{\mathbb{Q}(A\cup(t_i)_J)}$ we have the desired contradiction: $M_0\cup\{x\}$ contradicting the maximality of M_0. Now if J should be finite, then $\tilde{X}=X$ and we may, if necessary, pass from x to a vector $x+y$ with $y\in X^\perp$ and $\Phi(y, y)=\alpha-\Phi(x, x)$ and suitably chosen α. If card $J=\aleph_0$, then by varying the *rational* λ_i in the open unit interval we can arrange for $\Phi(x, x)$ to be any real number of the open interval $[0, t]$. Clearly then, there is a choice with $\Phi(x, x)$ outside the denumerable $\overline{\mathbb{Q}(A\cup(t_i)_J)}$. Q.E.D. We can do the same for hermitean forms over a complex Hilbert space. Thus

THEOREM 3. *There exist (infinite) positive definite symmetric (hermitean) forms over* $\mathbb{R}(\mathbb{C})$ *which satisfy* (**) *absolutely.*

Remark. We briefly indicate how to construct spaces which satisfy (**) but not absolutely so. Let k be nondenumerably infinite. Let $(f_\iota)_{\iota\in I}$, $(g_\iota)_{\iota\in I}$ be bases of k-vectorspaces F and G respectively, card$I=$card$k$. Choose subsets X and Y of k with $X\cap Y=\emptyset$ and $X\cup Y$ algebraically independent over the primefield k_0 of k. Define a symmetric bilinear form Φ on $E=F\oplus G$ as follows: $\Phi(f_\iota, f_\kappa)=-\Phi(g_\iota, g_\kappa)=\xi_{\iota\kappa}$. $\Phi(f_\iota, g_\kappa)=\Phi(f_\kappa, g_\iota)=\eta_{\iota\kappa}$ with $\xi_{\iota\kappa}\in X$, $\eta_{\iota\kappa}\in Y$ and $\xi_{\iota\kappa}=\xi_{\nu\mu}$ and $\eta_{\iota\kappa}=\eta_{\nu\mu}$ if and only if $\{\iota, \kappa\}=\{\nu, \mu\}$. If k is assumed orderable, then the reader proves by the method illustrated above that $E=F\oplus G$ satisfies (**). However, over the extension $k(\sqrt{-1})$ E decomposes orthogonally, $E=H\oplus L$ with H spanned by all $f_\iota+\sqrt{-1}\cdot g_\iota (\iota\in I)$ and L spanned by all $f_\iota-\sqrt{-1}\cdot g_\iota (\iota\in I)$.

§2. The Orthogonal Group

In this section we study the orthogonal group $\mathfrak{O}$ associated with certain infinite dimensional spaces (E, Φ) which satisfy (*). Here Φ will always be symmetric or anti-symmetric and tracevalued if it is symmetric.

Consider an isometry T such that there is an orthogonal decomposition $E=E_0\oplus E_1$ with $\dim E_1<\infty$ and $T=\pm\mathbf{1}$ on E_0. Any isometry T with $\mathrm{Ker}(T-\mathbf{1})$ or $\mathrm{Ker}(T+\mathbf{1})$ of finite codimension in E admits such an orthogonal decomposition of E. The set $\mathfrak{J}$ of all such isometries T is an invariant subgroup of the orthogonal group $\mathfrak{O}$ associated with the space E; it contains the subgroup $\mathfrak{J}_0$ of index $\leqslant 2$ of all T which are the identity on almost all of E. For symmetric Φ and char$k \neq 2$ [2] gives a detailed account of $\mathfrak{J}_0$; in that case $\mathfrak{J}_0$ is generated by all symmetries about semisimple hyperplanes. We shall show that for prescribed natural $n>1$ there are infinite spaces (E, Φ) with $\mathfrak{O}/\mathfrak{J}_0$ isomorphic to a product of n copies of $\mathbb{Z}_2$ (characteristic not 2).

It is natural to expect, that spaces with few orthogonal splittings in the sense of (**) admit 'few' isometries. A confirmation of this expection is provided by the first two theorems.

THEOREM 1. *If (E, Φ) satisfies (*), then every isometry on E is determined modulo a factor from $\mathfrak{J}_0$ by its action on any subspace of denumerably infinite dimension.*

THEOREM 2. *If (E, Φ) satisfies (*), and this absolutely so when the base field is not algebraically closed, then every locally algebraic isometry belongs to the group $\mathfrak{J}$ associated with (E, Φ).*

Proof of Theorem 1. Assume first that E is semisimple. For $\lambda \neq 0$ an element of the basefield k let $X(\lambda)$ be the eigenspace $\ker(T-\lambda 1)$ of the isometry T of E. $X(\lambda) \perp X(\mu)$ if $\lambda\mu \neq 1$. Thus we cannot have $\dim X(\lambda) = \dim E$ unless $\lambda^2 = 1$ by (*). $\operatorname{Im}(\lambda T - 1) \subset \subset X(\lambda)^{\perp}$ and $\operatorname{Ker}(\lambda T - 1) = X(\lambda^{-1})$ so

$$\dim E/X(\lambda^{-1}) \leqslant \dim X(\lambda)^{\perp}. \tag{1}$$

Assume that for some subspace U of E we have $T|_U = \mathbf{1}_U$, $\dim U = \aleph_0$. Since T preserves Φ we conclude that $\operatorname{Im}(T-\mathbf{1})$ is contained in $U^{\perp}$ and thus of dimension smaller than $\dim E$. Hence we must have $\dim X(1) = \dim E$ and therefore $\dim X(1)^{\perp} < \infty$ by (*). Hence $\dim E/X(1) < \infty$ by (1) and therefore $\dim X(1)^{\perp} \leqslant \dim E/X(1)$ as E is semisimple. Together with (1) $\dim X(1)^{\perp} = \dim E/X(1) < \infty$. From this we conclude that there exists a subspace $H \subset X(1)$ of finite codimension in E with $E = H \oplus H^{\perp}$. Since T is the identity on H we have $T \in \mathfrak{J}_0$. If E is not semisimple, then $\operatorname{rad} E$ is of finite dimension. Let E_0 be a linear complement of $\operatorname{rad} E$ in E. We can find T_0 in $\mathfrak{J}_0$ such that $T_0 T(E_0)) \subset E_0$. Since radicals are mapped onto themselves under isometries we must have $T_0 T(E_0) = E_0$. By what we have already proved it follows that the restriction of $T_0 T$ to E_0 is determined modulo $\mathfrak{J}_0$ by its action on $U \cap E_0 \cap \operatorname{Ker}(T_0 - 1)$. Hence the same holds for T. Q.E.D.

Proof of Theorem 2. *Case* 1: there is a λ with $\dim X(\lambda) = \dim E$. Hence $\lambda^2 = 1$ and $T \in \mathfrak{J}$ by Theorem 1.

Case 2: $\dim X(\lambda^{-1}) < \dim E$ for all $\lambda \in k \backslash \{0\}$. Thus $\dim X(\lambda)^{\perp} = \dim E$ by (1) and so $\dim X(\lambda) < \infty$ for all $\lambda \in k \backslash \{0\}$ by (*). For every member x of a Basis $\mathscr{B}$ of E we let f_x be the annihilating polynomial. f_x splits into linear factors over the algebraic closure k' of k, $f_x = \prod (Z - \lambda_i)$. Every linear factor provides an eigenvalue $\lambda_i \in k'$ of $T' : E' = k' \otimes E \to E'$. Since E' satisfies (*) by the assumptions of the theorem we see that the number l of different λ_i must be less than $\dim E$. Hence there are only $l < \dim E$ different annihilating polynomials $f_x (x \in \mathscr{B})$. We conclude that there is at least one f_x annihilating a subspace $G \subset E$ of dimension $\dim G = \dim E$. Let $f_x = \prod (Z - \lambda_i)$ be the splitting of this very polynomial. If some of the λ_i equal ± 1 we let G_0 be the image of G under the map $\prod_{\lambda_i = \pm 1} (T - \lambda_i 1)$. We have $\dim G_0 = \dim G$ in the present case. Let g be the product of the remaining linear factors $(Z - \lambda)$. Since $\dim G_0 = \dim E$ and since $g(T)$ annihilates G_0 and hence also $G_0' = k' \otimes G_0$, we conclude that the dimension of $\ker(T - \lambda)$ must equal $\dim E$ for at least one $\lambda \neq \pm 1$. This is a contradiction as G_0' satisfies property (*).

COROLLARY. *If (E, Φ) is as in Theorem 2, then the set of all locally algebraic isometries on E is a group. It coincides with the set of all algebraic isometries on E and it is generated by all $T \in \mathfrak{O}$ with $E/\mathrm{Ker}(T-1)$ or $E/\mathrm{Ker}(T+1)$ finite dimensional; hence it is a normal subgroup of $\mathfrak{O}$.*

LEMMA. *Assume that $E_1, \ldots, E_n$ all satisfy (*) and that $\dim E_i > \dim E_{i+1}$ $(i = 1, \ldots, n-1)$. If T is any endomorphism of the orthogonal sum $E = E_1 \oplus \cdots \oplus E_n$ that preserves orthogonality then the E_i are left almost invariant under T: $\dim(E_i + T(E_i))/E_i$ is finite for all i.*

Proof. Let $F_1 = E_2 \oplus \cdots \oplus E_n$. $\mathrm{Dim}\, E_1 > \dim F_1$ so that there is a subspace V_1 of E_1 with $T(V_1) \subset E_1$ and $\dim V_1 = \dim E_1$. By the assumptions of the lemma $\dim T(V_1) = \dim E_1$. Call K_1 the projection of $T(F_1)$ onto E_1 (for the decomposition $E = E_1 \oplus F_1$). $T(V_1) \perp K_1$ hence K_1 and $(F_1 + T(F_1))/F_1$ are finite dimensional. Setting $F_2 = E_3 \oplus \cdots \oplus E_n$ we have $\dim E_2 > \dim F_2$. As $F_1 + T(F_1))/F_1$ is finite dimensional we conclude that there exists $V_2 \subset E_2$ with $T(V_2) \subset E_2$ and $\dim V_2 = \dim E_2$. It is now clear how the argument may be repeated in order to conclude that there exist spaces $V_i \subset E_i$ with $T(V_i) \subset E_i$ and $\dim V_i = \dim E_i$. Let then K_{ij} be the projection of $T(E_i)$ on E_j. $K_{ij} \perp V_j$ for all $i \neq j$. Since $\dim T(V_i) = \dim E_i$ by the choice of the V_i and by the assumptions of the lemma, we conclude that K_{ij} is finite dimensional for all pairs $i \neq j$. This is what the lemma asserts.

We now consider the orthogonal sum of finitely many spaces (E_i, Φ_i) of the kind constructed in §1. For the sake of simplicity we choose Φ_i symmetric: For $i = 1, 2, \ldots n$ let $(e_\iota^i)_{\iota \in J(i)}$ be a basis of E_i, $\Phi_i(e_\iota^i, e_\nu^i) = \xi_{\iota\nu}^i$ where $\xi_{\iota\nu}^i = \xi_{\mu\kappa}^i$ if and only if $\{\iota, \nu\} = \{\mu, \kappa\}$ and where, for every fixed i, the set X^i of all $\xi_{\iota\nu}^i$ $\{\iota, \nu \in J(i)\}$ is algebraically independent over the prime field k_0 of the basefield k. We shall *not* assume that the sets $X^1, \ldots, X^n$ are disjoint. For these symmetric spaces we prove

THEOREM 3. *Assume that $\dim E_i > \dim E_{i+1} > \aleph_0$ $(i = 1, \ldots, n-1)$. Then every isometry of the orthogonal sum $E = E_1 \oplus \cdots \oplus E_n$ is locally algebraic.*

Proof. For the sake of simplicity we omit the superscript 1 when mentioning e_ι^1 and $x_{\iota\nu}^1$; furthermore let $J(1) = J$. Let us study the action of T on E_1 for T an isometry of E:

$$Te_\iota = \sum_J \alpha_{\iota\mu} e_\mu + g_\iota, \quad \text{where} \quad g_\iota \in E_2 \oplus \cdots \oplus E_n$$

By the previous lemma, $G = k(g_\iota)_{\iota \in J}$ is of finite dimension. Let $Q \subset J$ be such that $(g_\iota)_{\iota \in Q}$ is a basis of G. We introduce the finite sets $M(\iota) = \{\mu \in J \mid \alpha_{\iota\mu} \neq 0\}$. Let $M = \bigcup_{\iota \in J} [M(\iota) \backslash \{\iota\}]$. We show that M is finite. Assume by way of contradiction that M is infinite. There is a denumerably infinite subset $S \subset J$ and a map κ that assigns to every $\iota \in S$ a $\kappa(\iota) \in J$ with $\kappa(\iota) \in M(\iota) \backslash \{\iota\}$ and $\kappa(\iota) \neq \kappa(\nu)$ for all $\iota \neq \nu$ in S. There is a subset A of $X^1 \cup \cdots \cup X^n$ with $\operatorname{card} A \leqslant \aleph_0$ such that $\alpha_{\iota\kappa} \in \overline{k_0(A)}$ (the algebraic closure of $k_0(A)$ in k) for all $\kappa \in M(\iota)$, $\iota \in S$ and $\Phi(g_\iota, g_\kappa) \in \overline{k_0(A)}$ for all $\iota \in S$, $\kappa \in Q$. Let $N = \bigcup_{\iota \in S} M(\iota)$. $\operatorname{card} N = \aleph_0$. There is a $\nu \in J$ and for it a $\mu_0 \in M_\nu$ such that $\mu_0 \notin N \cup S$ and

$$\{\xi_{\iota\mu_0} \in X \mid \iota \in N\} \cap A = \emptyset \tag{2}$$

For $Te_\nu = \sum_{\mu \in M(\nu)} \alpha_{\nu\mu} \cdot e_\mu + \sum_{\kappa \in Q} \beta_{\nu\kappa} g_\kappa$ we have

$$(Te_\iota, Te_\nu) = \xi_{\iota\nu} = \alpha_{\iota\kappa(\iota)} \alpha_{\nu\mu_0} \xi_{\kappa(\iota)\mu_0} + \sum_{\kappa\mu} \alpha_{\iota\kappa} \alpha_{\nu\mu} \xi_{\kappa\mu} + \sum_{\kappa \in Q} \beta_{\mu\kappa} \Phi(g_\iota, g_\kappa) \tag{3}$$

The first sum in (3) extends over the set $[M(\iota) \times M(\nu)] \setminus \{\kappa(\iota), \mu_0\}$. There is a finite subset B of $X^1 \cup \cdots \cup X^n$ such that $\alpha_{\nu\mu} \in \overline{k_0(B)}$ for all $\mu \in M(\nu)$ and $\beta_{\nu\kappa} \in \overline{k_0(B)}$ for all $\kappa \in Q$. Since S is infinite, there is a $\sigma \in S$ with $\xi_{\kappa(\sigma)\mu_0} \notin B$. As $\kappa(\sigma) \neq \sigma$ by the choice of the map κ and since $\mu_0 \neq \sigma$ we have $\xi_{\sigma\nu} \neq \xi_{\kappa(\sigma)\mu_0}$. Let $C = A \cup B \cup \{\xi_{\sigma\nu}, \xi_{\kappa\mu} \mid (\kappa, \mu) \in [M(\sigma) \times M(\nu)] \setminus (\kappa(\sigma), \mu_0)\}$. By (2) we have $\xi_{\kappa(\sigma)\mu_0} \notin A$, hence $\xi_{\kappa(\sigma)\mu_0} \notin C$. All quantities in equation (3) equated for $\iota = \sigma$ are contained in $\overline{k_0(C)}$ with the exception of $\xi_{\kappa(\sigma)\mu_0}$. The coefficient of $\xi_{\kappa(\sigma)\mu_0}$ in (3) is not zero. Hence we should have $\xi_{\kappa(\sigma)\mu_0} \in \overline{k_0(C)}$; so $\xi_{\kappa(\sigma)\mu_0}$ is algebraically dependent over C which is a contradiction. We have thus shown that M is finite. G being finite dimensional, there is a subspace F_1 of E, spanned by finitely many e_ι^i, $\iota \in J(i)$, $i = 1, \ldots, n$ such that $Te_\nu^1 \in k(e_\nu^1) + F_1$ for all $\nu \in J(1)$. In the same manner we find for $i = 2, \ldots, n$ finite dimensional spaces F_i such that $Te_\nu^i \in k(e_\nu^i) + F_i$. Set $F = \sum_{i=1}^n F_i$. We have $Te_\mu^i \in k(e_\mu^i) + F$ for all $\mu \in J(i)$ and all $i = 1, \ldots, n$. In particular $T(F) \subset F$. Since F is finite dimensional we conclude that T is locally algebraic on all basis vectors e_μ and hence locally algebraic on each $x \in E$. Q.E.D.

Let us look at the proof for one more moment. We have shown that there is a subspace F of E, spanned by finitely many of the basisvectors e_ι^i such that $Te \in k(e) + F$ for all basis vectors $e = e_\iota^i$. Hence F is the orthogonal sum of its projections onto the summands E_i in the decomposition $E = E_1 + \cdots + E_n$. These projections, say G_i, are semissimple (as are all spans of collections of basisvectors of our particular bases $(e_\iota^i)_{\iota \in J(i)}$, $(i = 1, \ldots, n)$). Therefore $E_i = G_i \oplus (G_i^\perp \cap E_i)$. Since $T(F) = F$ it follows that the spaces $G_i^\perp \cap E_i$ are left invariant under T. If we extend $T^{-1}|_F$ to an isometry T_0 on E by letting T_0 act as the identity on $F^\perp$ we have $T_0 \in \mathfrak{J}_0(E)$ and $T_0 \circ T$ leaves each summand E_i of E invariant. The restriction of $T_0 \circ T$ to E_i is locally algebraic. Hence if $\operatorname{char} k \neq 2$ then we see by Theorem 2 that these restrictions are, up to a factor ± 1, a product of finitely many symmetries. We have thus shown that we can find altogether finitely many symmetries S on E such that $T_0 \circ T \circ \prod S$ acts on each E_i as $\mathbf{1}_{E_i}$ or $-\mathbf{1}_{E_i}$. Since $T_0 \in \mathfrak{J}_0(E)$ we obtain the

COROLLARY. *Let* $E = E_1 \oplus \cdots \oplus E_n$ *be as in Theorem* 3 *and* $\operatorname{char} k \neq 2$. *The* *quotient group* $\mathfrak{O}/\mathfrak{J}_0$ *of the full orthogonal group of E modulo the invariant subgroup* $\mathfrak{J}_0$ *is isomorphic to the direct product of n copies of* $\mathbb{Z}_2$. *In particular, if* $n=1$, *then* $\mathfrak{O}/\mathfrak{J}$ *is trivial.*

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