Partitions of Graphs into Coverings and Hypergraphs into Transversals

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Abstract

A covering of a multigraph G is a subset of edges which meet all vertices of G. Partitions of the edges of G into coverings $C_1, C_2, \ldots, C_k$ are considered.

In particular we examine how close the cardinalities of these coverings may be. A result concerning matchings is extended to the decomposition into coverings. Finally these considerations are generalized to the decompositions of the vertices of a hypergraph into transversals (a transversal is a set of vertices meeting all edges of the hypergraph).

Introduction

In this note a multigraph $G=(X, U)$ consists of a finite non-empty set X of vertices and a set U of edges.

A *covering* C in G is a subset of edges such that each vertex of G is adjacent to at least one edge of C. Given a multigraph G we will consider partitions of the edges of G into coverings $C_1, C_2, \ldots, C_k$. (Such a partition exists only if each vertex x has degree at least k, i.e. if any x is adjacent to at least k edges). The cardinality of C_i will be denoted by c_i.

We will first examine the following question: given a multigraph G when does a given finite sequence $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_k \geqslant 0$ represent the cardinalities of a partition of U into coverings?

A similar problem concerning partitions into matchings (i.e. subsets of nonadjacent edges) has been solved in [1] and [2].

In §2 the problem is formulated in terms of hypergraphs; we now have partitions of the vertices into transversals and we examine in particular how close the cardinalities of transversals in a partition can be.

All notions not defined here can be found in [3].

§1. Partitions into Coverings

Let us call *covering index* $i(G)$ of G the largest k for which there exists a partition of the edges of G into k coverings $C_1, C_2, \ldots, C_k$.

To each such partition we associate a sequence $c_1, c_2, \ldots, c_k$ where c_i is the cardinality of C_i and where the indices are chosen in such a way that $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_k$.

We may now formulate a theorem which is quite similar to the matching case.

THEOREM 1. *If the sequence* $c_1, c_2, \ldots, c_k$ *corresponds to a partition of the edges of G into coverings, then any sequence* $\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_k$ *with*

$$\bar{c}_1 \leqslant \bar{c}_2 \leqslant \cdots \leqslant \bar{c}_k$$

$$\sum_{i=1}^{r} \bar{c}_i \geqslant \sum_{i=1}^{r} c_i \qquad r=1, \ldots, k$$

$$\sum_{i=1}^{k} \bar{c}_i = \sum_{i=1}^{k} c_i$$

corresponds also to a partition of the edges of G into coverings.

Proof. We only have to prove that any couple of coverings C_i, C_j with $c_i - c_j = K \geqslant 2$ may be replaced by two disjoint coverings $\bar{C}_i$, $\bar{C}_j$ with $c_i - c_j = K-2$ and $\bar{C}_i \cup \bar{C}_j = C_i \cup C_j$; then by repeated transformations of this type we will obtain any sequence $\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_k$ satisfying the above condition.

Let G_{ij} be the graph formed by the edges of $C_i \cup C_j$; in G_{ij} we construct an alternating chain (i.e. its edges belong alternately to C_i and C_j) and extend it as far as possible (it may happen that this chain goes through the same vertex several times). We remove it from G_{ij} and we construct another alternating chain which is as long as possible in the remaining graph. We remove it and continue until we obtain an alternating chain Q starting and ending with edges in C_i (such a chain must exist since $c_i - c_j = K \geqslant 2$). We interchange the edges of $Q \cap C_i$ and $Q \cap C_j$ and obtain two subsets $\bar{C}_i$, $\bar{C}_j$ with $\bar{c}_i - \bar{c}_j = K-2$. $\bar{C}_i$ and $\bar{C}_j$ are still coverings: at each endpoint of Q there were (before the interchange) more edges of C_i than of C_j (i.e. at least 2 edges of C_i), so after the interchange $\bar{C}_i$ and $\bar{C}_j$ have at least one edge at each endpoint of Q as well as at any other vertex of G.

Now if we are interested in knowing how close the cardinalities of coverings in a partition can be, we have the following immediate consequence of the theorem.

COROLLARY. *For any* $k \leqslant i(G)$, *there exists a partition of the edges of G into coverings* $C_1, C_2, \ldots, C_k$ *with cardinalities* $c_1, c_2, \ldots, c_k$ *satisfying:* $|c_i - c_j| \leqslant 1$ $i, j = 1, \ldots, k$.

Remark. The proof of Theorem 1 may be adapted to the case of p-bounded colorations [4] for which a similar result holds.

§2. Transversals in Hypergraphs

A *hypergraph* $H = (X, U)$ consists of a finite set X of vertices and a family U of nonempty edges E_j ($j = 1, \ldots, m$) satisfying $\bigcup_{j=1}^{m} U_j = X$.

A *transversal* is a subset T of vertices such that $T \cap E_j \neq \emptyset$ for $j = 1, \ldots, m$.

$H(p)$ will denote any hypergraph in which any vertex belongs to at most p edges.

Let $T_1, T_2, \ldots, T_k$ be a partition of the vertices of a hypergraph $H(p)$ into transversals and let $t_1, t_2, \ldots, t_k$ be their cardinalities. If $p=1$, all edges are disjoint; in this case it is easy to obtain from $T_1, T_2, \ldots, T_k$ a partition $\bar{T}_1, \bar{T}_2, \ldots, \bar{T}_k$ with $|\bar{t}_i - \bar{t}_j| \leqslant 1$ $i, j = 1, \ldots, k$.

Hence we will assume that $p \geqslant 2$ in the remainder of the note.

LEMMA. *Any two transversals T_i, T_j of $H(p)$ with $t_j > (p-1)\, t_i + 1$ may be replaced by two transversals $\bar{T}_i$, $\bar{T}_j$ with $\bar{t}_i \leqslant \bar{t}_j \leqslant (p-1)\, \bar{t}_i + 1$.*

Proof. Consider the subhypergraph $H_{ij} = \langle T_i \cup T_j \rangle$ spanned by $T_i \cup T_j$ (its edges are $(T_i \cup T_j) \cap E_r$ for $r = 1, \ldots, m$).

We will associate to H_{ij} a graph G_{ij} whose vertices are those of $T_i \cup T_j$; its edges which will be called *heavy edges* are obtained as follows:

initially there are no heavy edges. We examine consecutively all edges E of H_{ij} (note that for each E, $T_i \cap E \neq \emptyset$ and $T_j \cap E \neq \emptyset$)

a) if in edge E no pair of vertices x, y with $x \in T_i \cap E$ and $y \in T_j \cap E$ is joined by a heavy edge, then we pick up one such pair (x, y) and it becomes a heavy edge.

b) if in edge E there is already a pair x, y with $x \in T_i \cap E$ and $y \in T_j \cap E$ which is a heavy edge, we simply examine the next edge of H_{ij}.

By construction, G_{ij} is bipartite; besides no vertex in G_{ij} has a degree greater than p (since no vertex belongs to more than p edges of H_{ij}).

Assume now that $t_j = t_i + M > (p-1)\, t_i + 1$. G_{ij} has at most $t_i \cdot p$ edges and $2t_i + M \geqslant t_i \cdot p + 2$ vertices, hence it cannot be connected.

So there must exist a connected component G'_{ij} of G_{ij} with $t'_i < t'_j = t'_i + L \leqslant (p-1) \times t'_i + 1$ where t'_i and t'_j are the cardinalities of the subsets T'_i and T'_j of vertices of G'_{ij} belonging to T_i and T_j respectively.

We now interchange the vertices of T'_i and T'_j, thus T_i and T_j are replaced by subsets $\bar{T}_i$, $\bar{T}_j$. We have to show that $\bar{T}_i$ and $\bar{T}_j$ are transversals of H_{ij} and consequently of $H(p)$.

Notice that each edge of H contains exactly one heavy edge of G_{ij} and possibly isolated vertices of G_{ij} (it may occur that a heavy edge belongs to several edges of H).

So changing the colour of an isolated vertex of G_{ij} will still give two transversals $\bar{T}_i$, $\bar{T}_j$. Furthermore by interchanging the colours of the vertices in a connected component of G_{ij} we also obtain transversals: all edges containing a heavy edge of G'_{ij} will still be met by $\bar{T}_i$ and $\bar{T}_j$ and the edges containing only nonadjacent vertices of G'_{ij} must contain a heavy edge of another component of G_{ij}; hence they will also be met by $\bar{T}_i$ and $\bar{T}_j$.

Finally observe that

$$0 < L \leqslant (p-2)\, t'_i + 1 \leqslant (p-2)\, t_i + 1 < M$$

So the cardinalities $\bar{t}_i$ and $\bar{t}_j$ satisfy

$$t_i < \bar{t}_i = \bar{t}_i + L < t_i + M = t_j$$
$$t_i = t_j - M < t_j - L = \bar{t}_j < t_j$$

which implies

$$\max(\bar{t}_i, \bar{t}_j) < t_j$$
$$\min(\bar{t}_i, \bar{t}_j) > t_i$$

Let us choose the indices so that $\bar{t}_j \geqslant \bar{t}_i$; if we still have $\bar{t}_j > (p-1)\,\bar{t}_i + 1$, we may repeat the interchange procedure; we will ultimately obtain transversals $\bar{T}_i$, $\bar{T}_j$ satisfying

$$\bar{t}_i \leqslant \bar{t}_j \leqslant (p-1)\,\bar{t}_i + 1.$$

We denote by q_H the greatest number k of transversals $T_1, T_2, \ldots, T_k$ in a partition of H.

THEOREM 2. *For any $k \leqslant q_H$, there exists a partition of the vertices of $H(p)$ into transversals $T_1, T_2, \ldots, T_k$ with cardinalities $t_1, t_2, \ldots, t_k$ satisfying:* $\max_i(t_i) \leqslant (p-1)\min_i(t_i) + 1$.

Proof. The theorem follows directly from the previous lemma: as long as we have in the partition two transversals T_i, T_j satisfying $t_j > (p-1)\,t_i + 1$ we perform the interchange procedure described in the lemma. Finally we will obtain a partition with cardinalities $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_k$ satisfying $(p-1)\,t_k + 1 \geqslant t_1$.

Remark. The partitioning problem of § 1 is in fact a problem of transversals in the dual hypergraph H of G: each edge of G is a vertex of H; to each vertex x of G we associate an edge E_x; it contains all vertices corresponding to edges of G which are adjacent to x. Clearly no vertex of H belongs to more than 2 edges. Coverings in G correspond to transversals in H.

Since $p=2$, interchanges may be performed whenever $|t_j - t_i| > 1$, this means that Theorem 1 holds.

REFERENCES

[1] FOLKMAN, J. and FULKERSON, D. R., *Edge colorings in bipartite graphs*, in: Combinatorial Mathematics and its applications (University of North Carolina Press, Chapel Hill, 1969).

[2] DE WERRA, D. *On some combinatorial problems arising in scheduling*, Canadian Op. Res. Soc. J. *8*, No. 3, Nov. 71, p. 165–175.

[3] BERGE, C. *Graphes et hypergraphes*, Dunod, Paris, 1970.

[4] DE WERRA, D. *Equitable colorations of graphs*, Revue française d'informatique et de recherche opérationnelle, No R-3, (1971), 3–8.

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