Scalar Curvature, Non-Abelian Group Actions, and the Degree of Symmetry of Exotic Spheres

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Abstract

It is proved that if a compact manifold admits a smooth action by a compact, connected, non-abelian Lie group, then it admits a metric of positive scalar curvature. This result is used to prove that if Σ^n is an exotic n-sphere which does not bound a spin manifold, then the only possible compact connected transformation groups of Σ^n are tori of dimension $\leqslant [(n+1)/2]$.

§1. Introduction and Statement of Results

It has been known for several years that if a compact spin manifold M admits either a non-trivial S^1 action or a metric of positive scalar curvature, then $\hat{A}(M)=0$; and it has been at times conjectured that these hypotheses are directly related, in particular, that the existence of an S^1-action implies the existence of a metric of positive scalar curvature. This conjecture turns out to be false because of the following two results.

THEOREM 1.1. (N. Hitchen [3].) *Let Σ^n be any exotic sphere which does not bound a spin manifold. Then Σ^n does not admit a riemannian metric of positive scalar curvature.*

For $n \equiv 1$ or 2 (mod 8), the exotic n-spheres which bound spin manifolds form a subgroup $\mathrm{B\,Spin}_n$ of index 2 in the group Θ_n of homotopy n-spheres.

THEOREM 1.2. (G. Bredon [2].) *For $n \equiv 2$ (mod 8), the spheres $\Sigma^n \in \Theta_n - \mathrm{B\,Spin}_n$ admit non-trivial S^1 actions.*

The idea of the proof of Theorem 1.1 is that by Atiyah and Singer [1] the dimension of the space of harmonic spinors (mod 2) on a compact, riemannian spin manifold M can be identified with a certain KO-Theory invariant $\alpha(M)$ of the spin-cobordism class of M. This invariant was introduced by Milnor and shown by Milnor and Adams to give a non-trivial homomorphism $\alpha: \Theta_n \to \mathbf{Z}_2$ for $n \equiv 1$ or 2 (mod 8). (See [0], [9].) However, by a result of Lichnerowicz [8], if the metric of M has positive scalar curvature (in fact, $\kappa \geqslant 0$ and $\not\equiv 0$), then there are no harmonic spinors.

In [9] Milnor actually constructes compact spin manifolds of type $M^{8n+1} = N^{8n} \times S^1$ for $n=1$ and 2 such that $\alpha(M^{8n+1}) \neq 0$. Consequently, it is not even true that a *free* S^1-action implies the existence of a metric of positive scalar curvature.

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This failure of the above conjecture motivates the principal result of this paper.

MAIN THEOREM. *If a compact manifold admits a smooth, effective action by any compact, connected, non-abelian Lie group (that is, if it admits a non-trivial S^3 action), then it admits a riemannian metric of strictly positive scalar curvature.*

Thus, we have the following diagram of results.

$$\begin{tikzcd}
& \exists \text{ an } S^1 \text{ action} \arrow[dr, Rightarrow] \arrow[dd, Rightarrow, "/" marking] & \\
\exists \text{ an } S^3 \text{ action} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \hat{A}=0 \quad \text{(for spin manifolds)} \\
& \exists \text{ a metric with } \kappa>0 \arrow[ur, Rightarrow] &
\end{tikzcd}$$

We now recall an elementary differential topological invariant.

DEFINITION 1.3. The *Hsiang index of symmetry* of a smooth n-manifold M^n is the integer

$$\mathfrak{S}(M^n)=\sup\{\dim_{\mathbf{R}} G : G \text{ is a compact subgroup of } \mathrm{Diff}(M^n)\}.$$

It is known that $\mathfrak{S}(M^n)\leqslant \frac{1}{2}n(n+1)$ with equality if and only if $M^n=S^n$ or $\mathbf{R}P^n$. Furthermore it has been proven by Wu-Yi Hsiang [5] that if $\Sigma^n\in\Theta_n$, $n\geqslant 40$, is an exotic sphere, then

$$\mathfrak{S}(\Sigma^n)<\tfrac{1}{8}n^2+1. \tag{1.1}$$

This result is sharp since from the Brieskorn representations one can easily see one that the Kervaire spheres Σ^n, $n=4k+1$, have $\mathfrak{S}(\Sigma^n)=\frac{1}{8}n^2+\frac{7}{8}$. However, if considers exotic spheres which do not bound parallelizable manifolds, the estimate (1.1) can be improved [4], [6]. Furthermore, R. Schultz [11], [12] has shown that there exists an infinite family of homotopy spheres for which $\mathfrak{S}(\Sigma^n)\leqslant\frac{30}{7}n$. As a consequence of our main theorem and Theorem 1.1 we have the following

THEOREM 1.4. *Let Σ^n be an exotic n-sphere which does not bound a spin manifold. Then the only compact, connected transformation groups of Σ^n are tori. In particular,*

$$\mathfrak{S}(\Sigma^n)\leqslant\left[\frac{n+1}{2}\right].$$

We reiterate that $\Theta_n/\mathrm{B\,Spin}_n=\mathbf{Z}_2$ for $n\equiv 1$ or $2 \pmod 8$.

Proof of Theorem 1.4. The first conclusion is an immediate consequence of the main theorem and the discussion above. The second conclusion can be seen as follows. If n is even, any toral transformation group T^k must have a fixed point set, and the induced linear action on the normal spaces to the fixed point set must be effective. Thus, $k \leqslant n/2$. For n odd, we refer to the work of Ku [7].

Theorem 1.4 raises the question of allowable torus actions on exotic spheres. There are results of this type due to R. Schultz who has a method of proving the non-existence of $(\mathbf{Z}_p)^r$ actions on exotic spheres in Θ_n for $n=2p^2-2p-2$ and p a prime [13]. In particular it can be shown that there are three exotic 10-spheres for which $1 \leqslant \mathfrak{S}(\Sigma^{10}) \leqslant 2$.

As a final note, we point out that the conclusion of Theorem 1.4 holds for any compact spin manifold M for which $\alpha(M) \neq 0$. Since the α-invariant is additive with respect to connected sums of manifolds, it is always possible to change the differentiable structure of M, in dimensions $\equiv 1$ or $2 \pmod 8$, to make $\alpha(M) \neq 0$.

§2. The Basic Construction

Let G be a compact, connected, non-abelian Lie group acting differentiably (and effectively) on a compact manifold M. The purpose of this section is to outline a method of using this action to construct a metric of positive scalar curvature on M.

We begin by considering the simplest possibility, namely, when the action is free. In this case we have a principal G-bundle $\pi: M \to M' = M/G$. Any invariant metric on M gives us a connection, i.e., an invariant field of horizontal planes, and we lift to these planes a fixed riemannian metric from M'. Let $\mathscr{G}$ be the Lie algebra of G with some Ad_G-invariant inner product, and carry this inner product over to M by the canonical identification $\mathscr{G} \subset \mathfrak{X}_M$. Now for each $t > 0$ we have a riemannian metric $g_t = g_H + t^2 g_V$ where g_H and g_V are the horizontal and vertical inner products defined above.

LEMMA 2.1. *The orbits of G in the metric g_t are totally geodesic submanifolds.*

Proof. Let B denote the second fundamental form of a fixed orbit. Choose any $X \in \mathscr{G} \subset \mathfrak{X}_M$ and let H be an invariant horizontal field. Then, since $\langle X, H \rangle = 0$ and $[X, H] = 0$,

$$\langle B(X, X), H \rangle = \langle \nabla_X X, H \rangle = -\langle X, \nabla_X H \rangle = -\langle X, \nabla_Y X \rangle = -\tfrac{1}{2} H \|X\|^2 = 0$$

(where $\langle .,. \rangle$ denotes any of these metrics and ∇ is the associated riemannian connection), and the statement is proved.

We shall now apply the O'Neill identities for the curvature of a riemannian submersion with totally geodesic fibers [10]. Let $\pi: M \to M'$ be any riemannian submersion[2]) and let $(\cdot)^h$, $(\cdot)^v$ denote orthogonal projection onto the horizontal and vertical subspaces respectively of T_xM at any point. Then the *fundamental tensor* of the submersion is a (2, 1) tensor which assigns to each $X \in \mathfrak{X}_M$ a section A_X of $\mathrm{Hom}(TM)$ given by

$$A_X(Y) = (\nabla_{X^h} Y^h)^v + (\nabla_{X^h} Y^v)^h \tag{2.1}$$

for $Y \in \mathfrak{X}_M$. If X and Y are both horizontal, then $A_X(Y) = -A_Y(X) = \frac{1}{2}[X, Y]^v$.

We now consider the family of metrics g_t constructed above on the principal G-bundle $\pi: M \to M'$, and for each t we let A^t denote the fundamental tensor of π for the metric g_t. For any $X, Y \in T_xM$ we let $K^t(X \wedge Y)$ denote the sectional curvature of the (X, Y)-plane in the metric g_t, and similarly we let $K'(\cdot)$ denote the sectional curvature of the common, submersed metric on M'. Let H, H' be local, orthonormal horizontal fields on M and let V, V' be canonical vertical fields which are orthonormal in the metric g_1. Set $\|\cdot\| = g_1(\cdot, \cdot)$. Then it follows easily from O'Neill [10] and the formula for curvature of a biinvariant metric on G, that:

$$K^t(H \wedge H') = K'(\pi_* H \wedge \pi_* H') - \tfrac{3}{4} t^2 \|[H, H']^v\|^2, \tag{2.2}$$

$$K^t(H \wedge V) = t^2 \|A^1_H(V)\|^2 \tag{2.3}$$

$$K^t(V \wedge V') = \frac{1}{t^2} \|[V, V']\|^2. \tag{2.4}$$

Since G is non-abelian, it is clear that for all t sufficiently small the metric g_t has positive scalar curvature.

For a general action of G on M the procedure is much more complicated and the estimates more delicate. The outline of our construction is as follows.

Step 1. Introduce a G-invariant metric on M.

Step 2. Let G carry a biinvariant metric b and consider the free G-action ϕ on $G \times M$ given by $\phi_g(h, x) = (g \cdot h, g(x))$.

There is a natural map $\pi^\phi: G \times M \to M$ given by projection along the orbits. ($\pi^\phi(g, x) = g^{-1}(x)$.) We now introduce a family of metrics g_t on $G \times M$ very much as we did above. Using the product metric on $G \times M$ we have defined an invariant field of normal planes to the orbits of the ϕ-action. We lift the metric of M to these planes via π^ϕ. Along the orbits we introduce the metric $t^2 b$ via the inclusion $\mathscr{G} \subset \mathfrak{X}_{G \times M}$ given by ϕ. By Lemma 2.1 the orbits of ϕ in the metric g_t are totally goedesic.

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[2]) This is defined as follows (cf. [10]). Let $\pi: M \to M'$ be a submersion between riemannian manifolds. For $x \in M$ there is an orthogonal splitting $T_xM = V_x \oplus H_x$ into vertical and horizontal subspaces where V_x is the tangent space to the fiber $\pi^{-1}(\pi(x))$ through x. Then π is called *riemannian* if $\pi_* \mid H_x: H_x \to T_{\pi x}M'$ is an isometry for all x.

Step 3. Each metric g_t on $G\times M$ is invariant under the G action ψ where $\psi_g(h, x)=(h\cdot g^{-1}, x)$. Hence, there is a metric $\tilde{g}_t$ on M for which the right hand projection $\pi: G\times M\to M$ is a riemannian submersion.

We shall show that if the original metric (Step 1) is appropriately chosen near the fixed-point set of G, then for all t sufficiently small the metric $\tilde{g}_t$ will have positive scalar curvature.

§3. Curvature Estimates away from the Fixed-Point Set

In this section we shall compute the scalar curvature of the metrics $\tilde{g}_t$ on M away from the fixed point set M^G. Actually, since sectional curvatures increase under a riemannian submersion (cf. Samelson [15], or [10, Cor. 1]), and since we are only interested in finding a positive lower bound, it will suffice for us to compute the average horizontal sectional curvature for the submersion $\pi: G\times M\to M$.

We assume we are in the situation set up in the beginning of Step 2 above. Fix a point $x\in M-M^G$. Then there is an orthogonal splitting $\mathscr{G}=\mathscr{G}_x\oplus\mathscr{P}_x$ where $\mathscr{G}_x$ is the Lie subalgebra of the isotropy subgroup G_x of x. There is a natural embedding $i_x:\mathscr{P}_x\subsetneq T_xM$ given by the action of G on M. Let t_x denote the orthogonal complement of $i_x\mathscr{P}_x$ in T_xM. Then

$$T_{(e,x)}(G\times M)\cong\mathscr{G}_x\oplus\mathscr{P}_x\oplus i_x\mathscr{P}_x\oplus t_x.$$

The canonical embedding $\mathscr{G}\subsetneq T_{(e,x)}(G\times M)$ is given, with respect to the above splittings of these spaces, by $(d, e)\mapsto(d, e, i_xe, 0)$. We now choose an orthonormal basis $\{e_1,\dots, e_l\}$ of $\mathscr{P}_x$ (in the biinvariant metric of G) so that $\langle i_xe_i, i_xe_j\rangle=\sigma_i^2\delta_{ij}$ where $\sigma_i>0$ for all i. Then for each $t>0$ there is a basis $\mathscr{B}_x^t$ of $T_{(e,x)}(G\times M)$ as follows

$$\mathscr{B}_x^t=\left\{\frac{1}{t}\eta_1,\dots,\frac{1}{t}\eta_k,\frac{1}{t}\xi_1,\dots,\frac{1}{t}\xi_l,\tilde{\xi}_1,\dots,\tilde{\xi}_l,\tilde{\eta}_1,\dots,\tilde{\eta}_m\right\}$$

where for each i:

$$\eta_i\in\mathscr{G}_x,\quad \tilde{\eta}_i\in t_x,\quad \xi_i=e_i+i_xe_i,\quad \tilde{\xi}_i=(\sigma_i^2e_i-i_xe_i)/\sigma_i(1+\sigma_i^2),$$

where the η_i and $\tilde{\eta}_j$ form orthonormal bases of $\mathscr{G}_x$ and t_x respectively in the product metric. Note that the η_i and ξ_j come from fields canonically associated by ϕ to an orthonormal basis of $\mathscr{G}$, and, furthermore, that $\langle\pi_*^\phi\tilde{\xi}_i, \pi_*^\phi\tilde{\xi}_j\rangle=\delta_{ij}$ for all i, j. (To check this second fact note that $\pi^\phi(g, x)=g^{-1}(x)$, and so for $g=$identity, we have $\pi_*^\phi(e, v)=$ $=-i_xe+v$.) Thus, we have the following.

Fact 3.1. $\mathscr{B}_x^t$ is an orthonormal basis of $T_{(e,x)}(G\times M)$ in the metric g_t constructed in Step 2. The elements in $\mathscr{B}_x^t$ denoted above with a tilda span the horizontal space for the submersion π^ϕ (and those without a tilda span the vertical space).

Notice that the splitting into horizontal and vertical spaces for the submersion π^{Φ} is independent of t. This is not true of the submersion π, which we must now consider. Let $\lambda_i=\sigma_i(1+\sigma_i^2)$, and set

$$\mathscr{V}^t_x=\left\{\frac{1}{t}\eta_1,\dots,\frac{1}{t}\eta_k, v_1^t,\dots,v_l^t\right\},$$
$$\mathscr{H}^t_x=\{\tilde{\eta}_1,\dots,\tilde{\eta}_m, h_1^t,\dots,h_l^t\}$$

where for each i,

$$v_i^t=(\xi_i+\lambda_i\tilde{\xi}_i)/\sqrt{t^2+\lambda_i^2},$$
$$h_i^t=(\lambda_i\xi_i-t^2\tilde{\xi}_i)/t\sqrt{\lambda_i^2+t^2}.$$

The following is straightforward to check.

Fact 3.2. $\mathscr{V}^t_x$ and $\mathscr{H}^t_x$ form orthonormal bases respectively of the vertical and horizontal subspaces of $T_{(e,x)}(G\times M)$ in the metric g_t for the submersion π defined in Step 3.

The remainder of this section is devoted to finding a positive lower bound for the average of the sectional curvatures of the metric g_t over the space $H_x^t=\operatorname{span}\mathscr{H}^t_x$.

To compute the curvature of the metric g_t we must know the riemannian connection ∇^t. Actually, it will suffice to relate the curvature for time t to those for time 1, and we now make the notational convention that: *items indexed by t will have the index deleted for the case* $t=1$. The first step in doing this relative computation is the following.

LEMMA 3.3. *Let* $C_X^t(Y)=\nabla_XY-\nabla_X^tY$ *for* $X, Y\in\mathfrak{X}_{G\times M}$. *Then*

$$C_X^t(Y)=(1-t^2)[\nabla_{X^h}Y^v+\nabla_{Y^h}X^v]^h$$

where $(\cdot)^h$ *and* $(\cdot)^v$ *denote orthogonal projection onto the horizontal and vertical subspaces respectively for the submersion* π^{Φ}.

Proof. It is straightforward to check that the connection $\nabla^t \overset{\text{def.}}{=} \nabla-C^t$ is torsion free and satisfies $\nabla^t g_t=0$.

Now for each $t>0$ we have the curvature transformation

$$R^t_{X,Y}=\nabla^t_{[X,Y]}-[\nabla^t_X,\nabla^t_Y],$$

and the fundamental tensor A^t (cf. §2) of the submersion π^{Φ} in the metric g_t. We note that

$$C_X^t(Y)=(1-t^2)\,(A_X(Y)+A_Y(X))^h.$$

A straightforward computation now gives the following.

COROLLARY 3.4. *For all* $t>0$,

$$R^t_{X,Y}=R_{X,Y}+(\nabla_X C^t)_Y-(\nabla_Y C^t)_X-[C^t_X, C^t_Y], \tag{3.1}$$

$$A^t_X(Y)=A_X(Y)-(1-t^2)\,(\nabla_{X^h}Y^v)^h. \tag{3.2}$$

It is not difficult to check that

$$[(\nabla_X C^t)_Y(Z)]^v=A_{X^h}(C^t_Y(Z)), \qquad [C^t_X, C^t_Y]^v=0.$$

Using these identities and Equation (3.1) or using O'Neill's identities and Equation (3.2) one can without difficulty establish the following result.

PROPOSITION 3.5. *Let* x, y, z, w *denote vectors which are vertical and* $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}$ *denote vectors which are horizontal for the submersion* π^ϕ. *Then for all* $t>0$ *we have the following identities.*

$$\langle R^t_{x,y}z, w\rangle_t=t^2\langle R_{x,y}z, w\rangle \tag{3.3}$$

$$\langle R^t_{x,y}z, \tilde{w}\rangle_t=0 \tag{3.4}$$

$$\langle R^t_{\tilde{x},\tilde{y}}z, w\rangle_t=t^2\langle R_{\tilde{x},\tilde{y}}z, w\rangle+t^2(1-t^2)\langle[A_{\tilde{x}}, A_{\tilde{y}}]\,(z), w\rangle \tag{3.5}$$

$$\langle R^t_{\tilde{x},y}\tilde{z}, w\rangle_t=t^2\langle R_{\tilde{x},y}\tilde{z}, w\rangle-t^2(1-t^2)\langle A_{\tilde{x}}(w), A_{\tilde{z}}(y)\rangle \tag{3.6}$$

$$\langle R^t_{\tilde{x},\tilde{y}}\tilde{z}, w\rangle_t=t^2\langle R_{\tilde{x},\tilde{y}}\tilde{z}, w\rangle \tag{3.7}$$

$$\begin{aligned}\langle R^t_{\tilde{x},\tilde{y}}\tilde{z}, \tilde{w}\rangle_t=\langle R_{\tilde{x},\tilde{y}}\tilde{z}, \tilde{w}\rangle&+(1-t^2)\,[2\langle A_{\tilde{x}}(\tilde{y}), A_{\tilde{z}}(\tilde{w})\rangle\\ &+\langle A_{\tilde{x}}(\tilde{z}), A_{\tilde{y}}(\tilde{w})\rangle-\langle A_{\tilde{y}}(\tilde{z}), A_{\tilde{x}}(\tilde{w})\rangle].\end{aligned} \tag{3.8}$$

In particular, from (3.8) we have the following identity on sectional curvature.

$$K^t(\tilde{x}\wedge\tilde{y})=K(\tilde{x}\wedge\tilde{y})+3(1-t^2)\,\|A_{\tilde{x}}(\tilde{y})\|^2. \tag{3.9}$$

Recall that we are interested in computing the "horizontal" scalar curvature of the metric g_t. Hence, we need to compute terms of the form: $K^t(h^t_i\wedge h^t_j)$, $K^t(h^t_i\wedge\tilde{\eta}_j)$ and $K^t(\tilde{\eta}_i\wedge\tilde{\eta}_j)$. We begin with the most complicated term.

$$\begin{aligned}K^t(h^t_i\wedge h^t_j)&=\langle R^t_{h_i{}^t, h_j{}^t}h^t_i, h^t_j\rangle_t\\
&=\frac{1}{t^4(\lambda_i^2+t^2)(\lambda_j^2+t^2)}\langle R^t_{\lambda_i\xi_i-t^2\tilde{\xi}_i,\,\lambda_j\xi_j-t^2\tilde{\xi}_j}\lambda_i\xi_i-t^2\tilde{\xi}_i, \lambda_j\xi_j-t\ \tilde{\xi}_j\rangle_t\\
&=\frac{1}{t^4(\lambda_i^2+t^2)(\lambda_j^2+t^2)}\{\lambda_i^2\lambda_j^2\langle R^t_{i,j}i, j\rangle_t+t^8\langle R^t_{\tilde{i},\tilde{j}}\tilde{i}, \tilde{j}\rangle_t\\
&\quad-2\lambda_i^2\lambda_jt^2\langle \cancelto{0}{R^t_{i,j}}i, \tilde{j}\rangle_t-2\lambda_j^2\lambda_it^2\langle \cancelto{0}{R^t_{i,j}}\tilde{i}, j\rangle_t+2\lambda_i\lambda_jt^4\langle R^t_{i,j}\tilde{i}, \tilde{j}\rangle_t\\
&\quad-2\lambda_jt^6\langle R^t_{\tilde{i},\tilde{j}}\tilde{i}, j\rangle_t-2\lambda_it^6\langle R^t_{\tilde{i},\tilde{j}}i, \tilde{j}\rangle_t+2\lambda_i\lambda_jt^4\langle R^t_{\tilde{i},j}i, \tilde{j}\rangle_t\\
&\quad+\lambda_i^2t^4\langle R^t_{i,\tilde{j}}i, \tilde{j}\rangle_t+\lambda_j^2t^4\langle R^t_{\tilde{i},j}\tilde{i}, j\rangle_t\}\end{aligned} \tag{3.10}$$

where for notational convenience we have replaced ξ_i by i and $\tilde{\xi}_i$ by $\tilde{i}$.

Now from Proposition 3.5 we have that the second two terms in this expansion are zero. Furthermore, as $t \downarrow 0$, the second curvature term in the expansion is $O(1)$ and all other curvature terms are $O(t^2)$. We combine this with the following elementary observations. For $t>0$,

$$\frac{\lambda_i^2}{\lambda_i^2+t^2}<1$$

and if $\lambda_i^2+t^2<1$, then

$$\lambda_i^2<\frac{\lambda_i^2}{\lambda_i^2+t^2}<1. \tag{3.11}$$

Furthermore,

$$\frac{\lambda_i t}{\lambda_i^2+t^2}\leqslant\tfrac{1}{2}, \tag{3.12}$$

$$\frac{t^2}{\lambda_i^2+t^2}\leqslant 1. \tag{3.13}$$

Finally, we observe that $\langle R_{i,j}i,j\rangle=\|[e_i, e_j]\|_*^2$, where $\|\cdot\|_*$ is the original biinvariant metric on G. Putting this all together, we have the following.

PROPOSITION 3.6. *For each* $i,j=1,\ldots,l$,

$$K^t(h_i^t\wedge h_j^t)=\frac{1}{t^2}\frac{\lambda_i^2\lambda_j^2}{(t^2+\lambda_i^2)(t^2+\lambda_j^2)}\|[e_i, e_j]\|_*^2+O(1)$$

as $t\downarrow 0$.

In a similar fashion, we have that

$$K^t(h_i^t\wedge\tilde{\eta}_j)=\frac{1}{t^2(\lambda_i^2+t^2)}\{\lambda_i^2\langle R_{i,\tilde{j}}^t i,\tilde{j}\rangle_t+t^4\langle R_{\tilde{i},\tilde{j}}^t\tilde{i},\tilde{j}\rangle_t-2\lambda_i t^2\langle R_{i,\tilde{j}}^t\tilde{i},\tilde{j}\rangle_t\}=O(1). \tag{3.14}$$

Combining this with Equation (3.9) we have proved:

PROPOSITION 3.7. *For all* i,j, $K^t(h_i^t\wedge\tilde{\eta}_j)=O(1)$ *and* $K^t(\tilde{\eta}_i\wedge\tilde{\eta}_j)=O(1)$ *as* $t\downarrow 0$.

Without any loss in generality we may assume that $G=SU(2)$ or $SO(3)$ since any connected, non-abelian Lie group has such a subgroup. We normalize the biinvariant metric b to have (constant) sectional curvature 1. Then the term $\|[e_i, e_j]\|_*^2$ in Proposition 3 equals 1 for all i,j. Moreover, at each point $x\in M-M^G$ we have $\dim\mathscr{P}_x\geqslant 2$ since G has no subgroups of codimension one. Consequently, if at each x we index the λ_i so that $\lambda_1\geqslant\lambda_2\geqslant\ldots$, then for each open neighborhood U of the fixed point set M^G

we have a constant $c=C(U)>0$ such that $\lambda_1 \geqslant \lambda_2 \geqslant c$ throughout $M-U$. Thus, from Propositions 3.6 and 3.7 we have the following.

THEOREM 3.8. *Let $G=SU(2)$ or $SO(3)$ and let U be any neighborhood of the fixed-point set of G in M. Then there exists $t(U)>0$ such that for all $t \leqslant t(U)$, the metric $\tilde{g}_t$ constructed in* §2 (Step 3) *has positive scalar curvature in $M-U$.*

§4. Time Independent Estimates near the Fixed-Point Set

In light of Theorem 3.8, it remains for us to construct a G-invariant metric γ on M with the property that there is some neighborhood U of M^G and some $t_0>0$ such that all the metrics g_t, $0<t<t_0$ constructed from γ as in §2, have positive scalar curvature in U.

To construct this metric we must consider the geometry of the fixed point set M^G. Let F be a component of M^G and let $p:N \to F$ be the normal bundle of F in M. Then G acts naturally on N by linear transformations in each fiber. Furthermore there is a natural G-equivariant diffeomorphism of N onto a neighborhood U_0 of F in M. Therefore, if we can construct a metric with the desired properties on the total space of N, we will be done, since we can extend the metric given on U_0 to all of M without changing it in a smaller neighborhood U of F, and then average the extended metric to make it G-invariant outside U. From here on we shall confine our attention to N.

We may assume that N carries an inner product for which the action of G in each fiber is orthogonal. If we fix a point $x \in F$ and an orthonormal basis $\mathscr{E}=\{e_1, \ldots, e_q\}$ in the fiber $N_x = p^{-1}(x)$, we get a natural homomorphism

$$i_{\mathscr{E}}: G \to O(q)$$

given by the action of G in N_x. It follows from the equivariance of $\exp_x$ that the conjugacy class of $i_{\mathscr{E}}(G)$ in $O(q)$ is independent of x and $\mathscr{E}$, and that, since G acts effectively, each $i_{\mathscr{E}}$ is an embedding.

We shall now introduce an explicit, invariant riemannian metric on N in which the fibers of $p:N \to F$ are totally geodesic and have positive sectional curvature near zero. To do this we must make some preliminary observations.

Let $P(N) \to F$ be the principal $O(q)$ bundle of orthonormal frames in N. G has a natural induced action on $P(N)$ which commutes with the standard action of $O(q)$. Hence we may introduce a G-invariant connection in $P(N)$. It is easy to see that at any $x \in F$, the action of G in N_x commutes with the holonomy transformations of this connection at x. Consequently, we can reduce the structure group of N to the centralizer of G in $O(q)$. Specifically, for a fixed frame $\mathscr{E}$ at x, let $i_{\mathscr{E}}: G \to O(q)$ be the embedding discussed above, and let $Z(G)$ be the centralizer of $i_{\mathscr{E}}(G)$ in $O(q)$. Then there is a principal $Z(G)$ bundle $p': P'(N) \to F$ such that N is the bundle associated to the representation of $Z(G)$ on $\mathbf{R}^q$ given by the inclusion $Z(G) \subset O(q)$. That is, $N = (P'(N) \times \mathbf{R}^q / Z(G))$ where the G-action on N comes from the representation i_g of G on $\mathbf{R}^q$ in the product.

We want to construct a G-invariant metric on N. This is done as follows. Introduce any metric on F and lift it to the horizontal spaces of the connection on $P'(N)$. Then carry a biinvariant metric on $Z(G)$ over to the vertical fields as before. Let $\mathbf{R}^q$ carry the metric σ_c of constant curvature c obtained by stereographic projection from S^q. That is,

$$\sigma_c = \frac{4}{c} \frac{|dx|^2}{(1+|x|^2)^2}. \tag{4.1}$$

We set $\hat{N} = P'(N) \times \mathbf{R}^q$ and give $\hat{N}$ the product metric. There is a natural action $\Phi_1 : Z(G) \to \mathrm{Isom}(\hat{N})$ given by

$$\Phi_1(g)(u, v) = (u \cdot g^{-1}, gv).$$

We give $N = \hat{N}/\Phi_1$ the submersed metric. Note that the fibers of the map $N \to F$ are totally geodesic since the fibers of $\hat{N} \to F$ are.

There is a natural action $\Phi_2 : G \to \mathrm{Isom}(G \times \hat{N})$ given by

$$\Phi_2(g)(h, u, v) = (g \cdot h, u, i_g(g)(v)).$$

This action commutes with Φ_1, and defines an action on $G \times N$ which is exactly the diagonal action ϕ used in Step 2 of the construction in §2.

We now introduce a family of metrics $\hat{g}_t$ on $G \times \hat{N}$ by modifying along the orbits of Φ_2 exactly as in Step 2. These metrics will be Φ_1 invariant and will therefore determine a family g_t of submersed metrics on $G \times N$. This is exactly the family of metrics obtained by modifying our original metric on $G \times N$ by the procedure of Step 2.

Now each of the metrics $\hat{g}_t$ is a product of a (modified) metric on $G \times \mathbf{R}^q$ with the fixed metric on $P'(N)$. Furthermore $p' : P'(N) \to F$ is a riemannian submersion with totally geodesic fibers. From this one can easily deduce the following about the metric $\tilde{g}_t$ on N obtained by projecting the metric g_t as in Step 3, §2.

LEMMA 4.1. *For all $t > 0$ the projection $p : N \to F$ with the metric $\tilde{g}_t$ on N is a riemannian submersion with totally geodesic fibers. Furthermore, the submersed metric on F is independent of t; and for fixed t, the fibers $N_x = p^{-1}(x)$, $x \in F$, are all isometric to each other.*

We are now in a position to state the main result of this section.

THEOREM 4.2. *Let $G = SU(2)$ or $SO(3)$. Then there exist numbers $c > 0$, $t_0 > 0$ and a neighborhood U of the zero-section of N such that the metric $\tilde{g}_t$ $(= \tilde{g}_t(c))$ has positive scalar curvature in U for all $t \in (0, t_0]$.*

Proof. From Lemma 4.1 together with the O'Neill formulas applied to the submersion $p: N \to F$ one can easily see that it suffices to prove the following.

LEMMA 4.3. *Let N_x be a fiber of $p: N \to F$ with the induced metric $\tilde{g}_t(=\tilde{g}_t(c))$, and let $\kappa_0 > 0$ be given. Then there exist $c > 0$, $t_0 > 0$ and a neighborhood U_x of 0 in N_x such that for all $t \in (0, t_0]$ the scalar curvature of $\tilde{g}_t$ is $> \kappa_0$ throughout U_x.*

Note. The scalar curvature in this lemma is that of the manifold N_x.

Proof of Lemma 4.3. The metric $\tilde{g}_t$ on N_x is obtained as follows. We begin with a product metric $b \times \tilde{\sigma}_c$ on $G \times \mathbf{R}^q$; we modify as in Step 2 to get g_t and then submerse this metric by right projection onto N_x. The metric $\tilde{\sigma}_c$ is obtained by submersing the product metric $b' \times \sigma_c$ on $Z(G) \times \mathbf{R}^q$, where b' is biinvariant and σ_c is given by (4.1), along the orbits of the $Z(G)$ action Φ_1.

We first observe that there is a bounded neighborhood U_0 of 0 in $\mathbf{R}^q$ in which $\tilde{\sigma}_c$ has all sectional curvature $\geqslant c/2$. To see this note that the vertical space above 0 in the projection $\pi^{\Phi_1}: Z(G) \times \mathbf{R}^q \to \mathbf{R}^q$ is just $Z(G) \times \{0\}$ since $Z(G)$ fixes 0 in $\mathbf{R}^q$. Hence, by the O'Neill formula (2.2) ([10, Cor. 1]) the sectional curvatures $\tilde{\sigma}_c$ at 0 are $\geqslant$ those of σ_c at 0, i.e., they are $\geqslant c$. So we can find U_0 as claimed.

Recall now that the metric g_t is constructed by lifting $\tilde{\sigma}_c$ to the normal spaces to the orbits of the diagonal G action ϕ on $G \times \mathbf{R}^q$ and introducing $t^2 b$ along the orbits. It follows again by formula (2.2) that there is some $t' > 0$ such that for all $t \in (0, t']$ the sectional curvatures of g_t in these normal spaces (i.e., the horizontal sectional curvatures for the submersion π^{ϕ}) are $\geqslant c/3$ throughout $G \times U_0$. In the terminology of §3, we have

$$K^t(\tilde{\imath} \wedge \tilde{\jmath}) = \langle R^t_{\tilde{\imath},\tilde{\jmath}} \tilde{\imath}, \tilde{\jmath} \rangle_t \geqslant \frac{c}{3}$$

in $G \times U_0$ for $0 < t \leqslant t'$.

We must now closely examine the formulas (3.10) and (3.14) for the horizontal sectional curvatures of the projection π. Again it will suffice to show that the average of these will be as large as desired in $G \times U_0$ since submersion increases curvature. We now fix the value of t and begin by examining Equation (3.10). Note that the values of the λ_i's go uniformly to zero as $x \to 0$ in $\mathbf{R}^q$. Furthermore, since G is acting linearly on $\mathbf{R}^q$, we see from the form of the metric $\tilde{\sigma}_c$ that the two largest eigenvalues $\lambda_1 \geqslant \lambda_2$ satisfy $\lambda_2/\lambda_1 \geqslant \varrho > 0$ in the neighborhood U_0 of 0 in $\mathbf{R}^q$. Now the first term of (3.10) for $(i,j) = (1,2)$ is

$$\frac{1}{t^2} \frac{\lambda_1^2 \lambda_2^2}{(t^2 + \lambda_1^2)(t^2 + \lambda_2^2)} \|[e_1, e_2]\|_*^2, \tag{4.2}$$

and since $G = SU(2)$ or $SO(3)$ with curvature 1, we have $\|[e_1, e_2]\|_*^2 = 1$. The expression (4.2) is greater than or equal to

$$f(\lambda_1)=\frac{\varrho^2\lambda_1^4}{t^2(t^2+\lambda_1^2)^2}.$$

Consequently, the term (4.2) will dominate all the possibly negative terms in the sum we are considering, plus κ_0, provided that

$$f(\lambda_1)\geqslant r^2 \overset{\text{def.}}{=} \sup_{G\times \bar{U}_0} \|R^1\|+\|A\|^2+\kappa_0 \tag{4.3}$$

where R^1 is the curvature tensor of g_1. The inequality (4.3) will hold provided that

$$\lambda_1^2\geqslant\frac{r}{\varrho}t^3\bigg/\left(1-\frac{r}{\varrho}t\right). \tag{4.4}$$

If we assume $t<\varrho/2r$, then (4.4) will hold provided that

$$\lambda_1^2\geqslant 2\frac{r}{\varrho}t^3.$$

Hence, we are reduced to the case where all λ_i satisfy

$$\lambda_i<\sqrt{\frac{2r}{\varrho}}\,t^{3/2}\leqslant t^{5/4} \tag{4.5}$$

where the second inequality holds for any t satisfying $t<(\varrho/2r)^2$.

We can now consider the sum (3.10) in detail. Observe first that the last two terms in the sum are positive by (2.3) (or [10, Cor. 1]) and can therefore be neglected. (This is also true of the first term, of course.) The third and forth terms are zero. Recall now that the remaining curvature expressions are all bounded by t^2r' in $G\times\bar{U}_0$ for some $r'>0$ and for all t. Thus, the sixth and seventh terms are bounded above by $2\lambda_i r'$ (or $2\lambda_j r'$)$\leqslant 2t^{5/4}r'$. The fifth and eighth terms are bounded by products of expressions of type

$$\frac{2\lambda_i t}{\lambda_i^2+t^2}r'\leqslant 2\frac{t^{5/4}t}{t^2}r'=2t^{1/4}r'.$$

Thus, for all t sufficiently small, these terms will be uniformly small in the critical region where (4.5) holds. However, in this region we clearly have the second term of (3.10) bounded *below* by

$$\tfrac{1}{4}\langle R^t_{\bar{i},\bar{j}}\bar{i},\bar{j}\rangle_t\geqslant\frac{c}{12}$$

for all $t \leqslant t'$. Consequently for c sufficiently large and for all $t>0$ sufficiently small, the contribution from terms $K^t(h_i^t \wedge h_j^t)$ is $\geqslant \kappa_0$ in the critical region.

Exactly the same analysis applies to Equation (3.14) to give a similar conclusion for the terms $K^t(h_i^t \wedge \tilde{\eta}_j)$. Of course, the terms $K^t(\tilde{\eta}_i \wedge \tilde{\eta}_j)$ are already $\geqslant c/3$ for $t \leqslant t'$. This concludes the proof of Theorem 4.2. Theorems 3.8 and 4.2 together give the main result of this paper, stated in the introduction.

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