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Functions of Bounded mean Oscillation and Quasiconformal Mappings

H. M. Reimann

1. Introduction

A locally integrable real valued function u is said to be of bounded mean oscillation (BMO) in \mathbb{R}^n , if

$$\frac{1}{|Q|} \int_{Q} \left| u(x) - \frac{1}{|Q|} \int_{Q} u(x) \, dx \right| \, dx \leq K$$

for every cube $Q \subset \mathbb{R}^n$ and some constant K. The notations

$$u_{Q} = \oint_{Q} u(x) dx = \frac{1}{|Q|} \int_{Q} u(x) dx \quad \text{with} \quad |Q| = \int_{Q} dx$$

will be used. On the space of BMO-functions modulo constants a norm can be defined by

$$\|u\|_{*} = \sup_{Q \in \mathbb{R}^{n}} \int_{Q} |u(x) - u_{Q}| \, dx \,. \tag{1.1}$$

With this norm BMO/R is a Banach space. The space of BMO-functions was introduced by John and Nirenberg [8]. We will make use of their fundamental lemma:

LEMMA 1. Assume that $u \in BMO$. Then, if $\mu(\sigma) = |\{x \in Q : |u(x) - u_Q| > \sigma\}|$ is the measure of the set of points in the cube Q where $|u(x) - u_Q| > \sigma$, we have

$$\mu(\sigma) \leq a \ e^{-b\sigma/\|u\|_*} |Q|, \qquad (1.2)$$

where a and b are constants depending on n only.

BMO-functions have been used in many different contexts, first in a paper of John on rotation and strain [7] and at the same time by Moser [9] in his work on the continuity of solutions of elliptic differential equations. Later on applications arose in connection with singular integral operators (Stein [12]) and as spaces of interpolation (Stampacchia [11], Stein and Zygmund [13]). Most recently, Fefferman and Stein[3] characterized the space of BMO-functions as the dual of the Hardy space H^1 .

It seems that BMO-functions also have their place in the theory of quasiconformal mappings. We propose to show that the logarithm of the Jacobian determinant of a

quasiconformal mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ is in BMO. We then proceed to study transformations of BMO-functions by quasiconformal mappings. It turns out that a quasiconformal mapping of \mathbb{R}^n onto itself induces a continuous bijective isomorphism $\varphi: u \to$ $\rightarrow u \circ f^{-1}$ of BMO/ \mathbb{R} . Moreover this situation is in a certain way typical for quasiconformal mappings: If $\varphi: u \to u \circ f^{-1}$ is a continuous bijective isomorphism of BMO/ \mathbb{R} which is induced by a homeomorphism f of \mathbb{R}^n satisfying certain regularity conditions, then f is quasiconformal.

2. The Jacobian of a Quasiconformal Mapping

For our considerations we adopt the so called analytic definition of quasiconformality. A function $f: G \to \mathbb{R}^n$ defined in a domain $G \subset \mathbb{R}^n$ is said to be absolutely continuous on lines (ACL), if it is continuous and if for each interval I = $= \{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i\} \subset G f$ is absolutely continuous on almost all line segments in I, parallel to the coordinate axes. The partial derivatives of an ACL-function f exist a.e. and the Jacobian matrix of f at x will be denoted by F(x), its determinant by $J_f(x)$. By definition a K-quasiconformal mapping is a homeomorphism $f: G \to \mathbb{R}^n$ such that $f \in ACL$, f is totally differentiable a.e. and

$$\sup_{\xi \in \mathbb{R}^n, |\xi|=1} |F(x)\xi|^n \leq K J_f(x) \quad \text{a.e.}$$
(2.1)

According to a theorem of Väisälä [14], in this definition the regularity conditions $f \in ACL$ and f differentiable a.e. can be replaced by the single hypothesis, that f has generalized derivatives which are locally L^n -integrable.

THEOREM 1. If f is a quasiconformal mapping of \mathbb{R}^n onto itself with Jacobian determinant J_f then $\log J_f \in BMO$.

The proof of Theorem 1 is based on a converse to the lemma of John and Nirenberg (Lemma 3) and on the following result due to Gehring [5]:

LEMMA 2. Assume that f is a K-quasiconformal mapping of G onto $G' \subset \mathbb{R}^n$ and that Q is a cube in the domain G with

dia
$$Q' < \text{dist}(Q', \partial G')$$
 (2.2)

(Q' = fQ). Then there exist constants c and p, p > n, which depend on K and n only, such that

$$\left(\oint_{Q} J_{f}^{p/n} dx \right)^{n/p} \leq c \oint_{Q} J_{f} dx \,. \tag{2.3}$$

We set

$$L_f(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|}.$$

Since f is K-quasiconformal, inequality (2.1) and the total differentiability imply $L_f^n \leq KJ_f$ a.e. and $J_f \leq L_f^n$ a.e. Hence according to Lemma 4 in [3] there exists a constant c_0 (depending on K and n) such that for every cube $Q \subset G$ with dia $Q' < dist(Q', \partial G')$

$$\oint_{Q} J_f \, dx \leqslant c_0 \left(\oint_{Q} J_f^{1/n} \, dx \right)^n, \tag{2.4}$$

If Q is such a cube (satisfying (2.3)), then (2.4) holds for any cube contained in Q and Lemma 3 in [5] shows that for some constants c and p, p > n,

$$\left(\int\limits_{Q} J_f^{p/n} \, dx\right)^{n/p} \leqslant c \int\limits_{Q} J_f \, dx \, .$$

For later reference let us note a simple consequence of this lemma (cf. [5] Theorem 2)

COROLLARY. Assume that f is a K-quasiconformal mapping of G onto $G' \subset \mathbb{R}^n$ and that Q is a cube in the domain G with dia $Q' < \text{dist } (Q', \partial G')$. Then

$$\frac{|A'|}{|Q'|} \leq c \left(\frac{|A|}{|Q|}\right)^{(p-n)/p}$$
(2.5)

for every measurable set $A \subset Q$ with image A' = fA. (As above |A| stands for the n-dimensional measure of the set A.)

If A is a measurable set, $A \subset Q$, then

$$\frac{|A'|}{|A|} = \int_{A} J_f \, dx \leq \left(\int_{A} J_f^{p/n} \, dx \right)^{n/p}$$
$$\leq \left(\frac{|A|}{|Q|} \right)^{-n/p} \left(\int_{Q} J_f^{p/n} \, dx \right)^{n/p} \leq c \left(\frac{|A|}{|Q|} \right)^{-n/p} \frac{|Q'|}{|Q|}$$

by Lemma 2 and Hölder's inequality.

Let us remark that for the case of plane quasiconformal mappings results similar to Lemma 2 have previously been established by Bojarski [1] and by Gehring and Reich [6].

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LEMMA 3. $f = \log u \in BMO$ if and only if for all cubes $Q \subset \mathbb{R}^n$

$$\left(\int_{Q} u^{a} dx\right)^{1/a} \leq k \left(\int_{Q} u^{-b} dx\right)^{-1/b}$$
(2.6)

for some positive constants a, b and k.

The fact, that inequality (2.6) is a consequence of Lemma 1 has already been pointed out by John and Nirenberg. (The result has been stated in this form by Moser [9].) Let us therefore assume that (2.6) holds for $u=e^{f}$. It is well known that

$$M_t = M_t(u) = \begin{cases} \left(\oint_Q u^t \, dx \right)^{1/t} & t \neq 0 \\ \exp \oint_Q \log u \, dx & t = 0 \end{cases}$$

is a monotone increasing function of t. Our assumption therefore implies $M_s \leq KM_0$ and $M_0 \leq KM_{-s}$ for $s = \min(a, b)$. If we set

$$Q_1 = \left\{ x \in Q : \log u(x) \ge \int_Q \log u \, dx \right\}$$

and $Q_2 = Q \setminus Q_1$ we obtain the inequalities

$$|Q|^{-1} \int_{Q_1} u^s \, dx \leq \oint_Q u^s \, dx = M_s^s \leq k^s M_0^s$$

and

$$|Q|^{-1} \int_{Q_2} u^{-s} dx \leq k^{-s} M_0^{-s}.$$

After adding these two inequalities and inserting the expressions

$$f = \log u$$
 and $f_Q = \oint_Q f \, dx = \log M_0$

we have

$$\int_{Q_1} e^{s(f-f_Q)} dx + \int_{Q_2} e^{-s(f-f_Q)} dx = \int_{Q} e^{s|f-f_Q|} dx \leq (k^s + k^{-s}) |Q|.$$

Finally,

$$\exp \oint_{Q} s |f - f_{Q}| \, dx \leq \oint_{Q} e^{s|f - f_{Q}|} \, dx$$

upon applying Jensen's inequality. This shows that $f \in BMO$ with

$$\|f\|_{*} \leq s^{-1} \log (k^{s} + k^{-s}). \tag{2.7}$$

We shall also need a distortion lemma for quasiconformal mappings, to the effect that the image of a cube can still be compared with a cube. This kind of result is typical for the geometric theory of quasiconformal mappings. We choose a formulation, which is particularly suited for our purposes.

LEMMA 4. Let $f: G \to \mathbb{R}^n$ be a K-quasiconformal mapping. There exists a constant k (which depends on K and n) such that to every cube $P' \subset G' = fG$ with

 $dist(P', \partial G') > 2 k dia P'$

there exists a cube $Q \subset G$ with $fQ = Q' \supset P'$, dia $Q' < \text{dist}(Q', \partial G')$ and

$$|Q'| \le k^n n^{n/2} |P'|.$$
(2.8)

The proof is based on the geometric definition of quasiconformality, according to which a homeomorphism $f: G \to \mathbb{R}^n$ is K-quasiconformal if and only if

 $\operatorname{mod} R' \leq K^{1/(n-1)} \operatorname{mod} R$

for every ring $R \subset G$. We refer the reader to the literature (see e.g. [10], [2]) for the precise definitions and for a proof of the equivalence of the analytic and geometric definitions of quasiconformality.

Using preliminary translations, we can assume that the given cube P' is centered at 0 and that f(0)=0. Consider the spherical ring $R' \subset G'$ with complementary components

 $C'_0 = \{z : |z| \le r' = 2^{-1} \operatorname{dia} P'\}$ and $C'_1 = \{z : |z| \ge s' = k2^{-1} \operatorname{dia} P'\}.$

The constant k > 1 will be determined later on. The modulus of the ring R' is given by

$$\operatorname{mod} R' = \log \frac{s'}{r'} = \log k \,. \tag{2.9}$$

We set $s = \inf_{z \in C'_1} |f^{-1}(z)|$ and $r = \sup_{z \in C'_0} |f^{-1}(z)|$ and observe that $|f^{-1}(z)| \leq r$ for all $z \in P'$.

The inner complementary component C_0 of $R = f^{-1}R'$ contains 0 and a point x_0 with $|x_0| = r$, the outer component C_1 a continuum connecting ∞ with a point x_1 , $|x_1| = s$. According to a theorem of Teichmüller and its space analogue (see [4], [2], [10])

$$\mod R \leq \log\left(\lambda^2 \left(\frac{s}{r} + 1\right)\right) \tag{2.10}$$

for some constant λ which depends on *n* only. Since K-quasiconformal mappings satisfy

$\operatorname{mod} R' \leq K_0 \operatorname{mod} R$

with $K_0 = K^{1/(n-1)}$, we then have by (2.9) and (2.10)

$$\log k \leq K_0 \log \left(\lambda^2 \left(\frac{s}{r} + 1 \right) \right)$$

which is equivalent to

$$k^{-K_0} \leq \lambda^2 \left(\frac{s}{r}+1\right).$$

This shows that $s/r \ge n^{1/2}$ if we choose $k = (\lambda^2(1+n^{1/2}))^{K_0}$. In this situation any cube $Q \subset G$ centered at the origin with side length 2r (and diameter $2r n^{1/2}$) satisfies the requirements of the lemma: The construction shows that dia Q' < k dia P' and

$$|Q'| \leq k^n n^{n/2} |P'|$$

since $|z| \leq k2^{-1} \operatorname{dia} P'$ for all $z \in Q'$. If we further assume that $\operatorname{dist}(P', \partial G') > 2k \operatorname{dia} P' = 4s'$, then it is clear that $\operatorname{dia} Q' \leq 2s' \leq \operatorname{dist}(Q', \partial G')$.

LEMMA 5. The Jacobian determinant $J = J_f$ of a K-quasiconformal mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\oint_{Q} J \, dx \leqslant c_1 \left(\oint_{Q} J^{-b} \, dx \right)^{-1/b} \tag{2.11}$$

for all cubes $Q \subset \mathbb{R}^n$. The constants b and c_1 depend on n and K only.

The inverse f^{-1} of a K-quasiconformal mapping is K^{n-1} -quasiconformal and its determinant J^{-1} satisfies Gehring's inequality (see (2.3)):

$$\left(\oint_{\mathbf{P}'} J^{-p/n} \, dy\right)^{n/p} \leq c \oint_{\mathbf{P}'} J^{-1} \, dy \tag{2.12}$$

for every cube $P' \subset \mathbb{R}^n$. To any cube $Q \subset \mathbb{R}^n$ let us choose in accordance with Lemma 4 a cube P' with $P = f^{-1} P' \supset Q$ and

$$|P| \le k^n n^{n/2} |Q| = k' |Q|. \tag{2.13}$$

A transformation of variables for the integrals in inequality (2.12) then shows that

$$\left(|P'|^{-1} \int_{P} J^{-p/n} J \, dx\right)^{n/p} \leq c |P| |P'|^{-1}.$$

Because $|P'| = \int_P J dx$ this can be rewritten in the form

$$\left(\int\limits_{P} J^{(n-p)/n} dx\right)^{n/p} \leq c |P| \left(\int\limits_{P} J dx\right)^{(n-p)/p}$$

and together with inequality (2.13) this leads to

$$\left(\oint_{Q} J^{(n-p)/n} dx\right)^{n/(n-p)} \ge (ck')^{p/(n-p)} \oint_{Q} J dx.$$

If the two Lemmata 3 and 5 are combined, a bound for $\|\log J\|_*$ can be given by

$$\|\log J\|_{*} \leq \frac{n}{p-n} \log((ck')^{p/n} + (ck')^{-p/n})$$

provided that $p \leq 2n$ and by

$$\|\log J\|_{*} \leq \log((ck')^{p/(p-n)} + (ck')^{p/(n-p)})$$

otherwise.

Remark 1. By definition, a K-quasiconformal mapping $f = (f_1, ..., f_n)$ of \mathbb{R}^n onto itself satisfies for i = 1, ..., n

$$K^{-n+1}J_f \leq |\operatorname{grad} f_i|^n \leq KJ_f$$
 a.e. in \mathbb{R}^n .

Hence there exist $g_i \in L^{\infty}(\mathbb{R}^n)$, $||g_i||_{\infty} \leq (n-1) \log K$, such that $n \log |\operatorname{grad} f_i| = \log J_f + g_i$ (i = 1, ..., n). But functions in $L^{\infty}(\mathbb{R}^n)$ are also in BMO and therefore $\log |\operatorname{grad} f_i| \in BMO$ (i = 1, ..., n).

Remark 2. Local variants of Theorem 1 can be obtained. If $f: G \to G'$ is a quasiconformal mapping and if $Q \subset G$ is a cube such that both dist $(Q, \partial G)$ and dist $(Q', \partial G')$ are big enough, then $\log J_f$ considered as a function in Q is in BMO.

3. The Invariance of the Space BMO

THEOREM 2. If f is a K-quasiconformal mapping of \mathbb{R}^n onto itself, then $\varphi: u \to u' = u \circ f^{-1}$ is a bijective isomorphism of BMO and

$$\|u'\|_* \leqslant C \|u\|_* \tag{3.1}$$

for all $u \in BMO$, where C is a constant depending on K and n only.

We note that since the inverse of a quasiconformal mapping is also a quasiconformal mapping, all that has to be shown is inequality (3.1). It then follows directly from the definition that φ is an isomorphism of BMO.

For the proof of Theorem 2 we assume that $u \in BMO$ and set $u' = u \circ f^{-1}$. To a given cube P' we determine Q as in Lemma 4 such that $P' \subset Q'$ and $|Q'| \leq k^n n^{n/2} |P'|$. The

set $A_{\sigma} = \{x \in Q : |u(x) - u_Q| > \sigma\}$ is mapped onto the set $A'_{\sigma} = \{z \in Q' : |u'(z) - u_Q| > \sigma\}$ and by the corollary to Lemma 2 one knows that

$$\frac{|A'_{\sigma}|}{|Q'|} \leq c \left(\frac{|A_{\sigma}|}{|Q|}\right)^{(p-n)/p}$$

Because of Lemma 1

$$|A_{\sigma}| \leq a e^{-b\sigma/||u||_*} |Q|$$

hence

$$\frac{A'_{\sigma}|}{|Q'|} \leq c a^{(p-n)/p} \exp\left(\frac{-b\sigma(p-n)}{\|u\|_* p}\right).$$

An integration of this inequality with respect to σ shows that

$$\oint_{Q'} |u'(z) - u_Q| dz = |Q'|^{-1} \int_{0}^{\infty} |A'_{\sigma}| d\sigma \leq c a^{(p-n)/p} b^{-1} p (p-n)^{-1} ||u||_{*}$$

and in combination with inequality (2.8) of Lemma 4

$$\oint_{P'} |u'(z) - u_Q| \, dz \leq k^n n^{n/2} \oint_{Q'} |u'(z) - u_Q| \, dz \leq \text{const} \, ||u||_*$$

One is left with the task of replacing u_Q by

$$u'_{P'} = \oint_{P'} u'(z) dz,$$

but

$$|u'_{P'}-u_{Q}| = \left| \oint_{P'} (u'(z)-u_{Q}) dz \right|,$$

so

$$\oint_{P'} |u'(z) - u'_{P'}| dz \leq |u'_{P'} - u_Q| + \oint_{P'} |u'(z) - u_Q| dz \leq 2 \oint_{P'} |u'(z) - u_Q| dz.$$

This shows that $||u'||_* \leq C ||u||_*$ with $C = 2k^n n^{n/2} c a^{(p-n)/p} b^{-1} p(p-n)^{-1}$.

THEOREM 3. Assume that f is a (orientation preserving) homeomorphism of \mathbb{R}^n onto itself, that $f \in ACL$ and that f is totally differentiable a.e. If the induced mapping $\varphi: u \rightarrow u' = u \circ f^{-1}$ is a bijective isomorphism of BMO and if

$$\|u'\|_* \leq C \|u\|_* \quad \text{for all} \quad u \in BMO \tag{3.1}$$

then f is a quasiconformal mapping.

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Let us make precise that the hypothesis of φ being a bijective isomorphism of BMO is meant to include the assumption that f and its inverse are absolutely continuous with respect to *n*-dimensional measure. If this were not the case, the isomorphism φ could not be defined properly: if the zero set N were mapped onto a set N' of positive measure, then both u'=0 and $u''=\chi_{N'}$, the characteristic function of N', were in BMO and both would satisfy

 $u'=u\circ f^{-1} \qquad u''=u\circ f^{-1}$

with u=0 a.e. in \mathbb{R}^n , $u \in BMO$.

Our first aim is to construct suitable functions $u \in BMO$.

LEMMA 6. (John-Nirenberg).

$$u(x) = \log^{+} \frac{1}{|x|} = \begin{cases} \log \frac{1}{|x|} & |x| \le 1\\ 0 & |x| \ge 1 \end{cases}$$

is in BMO.

For a proof see [9].

LEMMA 7. Assume that g is a continuous function defined on \mathbf{R} with

$$k_{g} = \sup_{x \in \mathbf{R}} |g(x)| + \sup_{x, y \in \mathbf{R}} |g(x) - g(y)| \left(1 + \log^{+} \frac{1}{|x - y|}\right) < \infty.$$
(3.2)

If $u \in BMO(\mathbb{R}^n)$ and if

$$k_{u} = \sup_{|\mathcal{Q}| \ge 1} \left| \int_{\mathcal{Q}} u(x) \, dx \right| < \infty \, ,$$

then $v(x, y) = u(x) g(y) \in BMO(\mathbb{R}^{n+1})$.

Let $Q_r \subset \mathbb{R}^n$ denote the cube with side length r, centred at the origin. If $u \in BMO$, then for $r \leq 1$

$$|u_{Q_1} - u_{Q_r}| \le (2^n + 1) \left(1 - \frac{\log r}{\log 2} \right) ||u||_*$$
(3.3)

This can be seen as follows: Set $u_s = u_{Q_{2-s}} s = 0, 1, ...$ Then

$$|u_{s}-u_{s-1}| = 2^{ns} \int_{Q_{2}-s} |u_{s}-u_{s-1}| dx$$

$$\leq 2^{ns} \int_{Q_{2}-s+1} |u(x)-u_{s-1}| dx + ||u||_{*} \leq (2^{n}+1) ||u||_{*},$$

$$|u_s - u_{Q_1}| \leq \sum_{k=1}^{s} |u_k - u_{k-1}| \leq s(2^n + 1) ||u||_*$$

For $r = 2^{-s}$ this is equivalent with

$$|u_{Q_r}-u_{Q_1}| \leq (2^n+1) \frac{-\log r}{\log 2} ||u||_*.$$

For arbitrary $r, 0 < r \le 1$, inequality (3.3) can now easily be derived.

A cube $Q \subset \mathbb{R}^{n+1}$ with sides parallel to the coordinate axes can be represented as a direct product $Q = P \times S$ of cubes $P \subset \mathbb{R}^n$, $S \subset \mathbb{R}$. Set $a_Q = u_P g_0$, where g_0 is the value of g at the center of S. Then

$$\oint_{Q} |v(x, y) - a_{Q}| dx dy \leq \oint_{S} |g(y)| dy \oint_{P} |u(x) - u_{P}| dx + |u_{P}| \oint_{S} |g(y) - g_{0}| dy.$$

If $|Q| \ge 1$ this gives immediately

$$\int_{Q} |v(x, y) - a_{Q}| \, dx \, dy \leq k_{g} (||u||_{*} + k_{u}).$$

If |Q| < 1, we make use of (3.3) and (3.2) to conclude that

$$|u_P| \leq (2^n+1) \left(1 - \frac{\log |P|}{n \log 2} \right) + k_u ||u||_*$$

and

$$\int_{S} |g(y) - g_0| \, dy \leq k_g (1 - n^{-1} \log |P|)^{-1}.$$

Therefore

$$\int_{Q} |v(x, y) - a_{Q}| \, dx \, dy \leq 2k_{g} (k_{u} + 2^{n+1} ||u||_{*})$$

for any cube with sides parallel to the coordinate axes. If $Q \subset \mathbb{R}^{n+1}$ is an arbitrary cube, there exists a cube $Q_0 \supset Q$ with sides parallel to the coordinate axes and with

$$|Q_0| \leq (n+1)^{(n+1)/2} |Q|.$$

So the estimate

$$\int_{Q} |v - v_Q| \, dx \, dy \leq 2 \int_{Q} |v - a_{Q_0}| \, dx \, dy \leq 2 \, (n+1)^{(n+1)/2} \int_{Q_0} |v - a_{Q_0}| \, dx \, dy$$

for the mean oscillation over Q shows that

$$\|v\|_{*} \leq 4(n+1)^{(n+1)/2} k_{g} (k_{u} + 2^{n+1} \|u\|_{*}).$$

As an application set $u(x) = \log^+ 1/|x|$, $x \in \mathbb{R}$ and let g(y) be the piecewise linear, continuous odd function defined for $y \in \mathbb{R}$ by

$$g(y) = \begin{cases} 1 - |y - 1| & 0 \le y \le 2, \\ 0 & 2 \le y, \\ -g(-y) & y \le 0. \end{cases}$$

Since $|g(x) - g(y)| \le \min\{2, |x - y|\}$, g satisfies the assumptions of Lemma 7 (with $k_g \le 3$). From the Lemmata 6 and 7 we conclude that

$$v(x_1, x_2) = g(x_2) \log^+ \frac{1}{|x_1|}$$

is in BMO. For dimensions n>2 let us define $v \in BMO$ by

$$v(x) = \log^+ 1/|x_1| h(x_2) \dots h(x_{n-1}) g(x_n)$$

with

$$h(x) = \begin{cases} 1 & |x| \leq 1, \\ 2 - |x| & 1 \leq |x| \leq 2, \\ 0 & |x| \geq 2, \end{cases}$$

v then has compact support and $v(x) = g(x_n) \log^+ 1/|x_1|$ for $|x_i| \le 1$, i = 1, ..., n. Finally, for r > 0 let v_r be defined by

$$v_r(x) = v\left(\frac{x}{r}\right).$$

Certainly $||v_r||_* = ||v||_*$, since the space of BMO-functions is invariant under dilations.

With $a = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n_+$ (i.e. $\alpha_i > 0$, i = 1, ..., n) we associate the sets $U_{a, r} = \{x: |x_1| \leq \alpha_i r, i = 1, ..., n\}$ and the functions

$$\varphi_{a,r} = \begin{cases} |U_{a,r}|^{-1} v_r(x) & x \in U_{a,r} \\ 0 & \text{otherwise} \end{cases}$$

and

 $\psi_{a,r} = |\varphi_{a,r}|.$

LEMMA 8. If $h \in L^1(\mathbb{R}^n)$, then there exists a sequence r_j converging to 0 such that a.e. in \mathbb{R}^n

$$\lim_{j\to\infty}\varphi_{a,r_j}*h(t)=\lim_{j\to\infty}\int_{\mathbb{R}^n}\varphi_{a,r_j}(x)h(t-x)\,dx=0$$

and

$$\lim_{j\to\infty}\psi_{a,r_j}*h(t)=c_ah(t)$$

with

$$c_{a} = \int_{\mathbb{R}^{n}} \psi_{a,r} \, dx = \int_{U_{a,1}} |v(x)| \, dx$$

 c_a is a continuous function of $a \in \mathbb{R}^n_+$. For $a = (\alpha_1, ..., \alpha_{n-1}, 1), \alpha_i \leq 1, c_a$ can easily be calculated:

$$c_a = \frac{1}{2} (1 - \log \alpha_1). \tag{3.4}$$

For the proof of Lemma 8 consider the differences

$$d_{1}(t) = \varphi_{a,r} * h(t) - 0 = \int_{\mathbb{R}^{n}} \varphi_{a,r}(x) \left(h(t-x) - h(t)\right) dx$$

and

$$d_{2}(t) = \psi_{a,r} * h(t) - c_{a}h(t) = \int_{\mathbb{R}^{n}} \psi_{a,r}(x, (h(t-x) - h(t))) dx.$$

They satisfy

$$\lim_{r \to 0} \int_{\mathbb{R}^n} |d_k(t)| \, dt \leq \lim_{r \to 0} \int_{\mathbb{R}^n} \psi_{a,r}(x) \int_{\mathbb{R}^n} |h(t-x) - h(t)| \, dt \, dx = 0 \, .$$

k = 1, 2 because

$$\lim_{x\to 0}\int_{\mathbb{R}^n}|h(t-x)-h(t)|\,dt=0$$

and $\psi_{a,r}$ has its support in $U_{a,r}$. For some sequence r_j with $\lim_{j\to\infty} r_j=0$ the differences d_1 and d_2 will therefore converge pointwise a.e.

For any rotation ρ of \mathbb{R}^n set $\varphi_{\rho,a,r}(x) = \varphi_{a,r}(\rho^{-1}x)$, $U_{\rho,a,r} = \rho^{-1}(U_{a,r})$ and $v_{\rho,r}(x) = v_r(\rho^{-1}x) = v(\rho^{-1}x/r)$. Observe that $||v_{\rho,r}||_* = ||v||_*$, since BMO is invariant under rotations. We think of ρ as being given by an element in O(n), the group of orthogonal $n \times n$ -matrices. As an immediate consequence of Lemma 8 let us note:

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COROLLARY. Let $\{\varrho_i\}$ and $\{a_m\}$ be countable dense subsets of O(n) and \mathbb{R}^n_+ respectively. If $h \in L^1(\mathbb{R}^n)$, then for any pair ϱ_i , a_m there exists a sequence $\{r_j\}$ with $\lim_{i \to \infty} r_i = 0$ such that

$$\lim_{j \to \infty} \varphi_{\varrho_l, a_m, r_j} * h(t) = 0 \tag{3.5}$$

and

$$\lim_{j \to \infty} \psi_{\boldsymbol{\varrho}_i, a_m, r_j} * h(t) = c_{a_m} h(t)$$
(3.6)

for all $t \in \mathbb{R}^n \setminus N$, where N is a set of measure zero which is independent of the pair ϱ_i , a_m .

The proof of Theorem 3 consists in showing that $\sup_{\xi \in \mathbb{R}^n, |\xi|=1} |F(x)\xi|^n \le \le K \det F(x)$ a.e. in \mathbb{R}^n . (F(x) is the Jacobian matrix of f at x.) This clearly is a local property. Based on our hypothesis and on the corollary to Lemma 8 we can assume that at x=0 the following conditions are satisfied:

- i) f is (totally) differentiable
- ii) $J_f = \det F \neq 0$, in view of the remark following Theorem 3

iii) (3.5) and (3.6) hold for
$$h(x) = \begin{cases} J_f(x) & |x| < 1\\ 0 & |x| \ge 1 \end{cases}$$

F can be written in the form $F = \rho D\sigma$, where $\rho, \sigma \in O(n)$ and $D = \begin{pmatrix} \lambda_1 & 0\\ \ddots & \lambda_n \end{pmatrix}$ is a di-

agonal matrix with $\lambda_1 \ge \lambda_2 \dots \ge \lambda_n > 0$ (this is true for any $n \times n$ -matrix M with det $M \ne 0$). Let us exclude the case det $\sigma = -1$ by possibly interchanging the order of the coordinates. If we compose f with the rotation σ^{-1} , then the resulting mapping $g = \sigma^{-1} \circ f$ still satisfies the assumptions of Theorem 3 and the three additional conditions above. Note that $J_g = J_f$ and that the Jacobian matrix of g at 0 is $G = \rho D$. The same is true if we consider the mapping cf, c > 0, instead of f (with $J_{cf} = c^n J_f$). There-

fore we can assume without loss of generality, that $F = \rho D$ with $D = \begin{pmatrix} \lambda_1 & 0 \\ \ddots & \\ 0 & \lambda_n \end{pmatrix}$ and

 $\lambda_1 \ge \lambda_2 \dots \ge \lambda_n = 1$. We then have to show that $\lambda_1^n \le K \lambda_1 \dots \lambda_n$.

With this in mind let us choose ϱ_i and $a_m = (\alpha_{m1}, ..., \alpha_{mn})$ in such a way that

$$|\lambda_k \alpha_{mk} - 1| < \varepsilon \qquad k = 1, \dots, n \tag{3.7}$$

$$|c_{a_m} - \frac{1}{2}(1 - \log \alpha_{m1})| < \varepsilon \tag{3.8}$$

(cf. (3.4)) and such that for r small enough, say $r < \delta_1$, $U'_r = f U_{\varrho_i, a_m, r}$ contains the cube $S = \{z : |z_i| \le r(1-\varepsilon)\}$ and is contained in the cube $Q = \{z : |z_i| \le r(1+\varepsilon)\}$. We remind that $U_{\varrho,a,r}$ was defined by $U_{\varrho,a,r} = \varrho^{-1}\{x : |x_i| \le r\alpha_i, i = 1, ..., n\}$.

By the main hypothesis (3.1) $w_r = v_{\varrho,r} \circ f^{-1}$ is in BMO and $||w_r||_* \leq C ||v||_*$. So with

$$w_{Q} = \oint_{Q} w_{r}(z) dz$$

$$\int_{U'r} |w_{r}(z) - w_{Q}| dz \leq |S|^{-1} \int_{Q} |w_{r}(z) - w_{Q}| dz$$

$$\leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{n} \oint_{Q} |w_{r}(z) - w_{Q}| dz \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{n} C ||v||_{*}.$$

In this inequality we can replace w_Q by the mean value

$$\tilde{v}_{r} = \int_{U'r} w_{r}(z) dz$$

if we instead write

$$\oint_{U'_r} |w_r(z) - \tilde{v}_r| \, dz \leq 2 \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n C \, ||v||_* \, .$$

Due to the absolute continuity of f with respect to n-dimensional Lebesgue measure

$$\tilde{v}_{r} = \frac{|U_{\varrho_{i}, a_{m}, r}|}{|U_{r}'|} \int_{U_{\varrho_{i}, a_{m}, r}} v_{\varrho_{i}, r}(x) J_{f}(x) dx$$
$$= \frac{|U_{\varrho_{i}, a_{m}, r}|}{|U_{r}'|} \varphi_{\varrho_{i}, a_{m}, r} * h(0)$$

so by the corollary to Lemma 6

$$\lim_{j\to\infty}\tilde{v}_{r_j}=0.$$

Hence there exists $\delta_2 > 0$ ($\delta_2 \leq \delta_1$) such that for $r = r_j < \delta_2$

$$\int_{U'_r} |w_r(z)| \, dz \leq \varepsilon + 2\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n C \, \|v\|_* \,. \tag{3.9}$$

On the other side

$$\int_{U'_{r}} |w_{r}(z)| dz = \frac{|U_{\varrho_{i}, a_{m}, r}|}{|U'_{r}|} \psi_{\varrho_{i}, a_{m}, r} * h(0)$$

so by (3.6)

$$\lim_{j \to \infty} \oint_{U'_{r_j}} |w_{r_j}(z)| \, dz = (J_f(0))^{-1} \, c_{a_m} J_f(0) = c_{a_m}.$$
(3.10)

Combining the two results (3.9) and (3.10) with (3.8) we conclude that for j big enough

$$\frac{1}{2}(1-\log \alpha_{m1})-\varepsilon \leqslant \int_{U'r_j} |w_{r_j}(z)| dz \leqslant \varepsilon + 2\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n C ||v||_*.$$

Together with (3.7) this implies

$$1 + \log \lambda_1 \leq 4 C \|v\|_*$$

since $\varepsilon > 0$ was arbitrary. The inequality $\lambda_1^n \leq K \lambda_1 \dots \lambda_n$ therefore holds with $K = e^{(n-1)(4C ||v||_* - 1)}$.

4. A Local Version

The object of this section is a local version of the Theorems 2 and 3. If G is a domain in \mathbb{R}^n we denote by BMO (G) the subspace of BMO consisting of all functions $u \in BMO$ with support in G (the support of a locally integrable function $u \in L^1_{loc}(\mathbb{R}^n)$ is the complement of the largest open set $O \subset \mathbb{R}^n$ with u(x) = 0 a.e. on O).

LEMMA 9. If
$$u \in L^1_{loc}(\mathbb{R}^n)$$
 and if $\operatorname{supp} u \subset G$, then
 $\|u\|_* \leq 4 \sup \oint_P |u(x) - u_P| dx$,

where the supremum is extended over all cubes P with dia $P \leq 4n^{1/2}$ dia G.

G is contained in a ball B with radius dia G. If $Q \cap G \neq \emptyset$ for some cube with dia $Q \ge 4n^{1/2}$ dia G, then there exists a cube $P \subset Q$ with side length $n^{-1/2}$ dia P = 2 dia B such that $P \cap B = Q \cap B \supset Q \cap G$. With $N = \{x \in P : u(x) = 0\}, |N| \ge |P| - |G| \ge \ge (1 - 2^{-n})|P|$, we obtain

$$\int_{P} |u - u_{P}| dx = |P|^{-1} \int_{N} |u_{P}| dx + |P|^{-1} \int_{P \setminus N} |u - u_{P}| dx$$

which shows that

$$|u_P| \leq (1-2^{-n})^{-1} \oint_P |u-u_P| dx.$$

Finally

$$\begin{split} \oint_{Q} |u - u_{Q}| \, dx &\leq 2 \oint_{Q} |u - u_{P}| \, dx \\ &\leq 2 |Q|^{-1} \int_{Q \setminus P} |u_{P}| \, dx + 2 |Q|^{-1} \int_{P} |u - u_{P}| \, dx \\ &\leq 2 \left(1 - 2^{-n}\right)^{-1} \oint_{P} |u - u_{P}| \, dx \end{split}$$

and the proof is complete.

THEOREM 4. Assume that $f: G \to \mathbb{R}^n$ is a (orientation preserving) homeomorphism, $f \in ACL$, and that f is differentiable a.e. Then f is a quasiconformal mapping if and only if every point $x \in G$ has a neighbourhood U such that $\varphi: u \to u' = u \circ f^{-1}$ is an isomorphism of BMO (U) onto BMO (fU) which satisfies

 $\|u'\|_* \leqslant C_0 \|u\|_*$

for all $u \in BMO(U)$ with a fixed constant C_0 independent of U.

The proof for the quasiconformality of a homeomorphism $f: G \to \mathbb{R}^n$ satisfying all the above hypotheses is contained in the proof of Theorem 3. In order to show that a quasiconformal mapping gives rise to local isomorphisms of BMO (U) onto BMO (U'), U' = fU, the full strength of the Lemmata 2 and 4 has to be used.

For $x \in G$ we choose a neighbourhood U in such a way that

 $4n^{1/2}(1+2k) \operatorname{dia} U' < \operatorname{dist} (U', \partial G'),$

where k is the constant of Lemma 4. If P' is a cube with dia $P' \leq 4n^{1/2}$ dia U' and with $P' \cap U' \neq \emptyset$, then

 $\operatorname{dist}(P', \partial G') \ge \operatorname{dist}(U', \partial G') - \operatorname{dia} P' > \operatorname{dia} P'(1+2k) - \operatorname{dia} P' = 2k \operatorname{dia} P'.$

Hence by Lemma 4 there exists a cube $Q \subset G$ with $fQ = Q' \supset P'$, $|Q'| \subseteq k^n n^{n/2} |P'|$ and with

$$\operatorname{dist}(Q', \partial G') > \operatorname{dia} Q'. \tag{4.1}$$

For a given function $u \in BMO(U)$ with $u' = u \circ f^{-1}$ we can then proceed as in the proof of Theorem 2. Condition (4.1) ensures the validity of Lemma 2. It follows (see (2.11)) that

$$\int_{P'} |u'(z) - u'_{P'}| \, dz \leq C \, \|u\|_*$$

and this inequality holds for all P' with dia $P' \leq 4n^{1/2}$ dia U'. Therefore by Lemma 9

$$||u'||_* \leq 4C ||u||_* = C_0 ||u||_*.$$

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