Algebraic L-Theory

IV. *Polynomial Extension Rings*

by A. A. RANICKI, Trinity College, Cambridge

Introduction

In Chapter XII of [1] Bass defines the notion of a *contracted functor*, as a functor

F: (rings) $\to$ (abelian groups)

such that the sequence

$$0 \to F(A) \xrightarrow{\binom{\bar{\varepsilon}_+}{-\bar{\varepsilon}_-}} F(A[x]) \oplus F(A[x^{-1}]) \xrightarrow{(\bar{E}_+ \bar{E}_-)} F(A[x, x^{-1}]) \xrightarrow{B} LF(A) \to 0$$

is naturally split exact for any ring A (associative with 1), where

$$\bar{\varepsilon}_\pm : A \to A[x^{\pm 1}] \quad \bar{E}_\pm : A[x^{\pm 1}] \to A[x, x^{-1}]$$

are inclusions in polynomial extensions of A, and

$$\begin{aligned} B: F(A[x, x^{-1}]) &\to LF(A) \\ &= \mathrm{coker}((\bar{E}_+ \bar{E}_-): F(A[x]) \oplus F(A[x^{-1}]) \to F(A[x, x^{-1}])) \end{aligned}$$

is the natural projection. Theorem 7.4 of Chapter XII of [1], the "Fundamental Theorem" of algebraic K-theory, states that

K_1: (rings) $\to$ (abelian groups)

is a contracted functor such that

$$LK_1(A) = K_0(A)$$

up to natural isomorphism. Here, we obtain analogous results for the groups of algebraic L-theory considered in the previous instalments of this series ([5], [6], [7] – we shall refer to these as Parts I, II, III respectively). In Part I we defined L-theoretic functors

U_n, V_n: (rings with involution) $\to$ (abelian groups)

for $n \pmod 4$, using quadratic forms on $\begin{cases}\text{f.g. projective} \\ \text{f.g. free}\end{cases}$ A-modules for the $\begin{cases}U\text{-} \\ V\text{-}\end{cases}$groups.

(The definitions are reviewed in §3 below, allowing this part to be read independently of the previous parts). It was shown in Part II that

$$V_n(A[x, x^{-1}]) = V_n(A) \oplus U_{n-1}(A)$$

if the involution $^{-}: A \to A; a \mapsto \bar{a}$ is extended to $A[x, x^{-1}]$ by $\bar{x} = x^{-1}$. The main result of this part of the paper (Theorem 4.1) is a split exact sequence

$$0 \to V_n(A) \xrightarrow{\binom{\bar{\varepsilon}_+}{-\bar{\varepsilon}_-}} V_n(A[x]) \oplus V_n(A[x^{-1}]) \xrightarrow{(E_+ E_-)} V_n(A[x, x^{-1}]) \xrightarrow{B} U_n(A) \to 0$$

for each $n \pmod 4$, with the involution on A extended to $A[x^{\pm 1}]$, $A[x, x^{-1}]$ by $\bar{x} = x$. The proof depends on L-theoretic analogues (Lemmas 4.2, 4.3) of the Higman linearization trick (quoted in Lemma 2.2) and of a result from [2] (quoted in Lemma 2.3) on the automorphisms of $A[x, x^{-1}]$-modules which are linear in x. A similar result has been obtained independently by Karoubi ([4]), using an L-theoretic analogue of the localization sequence of Chapter IX of [1].

Adopting the terminology of [1], we can say that each

$$V_n : (\text{rings with involution}) \to (\text{abelian groups})$$

is a contracted functor, with

$$LV_n(A) = U_n(A)$$

up to natural isomorphism. Corollary 4.4 generalizes this "Fundamental Theorem" of algebraic L-theory to describe the intermediate L-groups $V_n^Q(A[x, x^{-1}])$, as defined in Part III, for suitable subgroups $Q \subseteq \tilde{K}_1(A[x, x^{-1}])$. Corollary 4.5 identifies the "lower L-theories" of Part II with the functors

$$L^m U_n : (\text{rings with involution}) \to (\text{abelian groups}) \qquad (m > 0)$$

derived from U_n. (There is an obvious analogy here with the "lower K-theories" of Chapter XII of [1],

$$K_{-m} = L^m K_0 : (\text{rings}) \to (\text{abelian groups}).)$$

Corollary 4.6 describes the L-groups of polynomial extensions in several variables.

The work presented here was stimulated by a course of lectures on algebraic K-theory given by Hyman Bass at Cambridge University in the Lent Term of 1973.

§1. Contracted Functors

Let (rings) be the category of associative rings with 1, and 1-preserving ring morphisms. Let x be an invertible indeterminate over such a ring A commuting with every element of A, and define $A[x, x^{-1}]$, the ring of finite polynomials $\sum_{j=-\infty}^{\infty} x^j a_j$ in x, x^{-1} with coefficients $a_j \in A$. Let $A[x^{\pm 1}]$ be the subring of $A[x, x^{-1}]$ of polynomials involving only non-negative powers of $x^{\pm 1}$. Let

$$\bar{\varepsilon}_{\pm}:A\to A[x^{\pm 1}],\quad \bar{E}_{\pm}:A[x^{\pm 1}]\to A[x,x^{-1}],\quad \bar{\varepsilon}=\bar{E}_{\pm}\bar{\varepsilon}_{\pm}:A\to A[x,x^{-1}]$$

be the inclusions, and define left inverses

$$\varepsilon_{\pm}:A[x^{\pm 1}]\to A,\quad \varepsilon:A[x,x^{-1}]\to A$$

for $\bar{\varepsilon}_{\pm}$, $\bar{\varepsilon}$ by $x^{\pm 1}\mapsto 1$.

A functor

$$F:(\text{rings})\to(\text{abelian groups})$$

is *contracted* if the sequence

$$0\to F(A)\xrightarrow{\binom{\bar{\varepsilon}_+}{-\bar{\varepsilon}_-}}F(A[x])\oplus F(A[x^{-1}])\xrightarrow{(\bar{E}_+\,\bar{E}_-)}F(A[x,x^{-1}])\xrightarrow{B}LF(A)\to 0$$

is exact for each A, and there is given a natural right inverse

$$\bar{B}:LF(A)\to F(A[x,x^{-1}])$$

for the natural projection

$$\begin{aligned}B:F(A[x,x^{-1}])&\to LF(A)\\&=\operatorname{coker}((\bar{E}_+\bar{E}_-):F(A[x])\oplus F(A[x^{-1}])\to F(A[x,x^{-1}])),\end{aligned}$$

that is $B\bar{B}=1:LF(A)\to LF(A)$. (This is just Definition 7.1 of Chapter XII of [1]).

LEMMA 1.1. *Let*

$$F,G:(\text{rings})\to(\text{abelian groups})$$

be functors, and suppose given

i) *a natural left inverse*

$$E_+:F(A[x,x^{-1}])\to F(A[x])$$

for

$$\bar{E}_+:F(A[x])\to F(A[x,x^{-1}])$$

such that the square

$$\begin{array}{ccc}F(A[x^{-1}]) & \xrightarrow{\bar{E}_-} & F(A[x,x^{-1}])\\ \downarrow{\scriptstyle\varepsilon_-} & & \downarrow{\scriptstyle E_+}\\ F(A) & \xrightarrow[\bar{\varepsilon}_+]{} & F(A[x])\end{array}$$

commutes,

ii) *natural morphisms*

$$\bar{\eta}_+:G(A)\to L_+F(A)=\operatorname{coker}(\bar{E}_+:F(A[x])\to F(A[x,x^{-1}]))$$
$$\eta_+:L_+F(A)\to G(A)$$

such that $\eta_+\bar{\eta}_+=1$, *and such that the square*

$$\begin{tikzcd}
L_+F(A) \arrow[r, "\eta_+"] \arrow[d, "\Delta_+"'] & G(A) \arrow[d, "\bar{\eta}_-"] \\
F(A[x, x^{-1}]) \arrow[r, "\delta_-"] & L_-F(A)
\end{tikzcd}$$

commutes, where

$$\Delta_+ : L_+F(A) \to F(A[x, x^{-1}])$$

is the right inverse for the natural projection

$$\delta_+ : F(A[x, x^{-1}]) \to L_+F(A)$$

induced by

$$1-\bar{E}_+E_+ : F(A[x, x^{-1}]) \to F(A[x, x^{-1}]),$$

and $\delta_-, \bar{\eta}_-$ *are defined as* $\delta_+, \bar{\eta}_+$ *but with* x^{-1} *replacing* x.

Then F is a contracted functor, and

$$B=\eta_+\delta_+ : F(A[x, x^{-1}]) \to G(A)$$

induces a natural isomorphism

$$LF(A)=\operatorname{coker}((\bar{E}_+\bar{E}_-) : F(A[x])\oplus F(A[x^{-1}]) \to F(A[x, x^{-1}])) \to G(A).$$

Proof. The diagrams

$$\begin{tikzcd}
0 \arrow[rr, bend left] \arrow[dr] & & F(A[x]) \arrow[rr, bend left, "\delta_-E_+"] \arrow[dr, "E_+"] & & L_-F(A) \arrow[rr, bend left] \arrow[dr, "\eta_-"] & & 0 \\
& F(A) \arrow[ur, "\bar{\varepsilon}_+"] \arrow[dr, "\bar{\varepsilon}_-"'] & & F(A[x, x^{-1}]) \arrow[ur, "\delta_-"] \arrow[dr, "\delta_+"'] & & G(A) \arrow[ur] \arrow[dr] & \\
0 \arrow[rr, bend right] \arrow[ur] & & F(A[x^{-1}]) \arrow[rr, bend right, "\delta_+E_-"'] \arrow[ur, "E_-"'] & & L_+F(A) \arrow[rr, bend right] \arrow[ur, "\eta_+"'] & & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 & & F(A[x]) \arrow[ll, bend right] \arrow[dl, "\varepsilon_+"'] & & L_-F(A) \arrow[ll, bend right, "E_+\Delta_-"'] \arrow[dl, "\Delta_-"'] & & 0 \arrow[ll, bend right] \arrow[dl] \\
& F(A) \arrow[ul] \arrow[dl] & & F(A[x, x^{-1}]) \arrow[ul, "E_+"'] \arrow[dl, "E_-"] & & G(A) \arrow[ul, "\bar{\eta}_-"'] \arrow[dl, "\bar{\eta}_+"] & \\
0 & & F(A[x^{-1}]) \arrow[ll, bend left] \arrow[ul, "\varepsilon_-"] & & L_+F(A) \arrow[ll, bend left, "E_-\Delta_+"] \arrow[ul, "\Delta_+"] & & 0 \arrow[ll, bend left] \arrow[ul]
\end{tikzcd}$$

are commutative exact braids, where E_-, Δ_-, η_- are defined as E_+, Δ_+, η_+ but with x^{-1} replacing x. It follows that

$$0 \to F(A) \xrightarrow{\binom{\bar{\varepsilon}_+}{-\bar{\varepsilon}_-}} F(A[x]) \oplus F(A[x^{-1}]) \xrightarrow{(E_+ E_-)} F(A[x, x^{-1}]) \xrightarrow{B} G(A) \to 0$$

is an exact sequence, with

$$\bar{B} = \Delta_\pm \bar{\eta}_\pm : G(A) \to F(A[x, x^{-1}])$$

a natural right inverse for

$$B = \eta_\pm \delta_\pm : F(A[x, x^{-1}]) \to G(A).$$

Thus F is a contracted functor, with

$$LF(A) = G(A)$$

up to natural isomorphism. □

(The conditions of Lemma 1.1 are necessary, as well as sufficient, for a functor to be contracted. If

$$F: (\text{rings}) \to (\text{abelian groups})$$

is a contracted functor, then

$$F(A[x, x^{-1}]) = \bar{\varepsilon} F(A) \oplus \bar{E}_+ N_+ F(A) \oplus \bar{E}_- N_- F(A) \oplus \bar{B} LF(A)$$

where

$$N_\pm F(A) = \ker(\varepsilon_\pm : F(A[x^{\pm 1}]) \to F(A)),$$

and the morphisms

$$\begin{aligned} &E_+ : F(A[x, x^{-1}]) \to F(A[x]) = \bar{\varepsilon}_+ F(A) \oplus N_+ F(A); \\ &\qquad \bar{\varepsilon}(r) \oplus \bar{E}_+(s_+) \oplus \bar{E}_-(s_-) \oplus \bar{B}(t) \mapsto \bar{\varepsilon}_+(r) \oplus s_+ \\ &\bar{\eta}_+ : LF(A) \to L_+ F(A) = \bar{E}_- N_- F(A) \oplus \bar{B} LF(A);\ t \mapsto 0 \oplus \bar{B}(t) \\ &\eta_+ : L_+ F(A) \to LF(A);\ \bar{E}_-(s_-) \oplus \bar{B}(t) \mapsto t \end{aligned}$$

satisfy the conditions of Lemma 1.1, with $G = LF$.)

§2. K-Theory of Polynomial Extensions

Let $\mathbf{P}(A)$ be the category of finitely generated (f.g.) projective left A-modules. Write $|\mathbf{P}(A)|$ for the class of objects, and $\mathrm{Hom}_A(P, Q)$ for the additive group of morphisms $g:P\to Q\in\mathbf{P}(A)$. A ring morphism

$$f:A\to A'$$

induces a functor

$$f:\mathbf{P}(A)\to\mathbf{P}(A');\begin{cases}P\in|\mathbf{P}(A)|\mapsto fP=A'\otimes_A P\in|\mathbf{P}(A')|\\ g\in\mathrm{Hom}_A(P,Q)\mapsto fg=1\otimes g\in\mathrm{Hom}_{A'}(fP,fQ).\end{cases}$$

Given $P\in|\mathbf{P}(A)|$, let

$$P[x^{\pm1}]=\bar{\varepsilon}_\pm P\in|\mathbf{P}(A[x^{\pm1}])|,\ P_x=\bar{\varepsilon}P\in|\mathbf{P}(A[x,x^{-1}])|.$$

Defining complementary A-submodules

$$P^+=\sum_{j=0}^{\infty}x^jP,\qquad P^-=\sum_{j=-\infty}^{-1}x^jP$$

of P_x (where $x^jP=x^j\otimes P$) we shall identify

$$P^+=P[x],\qquad xP^-=P[x^{-1}]$$

in the obvious way.

Let $\mathbf{N}(A)$ be the category with objects pairs

$$(P\in|\mathbf{P}(A)|,\ v\in\mathrm{Hom}_A(P,P)\ \text{nilpotent})$$

and morphisms

$$f:(P,v)\to(P',v')\in\mathbf{N}(A)$$

isomorphisms $f\in\mathrm{Hom}_A(P,P')$ such that

$$v'f=fv\in\mathrm{Hom}_A(P,P').$$

As usual, there are defined functors

$$K_i:(\text{rings})\to(\text{abelian groups});\qquad A\mapsto K_i(\mathbf{P}(A))$$

for $i=0,1$. Theorem 7.4 of Chapter XII of [1], the "Fundamental Theorem" of algebraic K-theory, may be stated and proved as follows:

THEOREM 2.1 *The functor K_1 is contracted, with*

$$L_+K_1(A)=K_0\mathbf{N}(A),\qquad LK_1(A)=K_0(A)$$

up to natural isomorphism.

Proof. Given an automorphism

$$f: G_x \to G_x \in \mathbf{P}(A[x, x^{-1}]) \qquad (G \in |\mathbf{P}(A)|)$$

let $F = f(G) \subseteq G_x$, and define

$$(P, \nu) = (G^- / x^{-N} F^-, x^{-1}) \in |\mathbf{N}(A)|$$

for $N \geqslant 0$ so large that $x^{-N} F^- \subseteq G^-$. Then

$$E_+ : K_1(A[x, x^{-1}]) \to K_1(A[x]);$$
$$\tau(f: G_x \to G_x) \mapsto \bar{\varepsilon}_+ \tau(\varepsilon f: G \to G) \oplus \tau((1-\nu)^{-1}(1 - x\nu): P^+ \to P^+)$$

is a well-defined morphism.

LEMMA 2.2 *Every element of* $K_1(A[x])$ *can be represented by an automorphism*

$$f = f_0 + x f_1 : G^+ \to G^+ \in \mathbf{P}(A[x])$$

with $f_0, f_1 \in \mathrm{Hom}_A(G, G)$.

Proof. Given an automorphism

$$f = f_0 + x f_1 + x^2 f_2 + \cdots + x^r f_r \in \mathrm{Hom}_{A[x]}(G^+, G^+) \qquad (f_j \in \mathrm{Hom}_A(G, G), 0 \leqslant j \leqslant r)$$

we can apply the usual Higman linearization trick (first used in the proof of Theorem 15 of [3]), the identity

$$\begin{pmatrix} 1 & -x^{r-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x f_r & 1 \end{pmatrix}$$
$$= \begin{pmatrix} f_0 + x f_1 + \cdots + x^{r-1} f_{r-1} & -x^{r-1} \\ x f_r & 1 \end{pmatrix} : G^+ \oplus G^+ \to G^+ \oplus G^+$$

$(r-1)$ times, to obtain a representative automorphism for $\tau(f) \in K_1(A[x])$ which is linear in x (with $r = 1$). $\square$

Given an automorphism

$$f = f_0 + x f_1 \in \mathrm{Hom}_{A[x]}(G^+, G^+)$$

let $\gamma = (f_0 + f_1)^{-1} f_1 \in \mathrm{Hom}_A(G, G)$. Then

$$f = (f_0 + f_1)(1 + (x-1)\gamma) : G^+ \to G^+$$

and (up to isomorphism)

$$(G^- / x^{-1} f(G^-), x^{-1}) = (G^- / x^{-1}(1 + (x-1)\gamma) G^-, x^{-1}) = (G, -\gamma(1-\gamma)^{-1}) \in |\mathbf{N}(A)|.$$

It follows that

$$\begin{aligned}E_+\bar{E}_+\tau(f)&=\tau(f_0+f_1:G^+\to G^+)\oplus\tau((1+\gamma(1-\gamma)^{-1})^{-1}\\&\qquad\times(1+x\gamma(1-\gamma)^{-1}):G^+\to G^+)\\&=\tau(f_0+f_1:G^+\to G^+)\oplus\tau(1+(x-1)\,\gamma:G^+\to G^+)\\&=\tau(f)\in K_1(A[x]).\end{aligned}$$

Thus the composite

$$K_1(A[x])\xrightarrow{E_+}K_1(A[x,x^{-1}])\xrightarrow{E_+}K_1(A[x])$$

is the identity. Similarly, it can be shown that the square

$$\begin{tikzcd} K_1(A[x^{-1}]) \arrow[r, "E_-"] \arrow[d, "\varepsilon_-"'] & K_1(A[x,x^{-1}]) \arrow[d, "E_+"] \\ K_1(A) \arrow[r, "\bar{\varepsilon}_+"'] & K_1(A[x]) \end{tikzcd}$$

commutes.

Higman's trick also shows that every element of $K_1(A[x,x^{-1}])$ may be expressed as

$$\tau=\tau(f_0+xf_1:P_x\to P_x)\oplus\tau(x^N:Q_x\to Q_x)\in K_1(A[x,x^{-1}])$$

for some $P, Q\in|\mathbf{P}(A)|, f_0, f_1\in \mathrm{Hom}_A(P,P)$, $N\in\mathbf{Z}$.

LEMMA 2.3. *If* $\gamma\in\mathrm{Hom}_A(P,P)$ *is such that*

$$1+(x-1)\,\gamma\in\mathrm{Hom}_{A[x,x^{-1}]}(P_x,P_x)$$

is an isomorphism then there exist integers $r, s\geqslant 0$ *such that*

$$\gamma^r(1-\gamma)^s=0\in\mathrm{Hom}_A(P,P),$$

and $R=\ker\gamma^r$, $S=\ker(1-\gamma)^s$ *are complementary submodules of* P, *such that*

$$\gamma=\begin{pmatrix}\gamma_R & 0\\ 0 & \gamma_S\end{pmatrix}:P=R\oplus S\to P=R\oplus S$$

with $\gamma_R\in\mathrm{Hom}_A(R,R)$, $1-\gamma_S\in\mathrm{Hom}_A(S,S)$ *nilpotent.*

Proof. See Corollary 2.4 of [2] and pp. 232–34 of [8]. □

If $f_0, f_1\in\mathrm{Hom}_A(P,P)$ are such that

$$f=f_0+xf_1\in\mathrm{Hom}_{A[x,x^{-1}]}(P_x,P_x)$$

is an isomorphism, then

$$\varepsilon f = f_0 + f_1 \in \mathrm{Hom}_A(P, P)$$

is an isomorphism, and $\gamma = (f_0 + f_1)^{-1} f_1 \in \mathrm{Hom}_A(P, P)$ satisfies the hypothesis of Lemma 2.3. Hence

$$\begin{aligned}\tau(f) &= \bar{\varepsilon}\tau(f_0 + f_1 : P \to P) \oplus \tau(1 + (x-1)\,\gamma : P_x \to P_x)\\ &= \bar{\varepsilon}\tau(f_0 + f_1 : P \to P)\\ &\quad \oplus \bar{E}_+ \tau(1 + (x-1)\,\gamma_R : R[x] \to R[x])\\ &\quad \oplus \bar{E}_- \tau(1 + (x^{-1} - 1)(1 - \gamma_S) : S[x^{-1}] \to S[x^{-1}])\\ &\quad \oplus \tau(x : S_x \to S_x) \in K_1(A[x, x^{-1}])\end{aligned}$$

It is now easy to verify that

$$K_1(A[x]) \underset{\bar{E}_+}{\overset{E_+}{\rightleftarrows}} K_1(A[x, x^{-1}]) \underset{\Delta_+}{\overset{\delta_+}{\rightleftarrows}} K_0\mathrm{N}(A)$$

is a direct sum system, with

$$\begin{aligned}&\Delta_+ : K_0\mathrm{N}(A) \to K_1(A[x, x^{-1}]);\ [P, v] \mapsto \tau((1 - v)^{-1}(x - v) : P_x \to P_x)\\ &\delta_+ : K_1(A[x, x^{-1}]) \to K_0\mathrm{N}(A);\ \tau(f : G_x \to G_x) \mapsto [G^+/x^N F^+, x] - [F^+/x^N F^+, x]\end{aligned}$$

where $F = f(G) \subseteq G_x$ (as before) and $N \geqslant 0$ is so large that $x^N F^+ \subseteq G^+$, (so that, in particular,

$$\delta_+ \tau(f_0 + x f_1 : P_x \to P_x) = [S, -\gamma_S^{-1}(1 - \gamma_S)] \in K_0\mathrm{N}(A)).$$

Identifying

$$L_+ K_1(A) = K_0\mathrm{N}(A)$$

in this way, note that the morphisms

$$\begin{aligned}&\eta_+ : K_0\mathrm{N}(A) \to K_0(A);\ [P, v] \mapsto [P]\\ &\bar{\eta}_+ : K_0(A) \to K_0\mathrm{N}(A);\ [P] \mapsto [P, 0]\end{aligned}$$

are such that the conditions of Lemma 1.1 are satisfied. Hence

$$K_1 : (\text{rings}) \to (\text{abelian groups})$$

is a contracted functor, with

$$LK_1(A) = K_0(A)$$

up to natural isomorphism. This completes the proof of Theorem 2.1. □

§3. Review of the Definitions of the L-Groups

Let (rings with involution) be the category of rings A (as in §1) with involution $^{-}: A \to A; a \mapsto \bar{a}$ such that

$$\bar{1}=1,\ \overline{a+b}=\bar{a}+\bar{b},\ \overline{ab}=\bar{b}\cdot\bar{a},\ \bar{\bar{a}}=a \quad \text{for all} \quad a, b \in A.$$

As in Part I it will be assumed that f.g. free A-modules have a well-defined dimension.

Given a ring with involution A define a *duality* involution

$$*: \mathbf{P}(A) \to \mathbf{P}(A) \begin{cases} P \in |\mathbf{P}(A)| \mapsto P^* = \mathrm{Hom}_A(P, A), \text{ left } A\text{-action by} \\ \qquad A \times P^* \to P^*; (a, p^*) \mapsto (p \mapsto p^*(p) \cdot \bar{a}) \\ f \in \mathrm{Hom}_A(P, Q) \mapsto (f^*: Q^* \to P^*; q^* \mapsto (p \mapsto q^*(f(p)))), \end{cases}$$

using the natural isomorphisms

$$P \to P^{**}; p \mapsto (p^* \mapsto \overline{p^*(p)}) \quad (P \in |\mathbf{P}(A)|)$$

to identify

$$** = 1: \mathbf{P}(A) \to \mathbf{P}(A).$$

An *ε-hermitian product* (over A) is a morphism

$$\theta: Q \to Q^* \in \mathbf{P}(A)$$

such that

$$\theta^* = \varepsilon\theta \in \mathrm{Hom}_A(Q, Q^*),$$

where $\varepsilon = \pm 1$. A $\pm$*form* (over A) is a pair

$$(Q \in |\mathbf{P}(A)|, \varphi \in \mathrm{Hom}_A(Q, Q^*)),$$

and

$$\theta = \varphi \pm \varphi^* \in \mathrm{Hom}_A(Q, Q^*)$$

is the *associated* $\pm$ hermitian product. An *isomorphism* of $\pm$forms

$$(f, \chi): (Q, \varphi) \to (Q', \varphi')$$

is an isomorphism $f \in \mathrm{Hom}_A(Q, Q')$ together with a morphism $\chi \in \mathrm{Hom}_A(Q, Q^*)$ such that

$$f^*\varphi' f - \varphi = \chi \mp \chi^* \in \mathrm{Hom}_A(Q, Q^*).$$

Such an isomorphism preserves the associated $\pm$ hermitian products, in that

$$f^*(\varphi'\pm\varphi'^*)f=(\varphi\pm\varphi^*)\in \mathrm{Hom}_A(Q, Q^*).$$

A $\pm$ form (Q, φ) is *non-singular* if the associated $\pm$ hermitian product $(\varphi\pm\varphi^*)\in \mathrm{Hom}_A(Q, Q^*)$ is an isomorphism. The *hamiltonian* $\pm$ form on $P\in|\mathbf{P}(A)|$,

$$H\pm(P)=(P\oplus P^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$$

is non-singular. A *sublagrangian* of a non-singular $\pm$ form (Q, φ) is a direct summand L of Q such that

$$j^*\varphi j=\lambda\mp\lambda^*\in \mathrm{Hom}_A(L, L^*)$$

for some $\lambda\in \mathrm{Hom}_A(L, L^*)$, denoting by $j\in \mathrm{Hom}_A(L, Q)$ the inclusion. It was shown in Theorem 1.1 of Part I that if L is a sublagrangian of (Q, φ) there is defined a non-singular $\pm$ form $(L^\perp/L, \hat{\varphi})$ on a direct complement $L^\perp/L$ to L in the *annihilator* of L in (Q, φ),

$$L^\perp=\ker(j^*(\varphi\pm\varphi^*):Q\to L^*),$$

and that there is defined an isomorphism of $\pm$ forms

$$(f, \chi):(Q, \varphi)\to H\pm(L)\oplus(L^\perp/L, \hat{\varphi})$$

with f the identity on $L^\perp=L\oplus L^\perp/L$. A *lagrangian* is a sublagrangian L such that

$$L^\perp=L,$$

in which case there is defined an isomorphism of $\pm$ forms

$$(f, \chi):(Q, \varphi)\to H\pm(L).$$

A $\pm$ *formation* (over A), $(Q, \varphi; F, G)$, is a triple consisting of

i) a non-singular $\pm$ form over A, (Q, φ),

ii) a lagrangian F of (Q, φ),

iii) a sublagrangian G of (Q, φ).

An *isomorphism* of $\pm$ formations

$$(f, \chi):(Q, \varphi; F, G)\to(Q', \varphi'; F', G')$$

is an isomorphism of $\pm$ forms

$$(f, \chi):(Q, \varphi)\to(Q', \varphi')$$

such that $f(F)=F'$, $f(G)=G'$. A *stable isomorphism* of $\pm$ formations

$$[f,\chi]:(Q,\varphi;F,G)\to(Q',\varphi';F',G')$$

is an isomorphism of $\pm$ formations

$$(f,\chi):(Q,\varphi;F,G)\oplus(H_{\pm}(P);P,P^*)\to(Q',\varphi';F',G')\oplus(H_{\pm}(P');P',P'^*)$$

defined for some $P,P'\in|\mathbf{P}(A)|$.

Let $T\subseteq\tilde{K}_0(A)=\operatorname{coker}(K_0(\mathbf{Z})\to K_0(A))$ be a subgroup invariant under the duality involution

$$*:\tilde{K}_0(A)\to\tilde{K}_0(A);\ [P]\mapsto[P^*]\qquad(\text{that is,}\quad *(T)=T).$$

For $n \pmod 4$ define the abelian monoid $X_n^T(A)$ of $\begin{cases}\text{isomorphism}\\ \text{stable isomorphism}\end{cases}$ classes of $\begin{cases}\pm\text{ forms }(Q,\varphi)\\ \pm\text{ formations }(Q,\varphi:F,G)\end{cases}$ over A such that the projective class $\begin{cases}[Q]\\ [G]-[F^*]\end{cases}$ lies in $T\subseteq\tilde{K}_0(A)$, under the direct sum $\oplus$, with $\pm=(-)^i$ if $n=\begin{cases}2i\\ 2i+1.\end{cases}$
The monoid morphisms

$$\partial^T:X_n^T(A)\to X_{n-1}^T(A);\ \begin{cases}(Q,\varphi)\mapsto(H_{\mp}(Q);Q,\Gamma_{(Q,\varphi)})\\ (Q,\varphi;F,G)\mapsto(G^{\perp}/G,\hat{\varphi})\end{cases}\qquad n=\begin{cases}2i\\ 2i+1\end{cases}$$

are such that $(\partial^T)^2=0$, where

$$\Gamma_{(Q,\varphi)}=\{(x,(\varphi\pm\varphi^*)x)\mid x\in Q\}\subseteq Q\oplus Q^*.$$

Define an equivalence relation $\sim$ on $\ker(\partial^T:X_n^T(A)\to X_{n-1}^T(A))$ by $z_1\sim z_2$ if there exist $b_1,b_2\in X_{n+1}^T(A)$ such that $z_1\oplus\partial^Tb_1=z_2\oplus\partial^Tb_2\in X_n^T(A)$. It was shown in Theorem 2.1 of Part III that the quotient monoids

$$U_n^T(A)=\ker(\partial^T:X_n^T(A)\to X_{n-1}^T(A))/\overline{\operatorname{im}(\partial^T:X_{n+1}^T(A)\to X_n^T(A))}$$

of equivalence classes are abelian groups, generalizing the definitions in Part I of

$$U_n(A)=U_n^{\tilde{K}_0(A)}(A),\qquad V_n(A)=U_n^{\{0\}}(A).$$

Theorem 2.3 of Part III established an exact sequence

$$\cdots\to H^{n+1}(T'/T)\to U_n^T(A)\to U_n^{T'}(A)\to H^n(T'/T)\to U_{n-1}^T(A)\to\cdots$$

for $*$-invariant subgroups $T\subseteq T'\subseteq\tilde{K}_0(A)$, where

$$H^n(G)=\{g\in G\mid g^*=(-)^n g\}/\{h+(-)^n h^*\mid h\in G\}$$

are the Tate cohomology groups (abelian, of exponent 2).

There are analogous definitions and results for L-groups associated with subgroups $R \subseteq \tilde{K}_1(A) = \mathrm{coker}(K_1(\mathbf{Z}) \to K_1(A))$ invariant under the duality involution

$$*: \tilde{K}_1(A) \to \tilde{K}_1(A);\ \tau(f: \underset{\sim}{P} \to \underset{\sim}{Q}) \mapsto \tau(f^*: \underset{\sim}{Q}^* \to \underset{\sim}{P}^*)$$

denoting by $\underset{\sim}{P}$ a f.g. free A-module P with a prescribed base, and by $\underset{\sim}{P}^*$ the dual based A-module.

A *based* $\pm$ form $(\underset{\sim}{Q}, \varphi)$ is a $\pm$ form (Q, φ) on a based A-module $\underset{\sim}{Q}$. The *torsion* of a based $\pm$ form $(\underset{\sim}{Q}, \varphi)$ is

$$\tau(\underset{\sim}{Q}, \varphi) = \begin{cases} \tau(\varphi \pm \varphi^*: \underset{\sim}{Q} \to \underset{\sim}{Q}^*) \in \tilde{K}_1(A) & \text{if} \quad (Q, \varphi) \text{ is non-singular} \\ 0 \in \tilde{K}_1(A) & \text{otherwise.} \end{cases}$$

An *R-isomorphism* of based $\pm$ forms

$$(f, \chi): (\underset{\sim}{Q}, \varphi) \to (\underset{\sim}{Q}', \varphi')$$

is an isomorphism of the underlying forms

$$(f, \chi): (Q, \varphi) \to (Q', \varphi')$$

such that

$$\tau(f: \underset{\sim}{Q} \to \underset{\sim}{Q}') \in R \subseteq \tilde{K}_1(A).$$

A *based* $\pm$ formation $(Q, \varphi; \underset{\sim}{F}, \underset{\sim}{G})$ is a $\pm$ formation $(Q, \varphi; F, G)$ with bases for F, G and $G^\perp/G$. The *torsion* $\tau(Q, \varphi; \underset{\sim}{F}, \underset{\sim}{G}) \in \tilde{K}_1(A)$ of a based $\pm$ formation is the torsion of the isomorphism

$$f: \underset{\sim}{F} \oplus \underset{\sim}{F}^* \to \underset{\sim}{G} \oplus \underset{\sim}{G}^* \oplus \underset{\frown}{G^\perp/G}$$

in the isomorphism of $\pm$ forms

$$(f, \chi): H_{\pm}(F) \to H_{\pm}(G) \oplus (G^\perp/G, \hat{\varphi})$$

given by Theorem 1.1 of Part I. An *R-isomorphism* of based $\pm$ formations

$$(f, \chi): (Q, \varphi; \underset{\sim}{F}, \underset{\sim}{G}) \to (Q', \varphi'; \underset{\sim}{F}', \underset{\sim}{G}')$$

is an isomorphism of the underlying $\pm$ formations such that the restrictions

$$\underset{\sim}{F} \to \underset{\sim}{F}',\ \underset{\sim}{G} \to \underset{\sim}{G}',\ \underset{\frown}{G^\perp/G} \to \underset{\frown}{G'^\perp/G'}$$

of f have torsions in $R \subseteq \tilde{K}_1(A)$. A *stable R-isomorphism* of based $\pm$ formations

$$[f, \chi]: (Q, \varphi; F, \underset{\sim}{G}) \to (Q', \varphi'; \underset{\sim}{F}', \underset{\sim}{G}')$$

is an R-isomorphism

$$(f,\chi):(Q,\varphi;\underset{\sim}{F},\underset{\sim}{G})\oplus(H_\pm(P);\underset{\sim}{P},\underset{\sim}{P}^*)\to(Q',\varphi';\underset{\sim}{F}',\underset{\sim}{G}')\oplus(H_\pm(P');\underset{\sim}{P}',\underset{\sim}{P}'^*)$$

defined for some based A-modules $\underset{\sim}{P}$, $\underset{\sim}{P}'$.

For $n(\bmod 4)$ define the abelian monoid $Y_n^R(A)$ of $\begin{cases}R\text{-isomorphism}\\ \text{stable } R\text{-isomorphism}\end{cases}$ classes of based $\begin{cases}\pm\text{ forms}\\ \pm\text{ formations}\end{cases}$ over A with torsion in $R\subseteq\tilde{K}_1(A)$, under the direct sum $\oplus$, with $\pm=(-)^i$ if $n=\begin{cases}2i\\ 2i+1\end{cases}$. The monoid morphisms

$$\partial^R: Y_n^R(A)\to Y_{n-1}^R(A);\begin{cases}(\underset{\sim}{Q},\varphi)\mapsto(H_\mp(Q);\underset{\sim}{Q},\Gamma_{(\underset{\sim}{Q},\varphi)})\\ (\underset{\sim}{Q},\varphi;\underset{\sim}{F},\underset{\sim}{G})\mapsto(\underbrace{G^\perp/G},\hat{\varphi})\end{cases}\qquad n=\begin{cases}2i\\ 2i+1\end{cases}$$

are such that $(\partial^R)^2=0$, and the quotient monoids

$$V_n^R(A)=\ker(\partial^R:Y_n^R(A)\to Y_{n-1}^R(A))/\overline{\operatorname{im}(\partial^R:Y_{n+1}^R(A)\to Y_n^R(A))}$$

are abelian groups (by Theorem 3.1 of Part III) generalizing the definitions in Part I of

$$V_n(A)=V_n^{\tilde{K}_1(A)}(A)\,(=U_n^{\{0\}}(A)),\qquad W_n(A)=V_n^{\{0\}}(A).$$

Theorem 3.3 in Part III established an exact sequence

$$\cdots\to H^{n+1}(R'/R)\to V_n^R(A)\to V_n^{R'}(A)\to H^n(R'/R)\to V_{n-1}^R(A)\to\cdots$$

for $*$-invariant subgroups $R\subseteq R'\subseteq\tilde{K}_1(A)$.

A morphism of rings with involution

$$f:A\to A'$$

such that $f(T)\subseteq T'$ (for some $*$-invariant subgroups $T\subseteq\tilde{K}_0(A)$, $T'\subseteq\tilde{K}_0(A')$) induces abelian group morphisms

$$f:U_n^T(A)\to U_n^{T'}(A');\begin{cases}(Q,\varphi)\mapsto(fQ,f\varphi)\\ (Q,\varphi;F,G)\mapsto(fQ,f\varphi;fF,fG)\end{cases}\qquad n=\begin{cases}2i\\ 2i+1.\end{cases}$$

Similarly, if $f(R)\subseteq R'$ (for $*$-invariant subgroups $R\subseteq\tilde{K}_1(A)$, $R'\subseteq\tilde{K}_1(A')$) there are induced morphisms

$$f:V_n^R(A)\to V_n^{R'}(A')\qquad(n(\bmod 4)).$$

§4. L-Theory of Polynomial Extensions

Given a ring with involution A and an indeterminate x over A commuting with every element of A extend the involution on A to the involution

$$\bar{\ }: A[x, x^{-1}] \to A[x, x^{-1}]; \quad \sum_{j=-\infty}^{\infty} x^j a_j \mapsto \sum_{j=-\infty}^{\infty} x^j \bar{a}_j$$

on $A[x, x^{-1}]$. This restricts to involutions on the subrings $A[x]$, $A[x^{-1}]$ of $A[x, x^{-1}]$. F. g, free $A[x]$-modules have well-defined dimension, as do those over $A[x^{-1}]$, $A[x, x^{-1}]$. Thus the rings with involution $A[x^{\pm 1}]$, $A[x, x^{-1}]$ satisfy the conditions imposed on A in §3.

Call a functor

$$F: (\text{rings with involution}) \to (\text{abelian groups})$$

contracted if the sequence

$$0 \to F(A) \xrightarrow{\binom{\bar{\varepsilon}_+}{-\bar{\varepsilon}_-}} F(A[x]) \oplus F(A[x^{-1}]) \xrightarrow{(E_+ E_-)} F(A[x, x^{-1}]) \xrightarrow{B} LF(A) \to 0$$

is exact for every ring with involution A and there is given a natural right inverse

$$\bar{B}: LF(A) \to F(A[x, x^{-1}])$$

for the natural projection

$$\begin{aligned} B: F(A[x, x^{-1}]) &\to LF(A) \\ &= \operatorname{coker}((\bar{E}_+ \bar{E}_-): F(A[x] \oplus F(A[x^{-1}]) \to F(A[x, x^{-1}])). \end{aligned}$$

The obvious analogue to Lemma 1.1 holds for functors

$$(\text{rings with involution}) \to (\text{abelian groups})$$

as does the following analogue of Theorem 2.1 for the L-theoretic functors of §3:

THEOREM 4.1. *Each of the functors*

$$V_n: (\text{rings with involution}) \to (\text{abelian groups}) \quad (n (\operatorname{mod} 4))$$

is contracted, with

$$LV_n(A) = U_n(A), \quad L_{\pm} V_n(A) = U_n^{\tilde{K}_0(A)}(A[x^{\mp 1}])$$

up to natural isomorphism, where $\tilde{K}_0(A) \equiv \bar{\varepsilon}_{\mp} \tilde{K}_0(A) \subseteq \tilde{K}_0(A[x^{\mp 1}])$. □

The proof of Theorem 4.1 in the case $n = 2i$ will be similar to the proof of Theorem 2.1. The case $n = 2i + 1$ will follow by an application of the results of Part II on the L-theory of Laurent extensions (that is, of the ring $A[x, x^{-1}]$ with involution by $\bar{x} = x^{-1}$).

Recall from Part II that a *modular A-base* of an $A[x, x^{-1}]$-module Q is an A-submodule Q_0 of Q such that every element q of Q has a unique expression as

$$q = \sum_{j=-\infty}^{\infty} x^j q_j \quad (q_j \in Q_0, \{j \mid q_j \neq 0\} \text{ finite}),$$

so that $Q = A[x, x^{-1}] \otimes_A Q_0$ up to $A[x, x^{-1}]$-module isomorphism. For example the A-modules generated by the bases of free $A[x, x^{-1}]$-modules are modular A-bases.

Define a morphism

$$\begin{aligned}\delta_+ : V_{2i}(A[x, x^{-1}]) \to U_{2i}^{\tilde{K}_0(A)}(A[x^{-1}]);\\ (Q, \varphi) \mapsto (P[x^{-1}], [\varphi]_{-1} - x^{-1}[\varphi]_{-2})\end{aligned}$$

by choosing a modular A-base Q_0 for Q (which is a f.g. free $A[x, x^{-1}]$-module) and an integer $N \geqslant 0$ so large that

$$(\varphi \pm \varphi^*)(x^N Q_0^+) \subseteq x^{-N} Q_0^{*+} \quad (\pm = (-)^i),$$

defining

$$P = x^N Q_0^- \cap (\varphi \pm \varphi^*)^{-1}(x^{-N} Q_0^{*+}) \in |\mathbf{P}(A)|,$$

with $[\varphi]_j \in \mathrm{Hom}_A(P, P^*)$ given by

$$[\varphi]_j(y)(y') = a_j \in A \quad (y, y' \in P, j \in \mathbf{Z})$$

if

$$\varphi(y)(y') = \sum_{j=-\infty}^{\infty} x^j a_j \in A[x, x^{-1}] \quad (a_j \in A),$$

and writing $P[x^{-1}]$ for $\bar{\varepsilon}_- P = A[x^{-1}] \otimes_A P \in |\mathbf{P}(A[x^{-1}])|$.

The A-module isomorphism

$$[\varphi \pm \varphi^*]_{-1} : Q \to Q^*$$

may be expressed as

$$[\varphi \pm \varphi^*]_{-1} = \begin{pmatrix} [\varphi]_{-1} \pm ([\varphi]_{-1})^* & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \pm 1 & 0 \end{pmatrix} : P \oplus L \oplus L^* \to P^* \oplus L^* \oplus L$$

where $L = (\varphi \pm \varphi^*)^{-1}(x^{-N} Q_0^{*-})$, $L^* = x^N Q_0^+ \subseteq Q$, so that $(P, [\varphi]_{-1})$ is a non-singular $\pm$ form over A.

For any $y, y' \in P$

$$\begin{aligned}[\varphi \pm \varphi^*]_{-2}(y)(y') &= [\varphi \pm \varphi^*]_{-1}(xy)(y')\\ &= [\varphi \pm \varphi^*]_{-1}(xy - x^N y_{N-1})(y') \in A,\end{aligned}$$

where $y_{N-1} \in Q_0$ is such that

$$y - x^{N-1} y_{N-1} \in x^{N-1} Q_0^- \cap (\varphi \pm \varphi^*)^{-1} (x^{-N-1} Q_0^*) = x^{-1} P.$$

Thus

$$(P, ([\varphi \pm \varphi^*]_{-1})^{-1} ([\varphi \pm \varphi^*]_{-2})) = ((\varphi \pm \varphi^*)^{-1} (x^{-N} Q_0^{*+}) / x^N Q_0^+, x) \in |\mathbf{N}(A)|,$$

and $(P[x^{-1}], [\varphi]_{-1} - x^{-1}[\varphi]_{-2})$ is a non-singular $\pm$ form over $A[x^{-1}]$.

Suppose that Q_0' is a different modular A-base of Q. Let $M \geqslant 0$ be so large that

$$Q_0' \subseteq \sum_{j=-M}^{M} x^j Q_0, \quad Q_0 \subseteq \sum_{j=-M}^{M} x^j Q_0'.$$

Then $N' = N + M$ is so large that

$$(\varphi \pm \varphi^*) (x^{N'} Q_0'^+) \subseteq x^{-N'} Q_0'^{*+},$$

and

$$\begin{aligned} P' &= x^{N'} Q_0'^- \cap (\varphi \pm \varphi^*)^{-1} (x^{-N'} Q_0'^{*+}) \quad \text{(definition)} \\ &= x^N (x^M Q_0'^- \cap Q_0^+) \oplus P \oplus x^{-N} (\varphi \pm \varphi^*)^{-1} (Q_0^{*-} \cap x^{-M} Q_0'^{*+}). \end{aligned}$$

Now

$$L = (x^N (x^M Q_0'^- \cap Q_0^+)) [x^{-1}] \subseteq P'[x^{-1}]$$

is a sublagrangian of $(P'[x^{-1}], [\varphi]_{-1} - x^{-1}[\varphi]_{-2})$ with $L^\perp / L = P[x^{-1}]$, so that

$$\begin{aligned} (P'[x^{-1}], [\varphi]_{-1} - x^{-1}[\varphi]_{-2}) &= (P[x^{-1}], [\varphi]_{-1} - x^{-1}[\varphi]_{-2}) \oplus H_\pm (L) \\ &= (P[x^{-1}], [\varphi]_{-1} - x^{-1}[\varphi]_{-2}) \in U_{2i}^{\tilde{K}_0(A)} (A[x^{-1}]). \end{aligned}$$

Thus the choice of N and Q_0 is immaterial to the definition of δ_+.

Finally, suppose that

$$(Q, \varphi) = \bar{E}_+ (Q_0^+, \varphi_0) \in V_{2i} (A[x, x^{-1}])$$

for some $(Q_0^+, \varphi_0) \in V_{2i}(A[x])$. Then we can choose $N = 0$, and

$$\delta_+ (Q, \varphi) = 0 \in U_{2i}^{\tilde{K}_0(A)} (A[x^{-1}]).$$

Hence the morphism

$$\delta_+ : V_{2i}(A[x, x^{-1}]) \to U_{2i}^{\tilde{K}_0(A)} (A[x^{-1}])$$

is well-defined, and such that the composite

$$V_{2i}(A[x]) \xrightarrow{E_+} V_{2i}(A[x, x^{-1}]) \xrightarrow{\delta_+} U_{2i}^{\tilde{K}_0(A)} (A[x^{-1}])$$

is zero. Before going on to show that this sequence is in fact split exact, we need an L-theoretic analogue of Lemma 2.2 (the Higman linearization trick):

LEMMA 4.2. *Every element of* $U_{2i}^{\tilde{K}_0(A)}(A[x])$ (*resp.* $V_{2i}(A[x, x^{-1}])$) *can be represented by a linear* $\pm$*form,* $(Q^+, \varphi_0+x\varphi_1)$ *over* $A[x]$ (*resp.* $(Q_x, \varphi_0+x\varphi_1)$ *over* $A[x, x^{-1}]$) *where* $\varphi_0, \varphi_1 \in \mathrm{Hom}_A(Q, Q^*)$.

Proof. Given $(Q^+, \varphi) \in U_{2i}^{\tilde{K}_0(A)}(A[x])$, let

$$\varphi = \sum_{j=0}^{N} x^j\varphi_j \ \mathrm{Hom}_{A[x]}(Q^+, Q^{*+}) \qquad (\varphi_j \in \mathrm{Hom}_A(Q, Q^*)),$$

and suppose $N > 1$. Now

$$\left(\begin{pmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ \pm x^{N-1}\varphi_N^* & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -x^{N-1}\varphi_N & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$
$$: (Q^+, \varphi) \oplus H_\pm(Q^+) \to \left(Q^+ \oplus Q^+ \oplus Q^{*+}, \begin{pmatrix} \varphi - x^N\varphi_N & -x^{N-1}\varphi_N & x \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

is an isomorphism of $\pm$forms over $A[x]$, so that

$$(Q'^+, \varphi') = (Q^+, \varphi) \in U_{2i}^{\tilde{K}_0(A)}(A[x])$$

with $Q' = Q \oplus Q \oplus Q^*$ such that

$$\varphi' = \sum_{j=0}^{N-1} x^j\varphi'_j \in \mathrm{Hom}_{A[x]}(Q'^+, Q'^{*+}) \qquad (\varphi'_j \in \mathrm{Hom}_A(Q', Q'^*)).$$

Iterating this procedure $(N-1)$ times we obtain a representative for

$$(Q^+, \varphi) \in U_{2i}^{\tilde{K}_0(A)}(A[x]) \text{ with } N = 1.$$

The same method works for elements $(Q_x, \varphi) \in V_{2i}(A[x, x^{-1}])$ provided we can assume that

$$(\varphi \pm \varphi^*)(Q^+) \subseteq Q^{*+}.$$

Choosing $N \geqslant 0$ so large that

$$(\varphi \pm \varphi^*)(x^N Q^+) \subseteq x^{-N} Q^{*+},$$

note that

$$(x^N, 0): (Q_x, \varphi' = x^{2N}\varphi) \to (Q_x, \varphi)$$

as an isomorphism of $\pm$ forms over $A[x, x^{-1}]$, so that

$$(Q_x, \varphi')=(Q_x, \varphi)\in V_{2i}(A[x, x^{-1}]),$$

and that

$$(\varphi' \pm \varphi'^*)(Q^+) \subseteq Q^{*+}. \quad \square$$

The morphism

$$\begin{aligned}\Delta_+ : U_{2i}^{\tilde{K}_0(A)}(A[x^{-1}]) &\to V_{2i}(A[x, x^{-1}]);\\ (Q[x^{-1}], \varphi) &\mapsto (Q_x, x\varphi) \oplus \bar{\varepsilon}\varepsilon_-(Q[x^{-1}], -\varphi) \oplus H_\pm(-Q_x)\end{aligned}$$

is clearly well-defined, with $-Q \in |\mathbf{P}(A)|$ such that $Q \oplus -Q$ is f.g. free.

The composite

$$U_{2i}^{\tilde{K}_0(A)}(A[x^{-1}]) \xrightarrow{\Delta_+} V_{2i}(A[x, x^{-1}]) \xrightarrow{\delta_+} U_{2i}^{\tilde{K}_0(A)}(A[x^{-1}])$$

is the identity: by Lemma 4.2 it is sufficient to consider $\delta_+\Delta_+(Q[x^{-1}], \varphi)$ with

$$\varphi = \varphi_0 + x^{-1}\varphi_{-1} \in \mathrm{Hom}_{A[x^{-1}]}(Q[x^{-1}], Q^*[x^{-1}]) \; (\varphi_0, \varphi_{-1} \in \mathrm{Hom}_A(Q, Q^*)),$$

and

$$\begin{aligned}&\delta_+\Delta_+(Q[x^{-1}], \varphi_0 + x^{-1}\varphi_{-1})\\ &\quad = \delta_+((Q_x, x\varphi_0 + \varphi_{-1}) \oplus (Q_x, -(\varphi_0 + \varphi_{-1})) \oplus H_\pm(-Q_x))\\ &\quad = ((Q^- \cap (x(\varphi_0 \pm \varphi_0^*) + (\varphi_{-1} \pm \varphi_{-1}^*))^{-1}(Q^{*+}))[x^{-1}],\\ &\qquad [x\varphi_0 + \varphi_{-1}]_{-1} - x^{-1}[x\varphi_0 + \varphi_{-1}]_{-2})\\ &\quad = ((1 + x^{-1}\gamma)^{-1}(x^{-1}Q), [x\varphi_0 + \varphi_{-1}]_{-1} - x^{-1}[x\varphi_0 + \varphi_{-1}]_{-2})\end{aligned}$$

where $\gamma = (\varphi_0 \pm \varphi_0^*)^{-1}(\varphi_{-1} \pm \varphi_{-1}^*) \in \mathrm{Hom}_A(Q, Q)$ is nilpotent. Now

$$(1 + x^{-1}\gamma)^{-1} = \sum_{j=0}^{\infty} (-)^j x^{-j}\gamma^j \in \mathrm{Hom}_{A[x^{-1}]}(Q[x^{-1}], Q[x^{-1}]),$$

so that

$$\begin{aligned}&[x\varphi_0 + \varphi_{-1}]_j (1 + y^{-1}\gamma)^{-1}(x^{-1}y)(1 + x^{-1}\gamma)^{-1}(x^{-1}y')\\ &\qquad = \begin{cases}\varphi_0(y)(y') \\ ((\varphi_{-1} - \varphi_0\gamma - \gamma^*\varphi_0)(y)(y')\end{cases} \text{if } j = \begin{cases}-1\\ -2\end{cases} \quad (y, y' \in Q),\end{aligned}$$

and

$$\varphi_{-1} - \varphi_0\gamma - \gamma^*\varphi_0 = -\varphi_{-1} + \chi \mp \chi^* \in \mathrm{Hom}_A(Q, Q^*),$$

where $\chi = \varphi_{-1} - \gamma^*\varphi_0 \in \mathrm{Hom}_A(Q, Q^*)$. Thus

$$\begin{aligned}\delta_+\Delta_+(Q[x^{-1}], \varphi_0 + x^{-1}\varphi_{-1}) &= (Q[x^{-1}], \varphi_0 + x^{-1}(\varphi_{-1} - (\chi \mp \chi^*)))\\ &= (Q[x^{-1}], \varphi_0 + x^{-1}\varphi_{-1}) \in U_{2i}^{\tilde{K}_0(A)}(A[x^{-1}])\end{aligned}$$

and

$$\delta_+ \Delta_+ = 1 : U_{2i}^{\tilde{K}_0(A)}(A[x^{-1}]) \to U_{2i}^{\tilde{K}_0(A)}(A[x^{-1}]).$$

It is therefore sufficient to prove that $V_{2i}(A[x, x^{-1}])$ is generated by the images of $\bar{E}_+ : V_{2i}(A[x]) \to V_{2i}(A[x, x^{-1}])$, $\Delta_+ : U_{2i}^{\tilde{K}_0(A)}(A[x^{-1}]) \to V_{2i}(A[x, x^{-1}])$ for the exactness of

$$V_{2i}(A[x]) \xrightarrow{E_+} V_{2i}(A[x, x^{-1}]) \xrightarrow{\delta_+} U_{2i}^{\tilde{K}_0(A)}(A[x^{-1}]).$$

We shall do this using the following L-theoretic analogue of Lemma 2.3:

LEMMA 4.3. *Let (Q_x, φ) be a non-singular $\pm$form over $A[x, x^{-1}]$ such that*

$$\varphi = \mu + (x-1)\, \nu \in \mathrm{Hom}_{A[x, x^{-1}]}(Q_x, Q_x^*) \qquad (\mu, \nu \in \mathrm{Hom}_A(Q, Q^*)).$$

Then (Q_x, φ) is isomorphic to the sum

$$(R_x, \mu_R + (x-1)\, \nu_R) \oplus (S_x, \mu_S + (x-1)\, \nu_S)$$

of non-singular $\pm$forms over $A[x, x^{-1}]$ such that

$$(R[x], \mu_R + (x-1)\, \nu_R)$$

is a non-singular $\pm$form over $A[x]$, and

$$(S[x^{-1}], x^{-1}(\mu_S + (x-1)\, \nu_S))$$

is a non-singular $\pm$form over $A[x^{-1}]$.

Proof. The invertibility of

$$\varphi \pm \varphi^* = (\mu \pm \mu^*) + (x-1)\,(\nu \pm \nu^*) \in \mathrm{Hom}_{A[x, x^{-1}]}(Q_x, Q_x^*)$$

implies that

$$\begin{aligned} &\varepsilon(\varphi \pm \varphi^*) = \mu \pm \mu^* \in \mathrm{Hom}_A(Q, Q^*) \\ &(\mu \pm \mu^*)^{-1}(\varphi \pm \varphi^*) = 1 + (x-1)\, \gamma \in \mathrm{Hom}_{A[x, x^{-1}]}(Q_x, Q_x) \end{aligned}$$

are isomorphisms, where

$$\gamma = (\mu \pm \mu^*)^{-1}(\nu \pm \nu^*) \in \mathrm{Hom}_A(Q, Q).$$

Hence, by Lemma 2.3,

$$\gamma = \begin{pmatrix} \gamma_R & 0 \\ 0 & \gamma_S \end{pmatrix} : Q = R \oplus S \to Q = R \oplus S$$

with $\gamma_R \in \mathrm{Hom}_A(R, R)$, $1 - \gamma_S \in \mathrm{Hom}_A(S, S)$ nilpotent.

Adding on some $\mp$ hermitian products of type $\chi \mp \chi^* \in \mathrm{Hom}_A(Q, Q^*)$ to μ and ν if necessary, it may be assumed that $\mu(R)(S)=0$, $\nu(R)(S)=0$. Let

$$\mu=\begin{pmatrix}\mu_R & \mu_{RS}\\ 0 & \mu_S\end{pmatrix}: R\oplus S \to R^*\oplus S^*, \qquad \nu=\begin{pmatrix}\nu_R & \nu_{RS}\\ 0 & \nu_S\end{pmatrix}: R\oplus S \to R^*\oplus S^*$$

so that

$$\begin{pmatrix}\mu_R\pm\mu_R^* & \mu_{RS}\\ \pm\mu_{RS}^* & \mu_S\pm\mu_S^*\end{pmatrix}\begin{pmatrix}\gamma_R & 0\\ 0 & \gamma_S\end{pmatrix}=\begin{pmatrix}\nu_R\pm\nu_R^* & \nu_{RS}\\ \pm\nu_{RS}^* & \nu_S\pm\nu_S^*\end{pmatrix}: R\oplus S \to R^*\oplus S^*.$$

Working as in the calculation of $\delta_+ \Delta_+$ above,

$$\begin{aligned}\delta_+(Q_x, \varphi) &= ((Q^- \cap (\varphi\pm\varphi^*)^{-1}(Q^{*+}))[x^{-1}], [\varphi]_{-1} - x^{-1}[\varphi]_{-2})\\ &= ((1+(x-1)\gamma_S)^{-1}(S)[x^{-1}], [\mu_S+(x-1)\nu_S]_{-1} - x^{-1}[\mu_S+(x-1)\nu_S]_{-2})\\ &= (S[x^{-1}], x^{-1}(\mu_S+(x-1)\nu_S)) \in U_{2i}^{\tilde{K}_0(A)}(A[x^{-1}]).\end{aligned}$$

Thus $\varepsilon_-\delta_+(Q_x, \varphi)=(S, \mu_S)$ is a non-singular $\pm$ form over A, and hence so is (S, ν_S), because

$$(\nu_S\pm\nu_S^*)=(\mu_S\pm\mu_S^*)\gamma_S \in \mathrm{Hom}_A(S, S^*)$$

and $\gamma_S \in \mathrm{Hom}_A(S, S)$ is an isomorphism (being unipotent). Let

$$\begin{aligned} g &= \pm(\nu_S\pm\nu_S^*)^{-1}\nu_{RS}^* \in \mathrm{Hom}_A(R, S)\\ \mu' &= \begin{pmatrix}\mu_R'=\mu_R-g^*\mu_S g & 0\\ 0 & \mu_S\end{pmatrix}: R\oplus S\to R^*\oplus S^*\\ \nu' &= \begin{pmatrix}\nu_R'=\nu_R-g^*\nu_S g & 0\\ 0 & \nu_S\end{pmatrix}: R\oplus S\to R^*\oplus S^*.\end{aligned}$$

Now

$$\begin{aligned}(f, \chi) &= \left(\begin{pmatrix}1 & 0\\ g & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\ (\mu_S+(x-1)\nu_S)g & 0\end{pmatrix}\right)\\ &: (Q_x, \varphi)=(R_x\oplus S_x, \mu+(x-1)\nu) \to (Q_x, \varphi')=(R_x\oplus S_x, \mu'+(x-1)\nu')\end{aligned}$$

is an isomorphism of $\pm$ forms over $A[x, x^{-1}]$. It follows that

$$f^*(\varphi'\pm\varphi'^*)f=(\varphi\pm\varphi^*)\in \mathrm{Hom}_{A[x,x^{-1}]}(Q_x, Q_x^*)$$

and as f is defined over A

$$\begin{aligned} f^*(\mu'\pm\mu'^*)f &= (\mu\pm\mu^*)\in \mathrm{Hom}_A(Q, Q^*)\\ f^*(\nu'\pm\nu'^*)f &= (\nu\pm\nu^*)\in \mathrm{Hom}_A(Q, Q^*).\end{aligned}$$

Defining

$$\gamma'=(\mu'\pm\mu'^*)^{-1}(\nu'\pm\nu'^*)=\begin{pmatrix}\gamma'_R=(\mu'_R\pm\mu'^*_R)^{-1}(\nu_R\pm\nu^*_R) & 0\\ 0 & \gamma_S\end{pmatrix}:R\oplus S\to R\oplus S,$$

we have that

$$\gamma'=f\gamma f^{-1}=\begin{pmatrix}1 & 0\\ g & 1\end{pmatrix}\begin{pmatrix}\gamma_R & 0\\ 0 & \gamma_S\end{pmatrix}\begin{pmatrix}1 & 0\\ -g & 1\end{pmatrix}=\begin{pmatrix}\gamma_R & 0\\ g\gamma_R-\gamma_S g & \gamma_S\end{pmatrix}:R\oplus S\to R\oplus S.$$

Hence

$$\gamma'_R=\gamma_R\in \mathrm{Hom}_A(R,R)$$

is nilpotent, and $(R[x],\ \mu'_R+(x-1)\nu'_R)$ is a non-singular $\pm$form over $A[x]$. This completes the proof of Lemma 4.3. $\square$

Given $(Q_x,\varphi)\in V_{2i}(A[x,x^{-1}])$ it may be assumed, by Lemma 4.2, that $\varphi=\mu+(x-1)\nu\in\mathrm{Hom}_{A[x,x^{-1}]}(Q_x,Q_x^*)$ $(\mu,\nu\in\mathrm{Hom}_A(Q,Q^*))$. Applying the decomposition of Lemma 4.3,

$$\begin{aligned}(Q_x,\varphi)&=(R_x,\mu_R+(x-1)\nu_R)\oplus(S_x,\mu_S+(x-1)\nu_S\\ &=\{(R_x,\mu_R+(x-1)\nu_R)\oplus(S_x,\mu_S)\}\oplus\{(S_x,\mu_S+(x-1)\nu_S)\\ &\qquad\oplus(S_x,-\mu_S)\oplus H_\pm(-S_x)\}\\ &=\bar{E}_+((R[x],\mu_R+(x-1)\nu_R)\oplus(S[x],\mu_S))\\ &\qquad\oplus\Delta_+(S[x^{-1}],x^{-1}(\mu_S+(x-1)\nu_S))\in V_{2i}(A[x,x^{-1}]).\end{aligned}$$

As pointed out above, this suffices to prove the exactness of

$$V_{2i}(A[x])\xrightarrow{E_+}V_{2i}(A[x,x^{-1}])\xrightarrow{\delta_+}U_{2i}^{\tilde{K}_0(A)}(A[x^{-1}]).$$

Define next a morphism

$$\begin{aligned}E_+:V_{2i}(A[x,x^{-1}])&\to V_{2i}(A[x]);\\ (Q_x,\varphi)\mapsto&((\varphi\pm\varphi^*)^{-1}(x^{N_1+1}Q^{*-})\cap x^{-N_1}Q^{*+})[x],[\varphi]_0-x([\varphi]_1)\\ &\oplus((x^NQ^-\cap(\varphi\pm\varphi^*)^{-1}(x^{-N}Q^{*+}))[x],[\varphi]_{-1}-[\varphi]_{-2})\end{aligned}$$

for $N, N_1\geqslant 0$ so large that

$$(\varphi\pm\varphi^*)(Q)\subseteq\sum_{j=-2N}^{2N_1+1}x^jQ^*$$

with $Q\in|\mathbf{P}(A)|$ f.g. free. The verification that E_+ is well-defined is by analogy with that for δ_+. Moreover, if

$$(Q_x,\varphi)=(R_x,\mu_R+(x-1)\nu_R)\oplus(S_x,\mu_S+(x-1)\nu_S)$$

(as in Lemma 4.3), then

$$E_+(Q_x, \varphi) = (R[x], \mu_R + (x-1)\nu_R) \oplus (S[x], \mu_S) \in V_{2i}(A[x]),$$

so that the composites

$$U_{2i}^{\tilde{K}_0(A)}(A[x^{-1}]) \xrightarrow{\Delta_+} V_{2i}(A[x, x^{-1}]) \xrightarrow{E_+} V_{2i}(A[x])$$
$$V_{2i}(A[x]) \xrightarrow{E_+} V_{2i}(A[x, x^{-1}]) \xrightarrow{E_+} V_{2i}(A[x])$$

are 0, 1 respectively. Thus

$$V_{2i}(A[x]) \underset{E_+}{\overset{E_+}{\rightleftarrows}} V_{2i}(A[x, x^{-1}]) \underset{\Delta_+}{\overset{\delta_+}{\rightleftarrows}} U_{2i}^{\tilde{K}_0(A)}(A[x^{-1}])$$

defines a direct sum system, and we can identify

$$L_+V_{2i}(A) = U_{2i}^{\tilde{K}_0(A)}(A[x^{-1}]).$$

Similarly, replacing x with x^{-1}, there is defined a direct sum system

$$V_{2i}(A[x^{-1}]) \underset{E_-}{\overset{E_-}{\rightleftarrows}} V_{2i}(A[x, x^{-1}]) \underset{\Delta_-}{\overset{\delta_-}{\rightleftarrows}} U_{2i}^{\tilde{K}_0(A)}(A[x]),$$

allowing the identification

$$L_-V_{2i}(A) = U_{2i}^{\tilde{K}_0(A)}(A[x]).$$

The proof of Lemma 4.2 shows that every element $(Q[x^{-1}], \varphi) \in V_{2i}(A[x^{-1}])$ has a representative with

$$\varphi = \varphi_0 + x^{-1}\varphi_{-1} \in \mathrm{Hom}_{A[x^{-1}]}(Q[x^{-1}], Q^*[x^{-1}]) \quad (\varphi_0, \varphi_{-1} \in \mathrm{Hom}_A(Q, Q^*)).$$

The composite

$$V_{2i}(A[x^{-1}]) \xrightarrow{E_-} V_{2i}(A[x, x^{-1}]) \xrightarrow{E_+} V_{2i}(A[x])$$

sends such a representative to

$$\begin{aligned} E_+E_-(Q[x^{-1}], \varphi) &= (((\varphi \pm \varphi^*)^{-1}(xQ^{*-}) \cap Q^+)[x], [\varphi]_0 - [\varphi]_1) \\ &\quad \oplus ((xQ^- \cap (\varphi \pm \varphi^*)^{-1}(x^{-1}Q^*))[x], [\varphi]_{-1} - [\varphi]_{-2}) \\ &= (Q[x], \varphi_0) \oplus ((\varphi \pm \varphi^*)^{-1}(Q^* \oplus x^{-1}Q^*)[x], [\varphi]_{-1} \\ &\qquad - [\varphi]_{-2}) \in V_{2i}(A[x, x^{-1}]). \end{aligned}$$

The A-module isomorphism

$$Q \oplus Q \to (\varphi \pm \varphi^*)^{-1}(Q^* \oplus x^{-1}Q^*);$$
$$(y, y') \mapsto (\varphi \pm \varphi^*)^{-1}((\varphi_0 \pm \varphi_0^*)y, x^{-1}(((\varphi_0 \pm \varphi_0^*) + \varphi_{-1} \pm \varphi_{-1}^*))y + (\varphi_0 \pm \varphi_0^*)y'))$$

defines an isomorphism of $\pm$ forms over A

$$(Q\oplus Q, \begin{pmatrix}\varphi_0+\varphi_{-1} & 0 \\ 0 & -\varphi_0\end{pmatrix})\to((\varphi\pm\varphi^*)^{-1}(Q^*\oplus x^{-1}Q^*), [\varphi]_{-1}-[\varphi]_{-2}).$$

Therefore

$$\begin{aligned}E_+\bar{E}_-(Q[x^{-1}], \varphi_0+x^{-1}\varphi_{-1})&=(Q[x], \varphi_0+\varphi_{-1})\oplus(Q[x]\oplus Q[x], \varphi_0\oplus-\varphi_0)\\ &=(Q[x], \varphi_0+\varphi_{-1})\\ &=\bar{\varepsilon}_+\varepsilon_-(Q[x^{-1}], \varphi_0+x^{-1}\varphi_{-1})\in V_{2i}(A[x]),\end{aligned}$$

and the square

$$\begin{array}{ccc} V_{2i}(A[x^{-1}]) & \xrightarrow{E_-} & V_{2i}(A[x, x^{-1}]) \\ \varepsilon_-\downarrow & & \downarrow E_+ \\ V_{2i}(A) & \xrightarrow{\bar{\varepsilon}_+} & V_{2i}(A[x]) \end{array}$$

commutes. Similarly, we can verify that the square

$$\begin{array}{ccc} U_{2i}^{\tilde{K}_0(A)}(A[x^{-1}]) & \xrightarrow{\eta_+} & U_{2i}(A) \\ \Delta_+\downarrow & & \downarrow\bar{\eta}_- \\ V_{2i}(A[x, x^{-1}]) & \xrightarrow[\delta_-]{} & U_{2i}^{\tilde{K}_0(A)}(A[x]) \end{array}$$

commutes, where

$$\eta_\pm: U_{2i}^{\tilde{K}_0(A)}(A[x^{\mp1}])\to U_{2i}(A), \qquad \bar{\eta}_\pm: U_{2i}(A)\to U_{2i}^{\tilde{K}_0(A)}(A[x^{\mp1}])$$

are the morphisms induced by

$$\eta_\pm: A[x^{\mp1}]\to A;\ \sum_{j=0}^{\infty} x^{\mp j}a_j\mapsto a_0, \qquad \bar{\varepsilon}_\mp: A\to A[x^{\mp1}]$$

respectively (so that $\eta_\pm\bar{\eta}_\pm=1$). For

$$\begin{aligned}\delta_-\Delta_+(Q[x^{-1}], \varphi=\varphi_0+x^{-1}\varphi_{-1}) &\\ &=\delta_-((Q_x, x\varphi)\oplus(Q_x, -(\varphi_0+\varphi_{-1}))\oplus H_\pm(-Q_x))\\ &=((x^{-1}Q^+\cap(\varphi\pm\varphi^*)^{-1}(Q^{*-}))[x], [x\varphi]_{-1}-x[x\varphi]_0)\\ &=((x^{-1}Q)[x], [x\varphi]_{-1})=(Q[x], \varphi_0)\\ &=\bar{\eta}_-\eta_+(Q[x^{-1}], \varphi)\in U_{2i}^{\tilde{K}_0(A)}(A[x]).\end{aligned}$$

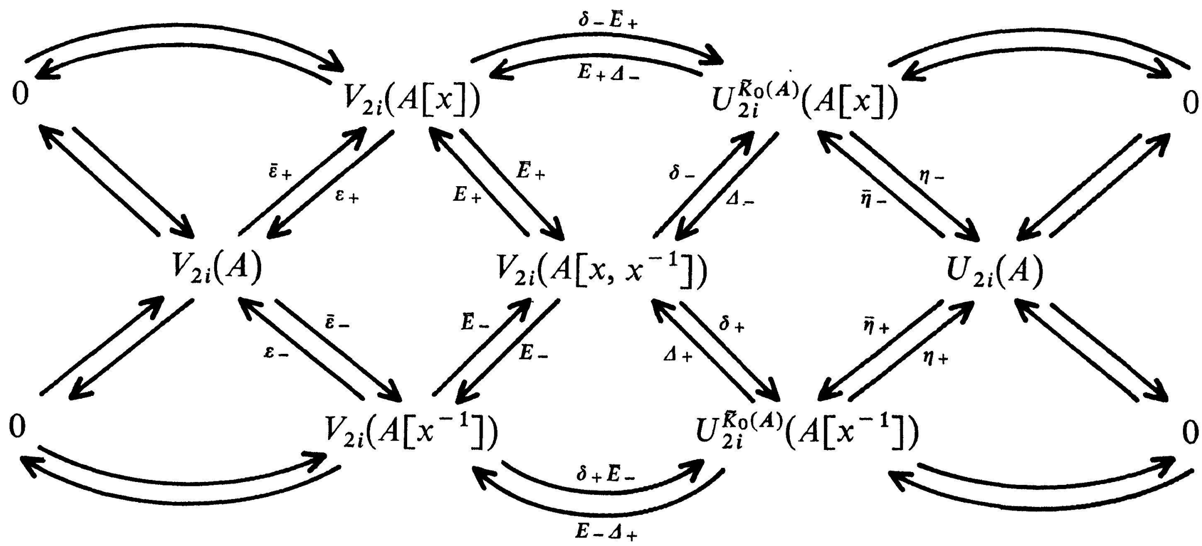
The conditions of Lemma 1.1 are now satisfied, and so

V_{2i}: (rings with involution) $\to$ (abelian groups)

is a contracted functor, with

$$L_\pm V_{2i}(A)=U_{2i}^{\tilde{K}_0(A)}(A[x^{\mp1}]), \qquad LV_{2i}(A)=U_{2i}(A)$$

(up to natural isomorphisms), and the diagram

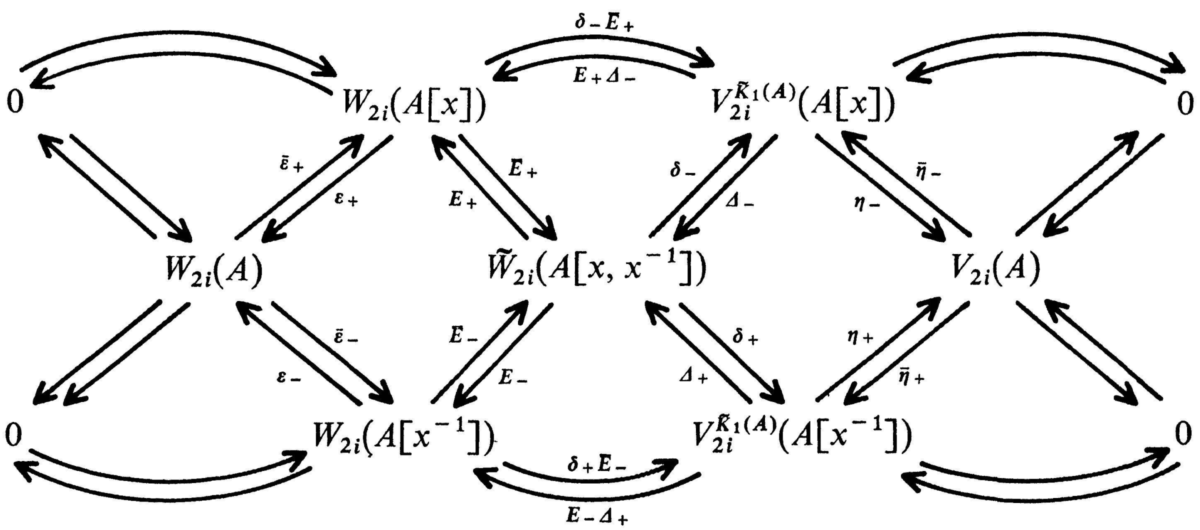


incorporates two commutative exact braids.

Let $S_0 \subseteq \tilde{K}_1(A[x, x^{-1}])$ be the infinite cyclic subgroup generated by $\bar{B}([A]) = \tau(x: A_x \to A_x)$, and define

$$\tilde{W}_n(A[x, x^{-1}]) = V_n^{S_0}(A[x, x^{-1}]) \quad (n \,(\mathrm{mod}\, 4)).$$

Working as for $V_{2i}(A[x, x^{-1}])$, it is possible to define morphisms to fit into a diagram



(with $E_+\bar{E}_+ = 1$ etc.) incorporating two commutative exact braids. For example,

$$\delta_+ : \tilde{W}_{2i}(A[x, x^{-1}]) \to V_{2i}^{\tilde{K}_1(A)}(A[x^{-1}]);\ (\underset{\sim}{Q}_x, \varphi) \mapsto (\underset{\sim}{P}[x^{-1}], [\varphi]_{-1} - x^{-1}[\varphi]_{-2})$$

$$E_+ : \tilde{W}_{2i}(A[x, x^{-1}]) \to W_{2i}(A[x]);$$

$$(Q_x, \varphi) \mapsto (\underset{\sim}{P}_1[x], [\varphi]_0 - x[\varphi]_1) \oplus (\underset{\sim}{P}[x], [\varphi]_{-1} - [\varphi]_{-2})$$

for any A-base $\underset{\sim}{P}$ of $P=x^NQ^-\cap(\varphi\pm\varphi^*)^{-1}(x^{-N}Q^{*+})$ (which is free for sufficiently large $N\geqslant 0$, as $\tau(\underset{\sim}{Q}_x,\varphi)\in S_0$ and $[P]=B\tau(\underset{\sim}{Q}_x,\varphi)=0\in\tilde{K}_0(A)$) with

$$\underset{\sim}{P}_1=(\varphi\pm\varphi^*)^{-1}(x^N\underset{\sim}{Q}^*)\oplus(\varphi\pm\varphi^*)^{-1}(\underset{\sim}{P}^*)$$

the corresponding A-base of $P_1=(\varphi\pm\varphi^*)^{-1}(x^{N+1}Q^{*-})\cap x^{-N}Q^+$, for N so large that

$$(\varphi\pm\varphi^*)(Q)\subseteq\sum_{j=-2N}^{2N+1}x^jQ^*.$$

Also, let

$$\Delta_+:V_{2i}^{\tilde{K}_1(A)}(A[x^{-1}])\to\tilde{W}_{2i}(A[x,x^{-1}]);(Q[x^{-1}],\varphi)\mapsto(\underset{\sim}{Q}_x,x\varphi)\oplus(\underset{\approx}{Q}_x,-\bar{\varepsilon}\varepsilon_-\varphi)$$

where $\underset{\approx}{Q}=(\varepsilon_-(\varphi\pm\varphi^*))^{-1}(\underset{\sim}{Q}^*)$.

Given an invertible indeterminate z over A commuting with every element of A define A_z as $A[z,z^{-1}]$ but with involution by $\bar{z}=z^{-1}$. Similarly, define $A[x^{\pm1}]_z$, $A[x,x^{-1}]_z$, and identify

$$A[x^{\pm1}]_z=A_z[x^{\pm1}],\quad A[x,x^{-1}]_z=A_z[x,x^{-1}].$$

Let $S_0'\subseteq\tilde{K}_1(A_z)$ be the infinite cyclic subgroup generated by $\tau(z:A_z\to A_z)$ and define

$$\begin{aligned}&\tilde{W}_n(A_z)=V_n^{S'_0}(A_z)\\&\tilde{W}_n(A[x^{\pm1}]_z)=V_n^{\bar{\varepsilon}_\pm(x)S'_0}(A[x^{\pm1}]_z)\\&\tilde{\tilde{W}}_n(A[x,x^{-1}]_z)=V_n^{\bar{\varepsilon}(z)S_0\oplus\bar{\varepsilon}(x)S'_0}(A[x,x^{-1}]_z)\end{aligned}$$

for $n\,(\mathrm{mod}\,4)$. By analogy with $\tilde{W}_{2i}(A[x,x^{-1}])$, $\tilde{\tilde{W}}_{2i}(A[x,x^{-1}]_z$ fits into a diagram incorporating two commutative exact braids (where $A_z=A[z,z^{-1}]$, with $\bar{z}=z^{-1}$).

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr, shift left, bend left] & & \tilde{W}_{2i}(A[x]_z) \arrow[ll, shift left, bend right] \arrow[dl, shift left, "\varepsilon_+"] \arrow[dr, shift left, "E_+"] \arrow[rr, shift right, bend right, "\delta_-E_+"'] & & V_{2i}^{\tilde{K}_1(A_z)}(A[x]_z) \arrow[ll, shift right, bend right, "E_+\Delta_-"'] \arrow[dl, shift left, "\delta_-"] \arrow[dr, shift left, "\eta_-"] \arrow[rr, shift left, bend left] & & 0 \arrow[ll, shift left, bend right] \arrow[dl, shift left] \\
 & \tilde{W}_{2i}(A_z) \arrow[ul, shift left] \arrow[ur, shift left, "\bar{\varepsilon}_+"] \arrow[dl, shift left] \arrow[dr, shift left, "\bar{\varepsilon}_-"] & & \tilde{\tilde{W}}_{2i}(A[x,x^{-1}]_z) \arrow[ul, shift left, "E_+"] \arrow[ur, shift left, "\Delta_-"] \arrow[dl, shift left, "E_-"] \arrow[dr, shift left, "\Delta_+"] & & V_{2i}(A_z) \arrow[ul, shift left, "\bar{\eta}_-"] \arrow[ur, shift left] \arrow[dl, shift left, "\bar{\eta}_+"] \arrow[dr, shift left] & \\
0 \arrow[ur, shift left] \arrow[rr, shift left, bend right] & & \tilde{W}_{2i}(A[x^{-1}]_z) \arrow[ll, shift left, bend left] \arrow[ul, shift left, "\varepsilon_-"] \arrow[ur, shift left, "E_-"] \arrow[rr, shift left, bend right, "E_-\Delta_+"] & & V_{2i}^{\tilde{K}_1(A_z)}(A[x^{-1}]_z) \arrow[ll, shift left, bend left, "\delta_+E_-"] \arrow[ul, shift left, "\delta_+"] \arrow[ur, shift left, "\eta_+"] \arrow[rr, shift left, bend right] & & 0 \arrow[ll, shift left, bend left] \arrow[ul, shift left]
\end{tikzcd}$$

We can now apply the decompositions

$$\begin{aligned}
\tilde{W}_{2i}(A_z)&=\bar{\varepsilon}(z)\,W_{2i}(A)\oplus\bar{B}(z)\,V_{2i-1}(A)\\
\tilde{W}_{2i}(A[x]_z)&=\bar{\varepsilon}(z)\,W_{2i}(A[x])\oplus\bar{B}(z)\,V_{2i-1}(A[x])\\
\approx{W}_{2i}(A[x,x^{-1}]_z)&=\bar{\varepsilon}(z)\,\tilde{W}_{2i}(A[x,x^{-1}])\oplus\bar{B}(z)\,V_{2i-1}(A[x,x^{-1}])\\
V_{2i}^{\tilde{K}_1(A_z)}(A[x]_z)&=\bar{\varepsilon}(z)\,V_{2i}^{\tilde{K}_1(A)}(A)\oplus\bar{B}(z)\,U_{2i}^{\tilde{K}_0(A)}(A)\\
V_{2i}(A_z)&=\bar{\varepsilon}(z)\,V_{2i}(A)\oplus\bar{B}(z)\,U_{2i-1}(A)
\end{aligned}$$

given by Theorem 1.1 of Part II (and extended to the intermediate L-groups in Part III). The above diagram splits naturally (via $\bar{\varepsilon}(z),\bar{B}(z)$) into two similar ones: the diagram for $\tilde{W}_{2i}(A[x,x^{-1}])$ and the diagram

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr, bend left] \arrow[dr, shift left] & & V_{2i}(A[x]) \arrow[ll, bend right] \arrow[rr, bend left, "E_+\varDelta_-"] \arrow[dl, shift left, "\varepsilon_+"] \arrow[dr, shift left, "E_+"] & & U_{2i-1}^{\tilde{K}_0(A)}(A[x]) \arrow[ll, bend left, "\delta_-E_+"] \arrow[rr, bend left] \arrow[dl, shift left, "\delta_-"] \arrow[dr, shift left, "\eta_-"] & & 0 \arrow[ll, bend right] \arrow[dl, shift left] \\
& V_{2i-1}(A) \arrow[ul, shift left] \arrow[ur, shift left, "\bar{\varepsilon}_+"] \arrow[dl, shift left] \arrow[dr, shift left, "\bar{\varepsilon}_-"] & & V_{2i-1}(A[x,x^{-1}]) \arrow[ul, shift left, "E_+"] \arrow[ur, shift left, "\varDelta_-"] \arrow[dl, shift left, "E_-"] \arrow[dr, shift left, "\delta_+"] & & U_{2i-1}(A) \arrow[ul, shift left, "\bar{\eta}_-"] \arrow[ur, shift left] \arrow[dl, shift left, "\bar{\eta}_+"] \arrow[dr, shift left] & \\
0 \arrow[rr, bend right] \arrow[ur, shift left] & & V_{2i-1}(A[x^{-1}]) \arrow[ll, bend left] \arrow[rr, bend right, "\delta_+E_-"'] \arrow[ul, shift left, "\varepsilon_-"] \arrow[ur, shift left, "E_-"] & & U_{2i-1}^{\tilde{K}_0(A)}(A[x^{-1}]) \arrow[ll, bend right, "E_-\varDelta_+"'] \arrow[rr, bend right] \arrow[ul, shift left, "\varDelta_+"] \arrow[ur, shift left, "\eta_+"] & & 0 \arrow[ll, bend left] \arrow[ul, shift left]
\end{tikzcd}$$

where

$$\begin{aligned}
E_+:V_{2i-1}(A[x,x^{-1}])&\xrightarrow{\bar{B}(z)}\approx{W}_{2i}(A[x,x^{-1}]_z)\xrightarrow{E_+}\tilde{W}_{2i}(A[x]_z)\xrightarrow{B(z)}V_{2i-1}(A[x])\\
\delta_+:V_{2i-1}(A[x,x^{-1}])&\xrightarrow{\bar{B}(z)}\approx{W}_{2i}(A[x,x^{-1}]_z)\\
&\xrightarrow{\delta_+}V_{2i}^{\tilde{K}_1(A_z)}(A[x^{-1}]_z)\xrightarrow{B(z)}U_{2i-1}^{\tilde{K}_0(A)}(A[x^{-1}])\\
\varDelta_+:U_{2i-1}^{\tilde{K}_0(A)}(A[x^{-1}])&\xrightarrow{\bar{B}(z)}V_{2i}^{\tilde{K}_1(A_z)}(A[x^{-1}]_z)\\
&\xrightarrow{\varDelta_+}\approx{W}_{2i}(A[x,x^{-1}]_z)\xrightarrow{B(z)}V_{2i-1}(A[x,x^{-1}])
\end{aligned}$$

(and similarly for E_-, δ_-, $\varDelta_-$). Thus the conditions of Lemma 1.1 are also satisfied in the odd-dimensional case, and

V_{2i-1}: (rings with involution) $\rightarrow$ (abelian groups) is a contracted functor, with identifications

$$L_\pm V_{2i-1}(A)=U_{2i-1}^{\tilde{K}_0(A)}(A[x^{\mp 1}]),\qquad LV_{2i-1}(A)=U_{2i-1}(A).$$

This completes the proof of Theorem 4.1 □

The groups

$$\mathrm{Nil}_\pm(A)=\ker(\varepsilon_\pm:K_1(A[x^{\pm 1}])\to K_1(A))$$

are such that

$$K_1(A[x^{\pm 1}])=\bar{\varepsilon}_\pm K_1(A)\oplus\mathrm{Nil}_\pm(A)$$
$$K_1(A[x,x^{-1}])=\bar{\varepsilon}K_1(A)\oplus\bar{E}_+\mathrm{Nil}_+(A)\oplus\bar{E}_-\mathrm{Nil}_-(A)\oplus\bar{B}K_0(A),$$

fitting into direct sum systems

$$\mathrm{Nil}_\pm(A)\ \underset{E_\pm\Delta_\pm}{\overset{\delta_\pm E_\pm}{\rightleftarrows}}\ K_0\mathrm{N}(A)\underset{\bar{\eta}_\pm}{\overset{\eta_\pm}{\rightleftarrows}}K_0(A)$$

(by Theorem 2.1).

Given $*$-invariant subgroups $S_\pm\subseteq\mathrm{Nil}_\pm(A)$, define

$$N_\pm V_n^{S_\pm}(A)=\ker(\varepsilon_\pm:V_n^{\bar{\varepsilon}_\pm\tilde{K}_1(A)\oplus S_\pm}(A[x^{\pm 1}])\to V_n(A))\qquad(n\,(\mathrm{mod}\,4))$$

writing $\begin{cases}N_\pm V_n(A)\\ N_\pm W_n(A)\end{cases}$ for $\begin{cases}N_\pm V_n^{\mathrm{Nil}_\pm(A)}(A)\\ N_\pm V_n^{\{0\}}(A)\end{cases}$.

COROLLARY 4.4. *Given $*$-invariant subgroups*

$$R\subseteq\tilde{K}_1(A),\qquad S_\pm\subseteq\mathrm{Nil}_\pm(A),\qquad \tilde{T}\subseteq\tilde{K}_0(A)$$

there are direct sum decompositions

$$V_n^{\bar{\varepsilon}_\pm R\oplus S_\pm}(A[x^{\pm 1}])=\bar{\varepsilon}_\pm V_n^R(A)\oplus N_\pm V_n^{S_\pm}(A)$$
$$U_n^{\bar{\varepsilon}_\pm\tilde{T}}(A[x^{\pm 1}])=\bar{\varepsilon}_\pm U_n^{\tilde{T}}(A)\oplus N_\pm V_n(A)$$
$$V_n^Q(A[x,x^{-1}])=\bar{\varepsilon}V_n^R(A)\oplus\bar{E}_+N_+V_n^{S_+}(A)\oplus\bar{E}_-N_-V_n^{S_-}(A)\oplus\bar{B}U_n^{\tilde{T}}(A)$$

for $n\,(\mathrm{mod}\,4)$, where

$$Q=\bar{\varepsilon}R\oplus\bar{E}_+S_+\oplus\bar{E}_-S_-\oplus\bar{B}T\subseteq\tilde{K}_1(A[x,x^{-1}])$$
$$=\bar{\varepsilon}\tilde{K}_1(A)\oplus\bar{E}_+\mathrm{Nil}_+(A)\oplus\bar{E}_-\mathrm{Nil}_-(A)\oplus\bar{B}K_0(A)$$

with $T\subseteq K_0(A)$ the preimage of $\tilde{T}$ under the natural projection $K_0(A)\to\tilde{K}_0(A)$.

Proof. The forgetful map

$$V_n(A[x^{\pm 1}])\to U_n^{\bar{\varepsilon}_\pm\tilde{T}}(A[x^{\pm 1}])$$

fits into the exact sequence of Theorem 2.3 of Part III, which splits, via $\bar{\varepsilon}_{\pm}$, $\varepsilon_{\pm}$ into two exact sequences

$$\begin{array}{ccccccccc}
\to & 0 & \to & N_{\pm}V_n(A) & \to & N_{\pm}V_n(A) & \to & 0 & \to \\
& \upharpoonleft\downharpoonright & & \upharpoonleft\downharpoonright & & \upharpoonleft\downharpoonright & & \upharpoonleft\downharpoonright & \\
\to & H^{n+1}(\bar{\varepsilon}_{\pm}\tilde{T}) & \to & V_n(A[x^{\pm 1}]) & \to & U_n^{\bar{\varepsilon}_{\pm}T}(A[x^{\pm 1}]) & \to & H^n(\bar{\varepsilon}_{\pm}\tilde{T}) & \to \\
& {\scriptstyle\bar{\varepsilon}_{\pm}}\upharpoonleft\downharpoonright{\scriptstyle\varepsilon_{\pm}} & & {\scriptstyle\bar{\varepsilon}_{\pm}}\upharpoonleft\downharpoonright{\scriptstyle\varepsilon_{\pm}} & & {\scriptstyle\bar{\varepsilon}_{\pm}}\upharpoonleft\downharpoonright{\scriptstyle\varepsilon_{\pm}} & & {\scriptstyle\bar{\varepsilon}_{\pm}}\upharpoonleft\downharpoonright{\scriptstyle\varepsilon_{\pm}} & \\
\to & H^{n+1}(\tilde{T}) & \to & V_n(A) & \to & U_n^{T}(A) & \to & H^n(\tilde{T}) & \to .
\end{array}$$

Hence $N_{\pm}V_n(A) \subseteq V_n(A[x^{\pm 1}])$ is mapped isomorphically to $\ker(\varepsilon_{\pm}: U_n^{\bar{\varepsilon}_{\pm}T}(A[x^{\pm 1}]) \to U_n^{T}(A))$ and so (up to isomorphism)

$$U_n^{\bar{\varepsilon}_{\pm}T}(A[x^{\pm 1}]) = \bar{\varepsilon}_{\pm}U_n^{T}(A) \oplus N_{\pm}V_n(A).$$

In particular,

$$U_n^{\tilde{K}_0(A)}(A[x^{\pm 1}]) = \bar{\varepsilon}_{\pm}U_n(A) \oplus N_{\pm}V_n(A),$$
$$V_n(A[x^{\pm 1}]) = \bar{\varepsilon}\pm V_n(A) \oplus N\pm V_n(A).$$

It now follows from Theorem 4.1 that

$$V_n(A[x, x^{-1}]) = \bar{\varepsilon}V_n(A) \oplus \bar{E}_+N_+V_n(A) \oplus \bar{E}_-N_-V_n(A) \oplus \bar{B}U_n(A).$$

The expressions for $V_n^{\bar{\varepsilon}_{\pm}R\oplus S_{\pm}}(A[x^{\pm 1}])$, $V_n^{Q}(A[x, x^{-1}])$ may be deduced from those for $V_n(A[x^{\pm 1}])$, $V_n(A[x, x^{-1}])$, working as for $U_n^{\bar{\varepsilon}_{\pm}T}(A[x^{\pm 1}])$ above. (In particular, for $R=0$, $S_+=0$, $S_-=0$, $\tilde{T}=0$ we have

$$Q = S_0 \subseteq \tilde{K}_1(A[x, x^{-1}])$$

and

$$W_n(A[x^{\pm 1}]) = \bar{\varepsilon}_{\pm}W_n(A) \oplus N_{\pm}W_n(A),$$
$$\tilde{W}_n(A[x, x^{-1}]) = \bar{\varepsilon}W_n(A) \oplus \bar{E}_+N_+W_n(A) \oplus \bar{E}_-N_-W_n(A) \oplus \bar{B}V_n(A).) \quad \square$$

In §4 of Part II there were defined lower L-theories, functors

$$L_n^{(m)}: (\text{rings with involution}) \to (\text{abelian groups})$$

for $m<0$, $n \pmod 4$ by

$$L_n^{(m)}(A) = \ker(\varepsilon: L_{n+1}^{(m+1)}(A_z) \to L_{n+1}^{(m+1)}(A))$$

with $L_n^{(0)}(A) = U_n(A)$. By convention, $L_n^{(1)}(A) = V_n(A)$.

COROLLARY 4.5. *The lower L-theories $L_n^{(m)}$ coincide (up to natural isomorphism)* *with the functors LV_n, L^2V_n,... derived from V_n, with*

$$L_n^{(m)}(A)=L^{1-m}V_n(A) \quad (m\leqslant 0,\ n(\mathrm{mod}\,4)).$$

Proof. By Theorem 4.1,

$$LV_n(A)=U_n(A)=L_n^{(0)}(A).$$

Assume inductively that

$$L_n^{(p)}(A)=L^{1-p}V_n(A) \quad (n(\mathrm{mod}\,4))$$

for $0\geqslant p>m$, for some $m\leqslant -1$. Then

$$\begin{aligned}L_n^{(m)}(A)&=\ker(\varepsilon:L_{n+1}^{(m+1)}(A_z)\to L_{n+1}^{(m+1)}(A))\\&=\ker(\varepsilon:L^{-m}V_{n+1}(A_z)\to L^{-m}V_{n+1}(A))\\&=L(\ker(\varepsilon:L^{-m-1}V_{n+1}(A_z)\to L^{-m-1}V_{n+1}(A))\\&=L(\ker(\varepsilon:L_{n+1}^{(m+2)}(A_z)\to L_{n+1}^{(m+2)}(A)))\\&=LL_n^{(m+1)}(A)\\&=LL^{-m}V_n(A)=L^{1-m}V_n(A)\end{aligned}$$

giving the induction step. □

Given a functor

$$F:(\text{rings with involution})\to(\text{abelian groups})$$

define

$$N_{\pm}F(A)=\ker(\varepsilon_{\pm}:F(A[x^{\pm 1}])\to F(A)).$$

(By Corollary 4.4, the previous definitions of $N_{\pm}V_n(A)$, $N_{\pm}W_n(A)$ agree with this, up to natural isomorphism).

By analogy with the first part of Corollary 7.6 of Chapter XII of [1] we have

COROLLARY 4.6. *Let $x_1, x_2,..., x_p$ be independent commuting indeterminates over A, with $\bar{x}_j=x_j$ $(1\leqslant j\leqslant p)$. Then*

$$\begin{aligned}&L_n^{(m)}(A[x_1, x_2,..., x_p])=(1\oplus N_+)^p\,L_n^{(m)}(A)\\&L_n^{(m)}(A[x_1, x_1^{-1}, x_2, x_2^{-1},..., x_p, x_p^{-1}])=(1\oplus N_+\oplus N_-\oplus L)^p\,L_n^{(m)}(A)\end{aligned}$$

up to natural isomorphism, for $m\leqslant 1$, $n(\mathrm{mod}\,4)$, $p\geqslant 1$. □

REFERENCES

[1] BASS H., *Algebraic K-theory*, Benjamin (1968).
[2] BASS H., HELLER A. and SWAN R. G., *The Whitehead group of a polynomial extension*, Publ. Math. IHES no. 22 (1964).
[3] HIGMAN, G., *The units of group-rings*, Proc. London Math. Soc. (2) *46* (1940), 231–48.
[4] KAROUBI, M., *Localisation de formes quadratiques*, (preprint).
[5] RANICKI, A. A., *Algebraic L-theory, I: Foundations*, Proc. London Math. Soc. (3) *27* (1973), 101–25.
[6] ——, *Algebraic L-theory, II: Laurent extensions*, Proc. London Math. Soc. (3) *27* (1973), 126–58.
[7] ——, *Algebraic L-theory, III: Twisted Laurent extensions*, in *Algebraic K-theory III*, Springer Lecture. Notes No. 343 (1973), 412–463.
[8] SWAN, R. G., *Algebraic K-theory*, Springer Lecture Notes No. 76 (1968).

Trinity College, Cambridge, England

Received August 17, 1973