# Functions of Bounded mean Oscillation and Quasiconformal Mappings

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## 1. Introduction

A locally integrable real valued function $u$ is said to be of bounded mean oscillation (BMO) in $\mathbf{R}^n$, if

$$\frac{1}{|Q|}\int_Q \left| u(x) - \frac{1}{|Q|}\int_Q u(x)\,dx \right| dx \leqslant K$$

for every cube $Q \subset \mathbf{R}^n$ and some constant $K$. The notations

$$u_Q = \fint_Q u(x)\,dx = \frac{1}{|Q|}\int_Q u(x)\,dx \quad \text{with} \quad |Q| = \int_Q dx$$

will be used. On the space of BMO-functions modulo constants a norm can be defined by

$$\|u\|_* = \sup_{Q \subset \mathbf{R}^n} \fint_Q |u(x) - u_Q|\,dx. \tag{1.1}$$

With this norm BMO/R is a Banach space. The space of BMO-functions was introduced by John and Nirenberg [8]. We will make use of their fundamental lemma:

LEMMA 1. *Assume that* $u \in$BMO. *Then, if* $\mu(\sigma) = |\{x \in Q : |u(x) - u_Q| > \sigma\}|$ *is the measure of the set of points in the cube* $Q$ *where* $|u(x) - u_Q| > \sigma$, *we have*

$$\mu(\sigma) \leqslant a\, e^{-b\sigma/\|u\|_*}\, |Q|, \tag{1.2}$$

*where a and b are constants depending on n only.*

BMO-functions have been used in many different contexts, first in a paper of John on rotation and strain [7] and at the same time by Moser [9] in his work on the continuity of solutions of elliptic differential equations. Later on applications arose in connection with singular integral operators (Stein [12]) and as spaces of interpolation (Stampacchia [11], Stein and Zygmund [13]). Most recently, Fefferman and Stein[3] characterized the space of BMO-functions as the dual of the Hardy space $H^1$.

It seems that BMO-functions also have their place in the theory of quasiconformal mappings. We propose to show that the logarithm of the Jacobian determinant of a quasiconformal mapping $f: \mathbf{R}^n \to \mathbf{R}^n$ is in BMO. We then proceed to study transformations of BMO-functions by quasiconformal mappings. It turns out that a quasiconformal mapping of $\mathbf{R}^n$ onto itself induces a continuous bijective isomorphism $\varphi: u \to u \circ f^{-1}$ of BMO/$\mathbf{R}$. Moreover this situation is in a certain way typical for quasiconformal mappings: If $\varphi: u \to u \circ f^{-1}$ is a continuous bijective isomorphism of BMO/$\mathbf{R}$ which is induced by a homeomorphism $f$ of $\mathbf{R}^n$ satisfying certain regularity conditions, then $f$ is quasiconformal.

## 2. The Jacobian of a Quasiconformal Mapping

For our considerations we adopt the so called analytic definition of quasiconformality. A function $f: G \to \mathbf{R}^n$ defined in a domain $G \subset \mathbf{R}^n$ is said to be absolutely continuous on lines (ACL), if it is continuous and if for each interval $I = \{x \in \mathbf{R}^n : a_i \leqslant x_i \leqslant b_i\} \subset G$ $f$ is absolutely continuous on almost all line segments in $I$, parallel to the coordinate axes. The partial derivatives of an ACL-function $f$ exist a.e. and the Jacobian matrix of $f$ at $x$ will be denoted by $F(x)$, its determinant by $J_f(x)$. By definition a $K$-quasiconformal mapping is a homeomorphism $f: G \to \mathbf{R}^n$ such that $f \in$ ACL, $f$ is totally differentiable a.e. and

$$\sup_{\xi \in R^n, |\xi| = 1} |F(x)\, \xi|^n \leqslant K\, J_f(x) \quad \text{a.e.} \tag{2.1}$$

According to a theorem of Väisälä [14], in this definition the regularity conditions $f \in$ ACL and $f$ differentiable a.e. can be replaced by the single hypothesis, that $f$ has generalized derivatives which are locally $L^n$-integrable.

THEOREM 1. *If $f$ is a quasiconformal mapping of $\mathbf{R}^n$ onto itself with Jacobian determinant $J_f$ then* $\log J_f \in$ BMO.

The proof of Theorem 1 is based on a converse to the lemma of John and Nirenberg (Lemma 3) and on the following result due to Gehring [5]:

LEMMA 2. *Assume that $f$ is a $K$-quasiconformal mapping of $G$ onto $G' \subset \mathbf{R}^n$ and that $Q$ is a cube in the domain $G$ with*

$$\text{dia } Q' < \text{dist } (Q', \partial G') \tag{2.2}$$

*($Q' = fQ$). Then there exist constants $c$ and $p$, $p > n$, which depend on $K$ and $n$ only, such that*

$$\left( \fint_Q J_f^{p/n}\, dx \right)^{n/p} \leqslant c \fint_Q J_f\, dx. \tag{2.3}$$

We set

$$L_f(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|}.$$

Since $f$ is $K$-quasiconformal, inequality (2.1) and the total differentiability imply $L_f^n \leqslant KJ_f$ a.e. and $J_f \leqslant L_f^n$ a.e. Hence according to Lemma 4 in [3] there exists a constant $c_0$ (depending on $K$ and $n$) such that for every cube $Q \subset G$ with dia $Q' < \operatorname{dist}(Q', \partial G')$

$$\fint_Q J_f \, dx \leqslant c_0 \left( \fint_Q J_f^{1/n} \, dx \right)^n, \tag{2.4}$$

If $Q$ is such a cube (satisfying (2.3)), then (2.4) holds for any cube contained in $Q$ and Lemma 3 in [5] shows that for some constants $c$ and $p$, $p > n$,

$$\left( \fint_Q J_f^{p/n} \, dx \right)^{n/p} \leqslant c \fint_Q J_f \, dx.$$

For later reference let us note a simple consequence of this lemma (cf. [5] Theorem 2)

COROLLARY. *Assume that $f$ is a $K$-quasiconformal mapping of $G$ onto $G' \subset \mathbf{R}^n$ and that $Q$ is a cube in the domain $G$ with dia $Q' <$ dist $(Q', \partial G')$. Then*

$$\frac{|A'|}{|Q'|} \leqslant c \left( \frac{|A|}{|Q|} \right)^{(p-n)/p} \tag{2.5}$$

*for every measurable set $A \subset Q$ with image $A' = fA$. (As above $|A|$ stands for the n-dimensional measure of the set $A$.)*

If $A$ is a measurable set, $A \subset Q$, then

$$\frac{|A'|}{|A|} = \fint_A J_f \, dx \leqslant \left( \fint_A J_f^{p/n} \, dx \right)^{n/p}$$
$$\leqslant \left( \frac{|A|}{|Q|} \right)^{-n/p} \left( \fint_Q J_f^{p/n} \, dx \right)^{n/p} \leqslant c \left( \frac{|A|}{|Q|} \right)^{-n/p} \frac{|Q'|}{|Q|}$$

by Lemma 2 and Hölder's inequality.

Let us remark that for the case of plane quasiconformal mappings results similar to Lemma 2 have previously been established by Bojarski [1] and by Gehring and Reich [6].

LEMMA 3. $f=\log u\in\mathrm{BMO}$ *if and only if for all cubes* $Q\subset\mathbf{R}^n$

$$\left(\fint_Q u^a\,dx\right)^{1/a}\leqslant k\left(\fint_Q u^{-b}\,dx\right)^{-1/b} \tag{2.6}$$

*for some positive constants a, b and k.*

The fact, that inequality (2.6) is a consequence of Lemma 1 has already been pointed out by John and Nirenberg. (The result has been stated in this form by Moser [9].) Let us therefore assume that (2.6) holds for $u=e^f$. It is well known that

$$M_t=M_t(u)=\begin{cases}\left(\fint_Q u^t\,dx\right)^{1/t} & t\neq 0\\ \exp\fint_Q \log u\,dx & t=0\end{cases}$$

is a monotone increasing function of $t$. Our assumption therefore implies $M_s\leqslant KM_0$ and $M_0\leqslant KM_{-s}$ for $s=\min(a,b)$. If we set

$$Q_1=\left\{x\in Q\colon \log u(x)\geqslant\fint_Q \log u\,dx\right\}$$

and $Q_2=Q\backslash Q_1$ we obtain the inequalities

$$|Q|^{-1}\int_{Q_1}u^s\,dx\leqslant\fint_Q u^s\,dx=M_s^s\leqslant k^sM_0^s$$

and

$$|Q|^{-1}\int_{Q_2}u^{-s}\,dx\leqslant k^{-s}M_0^{-s}.$$

After adding these two inequalities and inserting the expressions

$$f=\log u\quad\text{and}\quad f_Q=\fint_Q f\,dx=\log M_0$$

we have

$$\int_{Q_1}e^{s(f-f_Q)}\,dx+\int_{Q_2}e^{-s(f-f_Q)}\,dx=\int_Q e^{s|f-f_Q|}\,dx\leqslant(k^s+k^{-s})\,|Q|.$$

Finally,

$$\exp\fint_Q s|f-f_Q|\,dx\leqslant\fint_Q e^{s|f-f_Q|}\,dx$$

upon applying Jensen's inequality. This shows that $f \in \mathrm{BMO}$ with

$$\|f\|_* \leqslant s^{-1} \log (k^s + k^{-s}). \tag{2.7}$$

We shall also need a distortion lemma for quasiconformal mappings, to the effect that the image of a cube can still be compared with a cube. This kind of result is typical for the geometric theory of quasiconformal mappings. We choose a formulation, which is particularly suited for our purposes.

LEMMA 4. *Let $f: G \to \mathbf{R}^n$ be a $K$-quasiconformal mapping. There exists a constant $k$ (which depends on $K$ and $n$) such that to every cube $P' \subset G' = fG$ with*

$$\operatorname{dist}(P', \partial G') > 2\, k \operatorname{dia} P'$$

*there exists a cube $Q \subset G$ with $fQ = Q' \supset P'$, $\operatorname{dia} Q' < \operatorname{dist}(Q', \partial G')$ and*

$$|Q'| \leqslant k^n n^{n/2} |P'|. \tag{2.8}$$

The proof is based on the geometric definition of quasiconformality, according to which a homeomorphism $f: G \to \mathbf{R}^n$ is $K$-quasiconformal if and only if

$$\operatorname{mod} R' \leqslant K^{1/(n-1)} \operatorname{mod} R$$

for every ring $R \subset G$. We refer the reader to the literature (see e.g. [10], [2]) for the precise definitions and for a proof of the equivalence of the analytic and geometric definitions of quasiconformality.

Using preliminary translations, we can assume that the given cube $P'$ is centered at 0 and that $f(0) = 0$. Consider the spherical ring $R' \subset G'$ with complementary components

$$C_0' = \{z : |z| \leqslant r' = 2^{-1} \operatorname{dia} P'\} \quad \text{and} \quad C_1' = \{z : |z| \geqslant s' = k 2^{-1} \operatorname{dia} P'\}.$$

The constant $k > 1$ will be determined later on. The modulus of the ring $R'$ is given by

$$\operatorname{mod} R' = \log \frac{s'}{r'} = \log k. \tag{2.9}$$

We set $s = \inf_{z \in C'_1} |f^{-1}(z)|$ and $r = \sup_{z \in C'_0} |f^{-1}(z)|$ and observe that $|f^{-1}(z)| \leqslant r$ for all $z \in P'$.

The inner complementary component $C_0$ of $R = f^{-1} R'$ contains 0 and a point $x_0$ with $|x_0| = r$, the outer component $C_1$ a continuum connecting $\infty$ with a point $x_1$, $|x_1| = s$. According to a theorem of Teichmüller and its space analogue (see [4], [2], [10])

$$\operatorname{mod} R \leqslant \log\left(\lambda^2 \left(\frac{s}{r} + 1\right)\right) \tag{2.10}$$

for some constant $\lambda$ which depends on $n$ only. Since $K$-quasiconformal mappings satisfy

$$\operatorname{mod} R' \leqslant K_0 \operatorname{mod} R$$

with $K_0 = K^{1/(n-1)}$, we then have by (2.9) and (2.10)

$$\log k \leqslant K_0 \log\left(\lambda^2\left(\frac{s}{r}+1\right)\right)$$

which is equivalent to

$$k^{-K_0} \leqslant \lambda^2\left(\frac{s}{r}+1\right).$$

This shows that $s/r \geqslant n^{1/2}$ if we choose $k = (\lambda^2(1+n^{1/2}))^{K_0}$. In this situation any cube $Q \subset G$ centered at the origin with side length $2r$ (and diameter $2r\, n^{1/2}$) satisfies the requirements of the lemma: The construction shows that $\operatorname{dia} Q' < k \operatorname{dia} P'$ and

$$|Q'| \leqslant k^n n^{n/2} |P'|$$

since $|z| \leqslant k 2^{-1} \operatorname{dia} P'$ for all $z \in Q'$. If we further assume that $\operatorname{dist}(P', \partial G') > 2k \operatorname{dia} P' = 4s'$, then it is clear that $\operatorname{dia} Q' \leqslant 2s' \leqslant \operatorname{dist}(Q', \partial G')$.

LEMMA 5. *The Jacobian determinant $J = J_f$ of a $K$-quasiconformal mapping $f: \mathbf{R}^n \to \mathbf{R}^n$ satisfies*

$$⨍_Q J\, dx \leqslant c_1 \left(⨍_Q J^{-b}\, dx\right)^{-1/b} \tag{2.11}$$

*for all cubes $Q \subset \mathbf{R}^n$. The constants $b$ and $c_1$ depend on $n$ and $K$ only.*

The inverse $f^{-1}$ of a $K$-quasiconformal mapping is $K^{n-1}$-quasiconformal and its determinant $J^{-1}$ satisfies Gehring's inequality (see (2.3)):

$$\left(⨍_{P'} J^{-p/n}\, dy\right)^{n/p} \leqslant c ⨍_{P'} J^{-1}\, dy \tag{2.12}$$

for every cube $P' \subset \mathbf{R}^n$. To any cube $Q \subset \mathbf{R}^n$ let us choose in accordance with Lemma 4 a cube $P'$ with $P = f^{-1} P' \supset Q$ and

$$|P| \leqslant k^n n^{n/2} |Q| = k' |Q|. \tag{2.13}$$

A transformation of variables for the integrals in inequality (2.12) then shows that

$$\left(|P'|^{-1} \int_P J^{-p/n} J\, dx\right)^{n/p} \leqslant c\,|P|\,|P'|^{-1}.$$

Because $|P'|=\int_P J\,dx$ this can be rewritten in the form

$$\left(\int_P J^{(n-p)/n}\,dx\right)^{n/p}\leqslant c\,|P|\left(\int_P J\,dx\right)^{(n-p)/p}$$

and together with inequality (2.13) this leads to

$$\left(\fint_Q J^{(n-p)/n}\,dx\right)^{n/(n-p)}\geqslant (ck')^{p/(n-p)}\fint_Q J\,dx.$$

If the two Lemmata 3 and 5 are combined, a bound for $\|\log J\|_*$ can be given by

$$\|\log J\|_*\leqslant\frac{n}{p-n}\log\left((ck')^{p/n}+(ck')^{-p/n}\right)$$

provided that $p\leqslant 2n$ and by

$$\|\log J\|_*\leqslant\log\left((ck')^{p/(p-n)}+(ck')^{p/(n-p)}\right)$$

otherwise.

*Remark* 1. By definition, a $K$-quasiconformal mapping $f=(f_1,\dots,f_n)$ of $\mathbf{R}^n$ onto itself satisfies for $i=1,\dots,n$

$$K^{-n+1}J_f\leqslant|\operatorname{grad}f_i|^n\leqslant KJ_f\qquad\text{a.e. in } R^n.$$

Hence there exist $g_i\in L^\infty(\mathbf{R}^n)$, $\|g_i\|_\infty\leqslant(n-1)\log K$, such that $n\log|\operatorname{grad}f_i|=\log J_f+g_i$ $(i=1,\dots,n)$. But functions in $L^\infty(\mathbf{R}^n)$ are also in BMO and therefore $\log|\operatorname{grad}f_i|\in\mathrm{BMO}$ $(i=1,\dots,n)$.

*Remark* 2. Local variants of Theorem 1 can be obtained. If $f\colon G\to G'$ is a quasiconformal mapping and if $Q\subset G$ is a cube such that both $\operatorname{dist}(Q,\partial G)$ and $\operatorname{dist}(Q',\partial G')$ are big enough, then $\log J_f$ considered as a function in $Q$ is in BMO.

## 3. The Invariance of the Space BMO

THEOREM 2. *If $f$ is a $K$-quasiconformal mapping of* $\mathbf{R}^n$ *onto itself, then $\varphi\colon u\to u'=u\circ f^{-1}$ is a bijective isomorphism of* BMO *and*

$$\|u'\|_*\leqslant C\|u\|_* \tag{3.1}$$

*for all $u\in$* BMO*, where $C$ is a constant depending on $K$ and $n$ only.*

We note that since the inverse of a quasiconformal mapping is also a quasiconformal mapping, all that has to be shown is inequality (3.1). It then follows directly from the definition that $\varphi$ is an isomorphism of BMO.

For the proof of Theorem 2 we assume that $u\in$ BMO and set $u'=u\circ f^{-1}$. To a given cube $P'$ we determine $Q$ as in Lemma 4 such that $P'\subset Q'$ and $|Q'|\leqslant k^n n^{n/2}|P'|$. The set $A_\sigma=\{x\in Q:|u(x)-u_Q|>\sigma\}$ is mapped onto the set $A'_\sigma=\{z\in Q':|u'(z)-u_Q|>\sigma\}$ and by the corollary to Lemma 2 one knows that

$$\frac{|A'_\sigma|}{|Q'|}\leqslant c\left(\frac{|A_\sigma|}{|Q|}\right)^{(p-n)/p}.$$

Because of Lemma 1

$$|A_\sigma|\leqslant a\, e^{-b\sigma/\|u\|_*}\,|Q|$$

hence

$$\frac{|A'_\sigma|}{|Q'|}\leqslant ca^{(p-n)/p}\exp\left(\frac{-b\sigma(p-n)}{\|u\|_*\,p}\right).$$

An integration of this inequality with respect to $\sigma$ shows that

$$\fint_{Q'}|u'(z)-u_Q|\,dz=|Q'|^{-1}\int_0^\infty |A'_\sigma|\,d\sigma\leqslant ca^{(p-n)/p}b^{-1}p(p-n)^{-1}\,\|u\|_*$$

and in combination with inequality (2.8) of Lemma 4

$$\fint_{P'}|u'(z)-u_Q|\,dz\leqslant k^n n^{n/2}\fint_{Q'}|u'(z)-u_Q|\,dz\leqslant \mathrm{const}\,\|u\|_*$$

One is left with the task of replacing $u_Q$ by

$$u'_{P'}=\fint_{P'}u'(z)\,dz,$$

but

$$|u'_{P'}-u_Q|=\left|\fint_{P'}(u'(z)-u_Q)\,dz\right|,$$

so

$$\fint_{P'}|u'(z)-u'_{P'}|\,dz\leqslant|u'_{P'}-u_Q|+\fint_{P'}|u'(z)-u_Q|\,dz\leqslant 2\fint_{P'}|u'(z)-u_Q|\,dz.$$

This shows that $\|u'\|_*\leqslant C\|u\|_*$ with $C=2k^n\,n^{n/2}c\,a^{(p-n)/p}\,b^{-1}p(p-n)^{-1}$.

THEOREM 3. *Assume that $f$ is a (orientation preserving) homeomorphism of $\mathbf{R}^n$ onto itself, that $f\in$ ACL and that $f$ is totally differentiable a.e. If the induced mapping $\varphi:u\to u'=u\circ f^{-1}$ is a bijective isomorphism of* BMO *and if*

$$\|u'\|_*\leqslant C\|u\|_* \quad \text{for all} \quad u\in\mathrm{BMO} \tag{3.1}$$

*then $f$ is a quasiconformal mapping.*

Let us make precise that the hypothesis of $\varphi$ being a bijective isomorphism of BMO is meant to include the assumption that $f$ and its inverse are absolutely continuous with respect to $n$-dimensional measure. If this were not the case, the isomorphism $\varphi$ could not be defined properly: if the zero set $N$ were mapped onto a set $N'$ of positive measure, then both $u'=0$ and $u''=\chi_{N'}$, the characteristic function of $N'$, were in BMO and both would satisfy

$$u'=u\circ f^{-1} \qquad u''=u\circ f^{-1}$$

with $u=0$ a.e. in $\mathbf{R}^n$, $u\in$BMO.

Our first aim is to construct suitable functions $u\in$BMO.

LEMMA 6. (John-Nirenberg).

$$u(x)=\log^+\frac{1}{|x|}=\begin{cases}\log\dfrac{1}{|x|} & |x|\leqslant 1\\ 0 & |x|\geqslant 1\end{cases}$$

*is in* BMO.

For a proof see [9].

LEMMA 7. *Assume that g is a continuous function defined on* $\mathbf{R}$ *with*

$$k_g=\sup_{x\in\mathbf{R}}|g(x)|+\sup_{x,y\in\mathbf{R}}|g(x)-g(y)|\left(1+\log^+\frac{1}{|x-y|}\right)<\infty. \tag{3.2}$$

*If* $u\in$BMO $(\mathbf{R}^n)$ *and if*

$$k_u=\sup_{|Q|\geqslant 1}\left|\int_Q u(x)\,dx\right|<\infty,$$

*then* $v(x,y)=u(x)\,g(y)\in$BMO $(\mathbf{R}^{n+1})$.

Let $Q_r\subset\mathbf{R}^n$ denote the cube with side length $r$, centred at the origin. If $u\in$BMO, then for $r\leqslant 1$

$$|u_{Q_1}-u_{Q_r}|\leqslant(2^n+1)\left(1-\frac{\log r}{\log 2}\right)\|u\|_* \tag{3.3}$$

This can be seen as follows: Set $u_s=u_{Q_{2^{-s}}}$ $s=0,1,\dots$ Then

$$|u_s-u_{s-1}|=2^{ns}\int_{Q_{2^{-s}}}|u_s-u_{s-1}|\,dx$$
$$\leqslant 2^{ns}\int_{Q_{2^{-s+1}}}|u(x)-u_{s-1}|\,dx+\|u\|_*\leqslant(2^n+1)\,\|u\|_*,$$

$$|u_s-u_{Q_1}|\leqslant\sum_{k=1}^{s}|u_k-u_{k-1}|\leqslant s(2^n+1)\,\|u\|_*.$$

For $r=2^{-s}$ this is equivalent with

$$|u_{Q_r}-u_{Q_1}|\leqslant(2^n+1)\,\frac{-\log r}{\log 2}\,\|u\|_*.$$

For arbitrary $r$, $0<r\leqslant 1$, inequality (3.3) can now easily be derived.

A cube $Q\subset\mathbf{R}^{n+1}$ with sides parallel to the coordinate axes can be represented as a direct product $Q=P\times S$ of cubes $P\subset\mathbf{R}^n$, $S\subset\mathbf{R}$. Set $a_Q=u_P g_0$, where $g_0$ is the value of $g$ at the center of $S$. Then

$$\begin{aligned}&\fint_Q|v(x,y)-a_Q|\,dx\,dy\\&\quad\leqslant\fint_S|g(y)|\,dy\fint_P|u(x)-u_P|\,dx+|u_P|\fint_S|g(y)-g_0|\,dy.\end{aligned}$$

If $|Q|\geqslant 1$ this gives immediately

$$\fint_Q|v(x,y)-a_Q|\,dx\,dy\leqslant k_g(\|u\|_*+k_u).$$

If $|Q|<1$, we make use of (3.3) and (3.2) to conclude that

$$|u_P|\leqslant(2^n+1)\left(1-\frac{\log|P|}{n\log 2}\right)+k_u\,\|u\|_*$$

and

$$\fint_S|g(y)-g_0|\,dy\leqslant k_g(1-n^{-1}\log|P|)^{-1}.$$

Therefore

$$\fint_Q|v(x,y)-a_Q|\,dx\,dy\leqslant 2k_g(k_u+2^{n+1}\|u\|_*)$$

for any cube with sides parallel to the coordinate axes. If $Q\subset\mathbf{R}^{n+1}$ is an arbitrary cube, there exists a cube $Q_0\supset Q$ with sides parallel to the coordinate axes and with

$$|Q_0|\leqslant(n+1)^{(n+1)/2}|Q|.$$

So the estimate

$$\fint_Q |v-v_Q|\,dx\,dy \leqslant 2 \fint_Q |v-a_{Q_0}|\,dx\,dy \leqslant 2\,(n+1)^{(n+1)/2} \fint_{Q_0} |v-a_{Q_0}|\,dx\,dy$$

for the mean oscillation over $Q$ shows that

$$\|v\|_* \leqslant 4(n+1)^{(n+1)/2} k_g\,(k_u + 2^{n+1}\|u\|_*).$$

As an application set $u(x)=\log^+ 1/|x|$, $x\in\mathbf{R}$ and let $g(y)$ be the piecewise linear, continuous odd function defined for $y\in\mathbf{R}$ by

$$g(y)=\begin{cases} 1-|y-1| & 0\leqslant y\leqslant 2,\\ 0 & 2\leqslant y,\\ -g(-y) & y\leqslant 0.\end{cases}$$

Since $|g(x)-g(y)|\leqslant \min\{2, |x-y|\}$, $g$ satisfies the assumptions of Lemma 7 (with $k_g\leqslant 3$). From the Lemmata 6 and 7 we conclude that

$$v(x_1, x_2)=g(x_2)\log^+\frac{1}{|x_1|}$$

is in BMO. For dimensions $n>2$ let us define $v\in\mathrm{BMO}$ by

$$v(x)=\log^+ 1/|x_1|\; h(x_2)\dots h(x_{n-1})\, g(x_n)$$

with

$$h(x)=\begin{cases} 1 & |x|\leqslant 1,\\ 2-|x| & 1\leqslant |x|\leqslant 2,\\ 0 & |x|\geqslant 2,\end{cases}$$

$v$ then has compact support and $v(x)=g(x_n)\log^+ 1/|x_1|$ for $|x_i|\leqslant 1$, $i=1,\dots,n$. Finally, for $r>0$ let $v_r$ be defined by

$$v_r(x)=v\left(\frac{x}{r}\right).$$

Certainly $\|v_r\|_*=\|v\|_*$, since the space of BMO-functions is invariant under dilations.

With $a=(\alpha_1,\dots,\alpha_n)\in R^n_+$ (i.e. $\alpha_i>0$, $i=1,\dots,n$) we associate the sets $U_{a,r}=\{x: |x_1|\leqslant \alpha_i r,\ i=1,\dots,n\}$ and the functions

$$\varphi_{a,r}=\begin{cases} |U_{a,r}|^{-1}\, v_r(x) & x\in U_{a,r}\\ 0 & \text{otherwise}\end{cases}$$

and

$$\psi_{a,r}=|\varphi_{a,r}|.$$

LEMMA 8. *If* $h \in L^1(\mathbf{R}^n)$, *then there exists a sequence* $r_j$ *converging to* 0 *such that a.e. in* $\mathbf{R}^n$

$$\lim_{j\to\infty} \varphi_{a,r_j} * h(t) = \lim_{j\to\infty} \int_{\mathbf{R}^n} \varphi_{a,r_j}(x)\, h(t-x)\, dx = 0$$

*and*

$$\lim_{j\to\infty} \psi_{a,r_j} * h(t) = c_a h(t)$$

*with*

$$c_a = \int_{\mathbf{R}^n} \psi_{a,r}\, dx = \fint_{U_{a,1}} |v(x)|\, dx\,.$$

$c_a$ is a continuous function of $a \in \mathbf{R}^n_+$. For $a = (\alpha_1, \ldots, \alpha_{n-1}, 1)$, $\alpha_i \leqslant 1$, $c_a$ can easily be calculated:

$$c_a = \tfrac{1}{2}(1 - \log \alpha_1). \tag{3.4}$$

For the proof of Lemma 8 consider the differences

$$d_1(t) = \varphi_{a,r} * h(t) - 0 = \int_{\mathbf{R}^n} \varphi_{a,r}(x)\,(h(t-x) - h(t))\, dx$$

and

$$d_2(t) = \psi_{a,r} * h(t) - c_a h(t) = \int_{\mathbf{R}^n} \psi_{a,r}(x,\,(h(t-x) - h(t))\, dx\,.$$

They satisfy

$$\lim_{r\to 0} \int_{\mathbf{R}^n} |d_k(t)|\, dt \leqslant \lim_{r\to 0} \int_{\mathbf{R}^n} \psi_{a,r}(x) \int_{\mathbf{R}^n} |h(t-x) - h(t)|\, dt\, dx = 0\,.$$

$k = 1, 2$ because

$$\lim_{x\to 0} \int_{\mathbf{R}^n} |h(t-x) - h(t)|\, dt = 0$$

and $\psi_{a,r}$ has its support in $U_{a,r}$. For some sequence $r_j$ with $\lim_{j\to\infty} r_j = 0$ the differences $d_1$ and $d_2$ will therefore converge pointwise a.e.

For any rotation $\varrho$ of $R^n$ set $\varphi_{\varrho,a,r}(x) = \varphi_{a,r}(\varrho^{-1}x)$, $U_{\varrho,a,r} = \varrho^{-1}(U_{a,r})$ and $v_{\varrho,r}(x) = v_r(\varrho^{-1}x) = v(\varrho^{-1}x/r)$. Observe that $\|v_{\varrho,r}\|_* = \|v\|_*$, since BMO is invariant under rotations. We think of $\varrho$ as being given by an element in $O(n)$, the group of orthogonal $n \times n$-matrices. As an immediate consequence of Lemma 8 let us note:

COROLLARY. *Let* $\{\varrho_i\}$ *and* $\{a_m\}$ *be countable dense subsets of* $O(n)$ *and* $\mathbf{R}^n_+$ *respectively. If* $h \in L^1(\mathbf{R}^n)$, *then for any pair* $\varrho_i$, $a_m$ *there exists a sequence* $\{r_j\}$ *with* $\lim_{j\to\infty} r_j = 0$ *such that*

$$\lim_{j\to\infty} \varphi_{\varrho_i, a_m, r_j} * h(t) = 0 \tag{3.5}$$

*and*

$$\lim_{j\to\infty} \psi_{\varrho_i, a_m, r_j} * h(t) = c_{a_m} h(t) \tag{3.6}$$

*for all* $t \in \mathbf{R}^n \setminus N$, *where* $N$ *is a set of measure zero which is independent of the pair* $\varrho_i$, $a_m$.

The proof of Theorem 3 consists in showing that $\sup_{\xi \in \mathbf{R}^n, |\xi|=1} |F(x)\,\xi|^n \leqslant K \det F(x)$ a.e. in $\mathbf{R}^n$. ($F(x)$ is the Jacobian matrix of $f$ at $x$.) This clearly is a local property. Based on our hypothesis and on the corollary to Lemma 8 we can assume that at $x=0$ the following conditions are satisfied:

i) $f$ is (totally) differentiable

ii) $J_f = \det F \neq 0$, in view of the remark following Theorem 3

iii) (3.5) and (3.6) hold for $h(x) = \begin{cases} J_f(x) & |x|<1 \\ 0 & |x|\geq 1 \end{cases}$

$F$ can be written in the form $F = \varrho D \sigma$, where $\varrho$, $\sigma \in O(n)$ and $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ is a diagonal matrix with $\lambda_1 \geqslant \lambda_2 \ldots \geqslant \lambda_n > 0$ (this is true for any $n \times n$-matrix $M$ with $\det M \neq 0$). Let us exclude the case $\det \sigma = -1$ by possibly interchanging the order of the coordinates. If we compose $f$ with the rotation $\sigma^{-1}$, then the resulting mapping $g = \sigma^{-1} \circ f$ still satisfies the assumptions of Theorem 3 and the three additional conditions above. Note that $J_g = J_f$ and that the Jacobian matrix of $g$ at 0 is $G = \varrho D$. The same is true if we consider the mapping $cf$, $c>0$, instead of $f$ (with $J_{cf} = c^n J_f$). Therefore we can assume without loss of generality, that $F = \varrho D$ with $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ and $\lambda_1 \geqslant \lambda_2 \ldots \geqslant \lambda_n = 1$. We then have to show that $\lambda_1^n \leqslant K \lambda_1 \ldots \lambda_n$.

With this in mind let us choose $\varrho_i$ and $a_m = (\alpha_{m1}, \ldots, \alpha_{mn})$ in such a way that

$$|\lambda_k \alpha_{mk} - 1| < \varepsilon \qquad k = 1, \ldots, n \tag{3.7}$$

$$|c_{a_m} - \tfrac{1}{2}(1 - \log \alpha_{m1})| < \varepsilon \tag{3.8}$$

(cf. (3.4)) and such that for $r$ small enough, say $r < \delta_1$, $U_r' = f U_{\varrho_i, a_m, r}$ contains the cube $S = \{z : |z_i| \leqslant r(1-\varepsilon)\}$ and is contained in the cube $Q = \{z : |z_i| \leqslant r(1+\varepsilon)\}$. We remind that $U_{\varrho, a, r}$ was defined by $U_{\varrho, a, r} = \varrho^{-1}\{x : |x_i| \leqslant r\alpha_i,\ i = 1, \ldots, n\}$.

By the main hypothesis (3.1) $w_r = v_{\varrho, r} \circ f^{-1}$ is in BMO and $\|w_r\|_* \leqslant C\|v\|_*$. So with

$$w_Q = \fint_Q w_r(z)\, dz$$

$$\fint_{U'_r} |w_r(z) - w_Q|\, dz \leqslant |S|^{-1} \int_Q |w_r(z) - w_Q|\, dz$$

$$\leqslant \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n \fint_Q |w_r(z) - w_Q|\, dz \leqslant \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n C\, \|v\|_*.$$

In this inequality we can replace $w_Q$ by the mean value

$$\tilde{v}_r = \fint_{U'_r} w_r(z)\, dz$$

if we instead write

$$\fint_{U'_r} |w_r(z) - \tilde{v}_r|\, dz \leqslant 2\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n C\, \|v\|_*.$$

Due to the absolute continuity of $f$ with respect to $n$-dimensional Lebesgue measure

$$\tilde{v}_r = \frac{|U_{\varrho i, a_m, r}|}{|U'_r|} \fint_{U_{\varrho i, a_m, r}} v_{\varrho i, r}(x)\, J_f(x)\, dx$$

$$= \frac{|U_{\varrho i, a_m, r}|}{|U'_r|}\, \varphi_{\varrho i, a_m, r} * h(0)$$

so by the corollary to Lemma 6

$$\lim_{j \to \infty} \tilde{v}_{r_j} = 0.$$

Hence there exists $\delta_2 > 0$ ($\delta_2 \leqslant \delta_1$) such that for $r = r_j < \delta_2$

$$\fint_{U'_r} |w_r(z)|\, dz \leqslant \varepsilon + 2\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n C\, \|v\|_*. \tag{3.9}$$

On the other side

$$\fint_{U'_r} |w_r(z)|\, dz = \frac{|U_{\varrho i, a_m, r}|}{|U'_r|}\, \psi_{\varrho i, a_m, r} * h(0)$$

so by (3.6)

$$\lim_{j\to\infty} \fint_{U'r_j} |w_{r_j}(z)|\, dz = (J_f(0))^{-1}\, c_{a_m} J_f(0) = c_{a_m}. \tag{3.10}$$

Combining the two results (3.9) and (3.10) with (3.8) we conclude that for $j$ big enough

$$\tfrac{1}{2}(1-\log\alpha_{m1}) - \varepsilon \leqslant \fint_{U'r_j} |w_{r_j}(z)|\, dz \leqslant \varepsilon + 2\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n C\, \|v\|_*.$$

Together with (3.7) this implies

$$1+\log\lambda_1 \leqslant 4\, C\|v\|_*$$

since $\varepsilon>0$ was arbitrary. The inequality $\lambda_1^n \leqslant K\lambda_1 \ldots \lambda_n$ therefore holds with $K = e^{(n-1)(4C\|v\|_*-1)}$.

## 4. A Local Version

The object of this section is a local version of the Theorems 2 and 3. If $G$ is a domain in $\mathbf{R}^n$ we denote by BMO $(G)$ the subspace of BMO consisting of all functions $u\in$BMO with support in $G$ (the support of a locally integrable function $u\in L^1_{\text{loc}}(\mathbf{R}^n$ is the complement of the largest open set $O\subset\mathbf{R}^n$ with $u(x)=0$ a.e. on $O$).

LEMMA 9. *If $u\in L^1_{\text{loc}}(\mathbf{R}^n)$ and if* $\operatorname{supp} u\subset G$, *then*

$$\|u\|_* \leqslant 4 \sup \fint_P |u(x)-u_P|\, dx,$$

*where the supremum is extended over all cubes $P$ with* $\operatorname{dia} P \leqslant 4n^{1/2} \operatorname{dia} G$.

$G$ is contained in a ball $B$ with radius $\operatorname{dia} G$. If $Q\cap G\neq\emptyset$ for some cube with $\operatorname{dia} Q \geqslant 4n^{1/2} \operatorname{dia} G$, then there exists a cube $P\subset Q$ with side length $n^{-1/2} \operatorname{dia} P = 2 \operatorname{dia} B$ such that $P\cap B = Q\cap B \supset Q\cap G$. With $N=\{x\in P: u(x)=0\}$, $|N|\geqslant |P|-|G| \geqslant$ $\geqslant (1-2^{-n})|P|$, we obtain

$$\fint_P |u-u_P|\, dx = |P|^{-1}\int_N |u_P|\, dx + |P|^{-1}\int_{P\setminus N} |u-u_P|\, dx$$

which shows that

$$|u_P| \leqslant (1-2^{-n})^{-1} \fint_P |u-u_P|\, dx.$$

Finally

$$\fint_Q |u-u_Q|\,dx \leqslant 2 \fint_Q |u-u_P|\,dx$$

$$\leqslant 2\,|Q|^{-1} \int_{Q\setminus P} |u_P|\,dx + 2\,|Q|^{-1} \int_P |u-u_P|\,dx$$

$$\leqslant 2\,(1-2^{-n})^{-1} \fint_P |u-u_P|\,dx$$

and the proof is complete.

THEOREM 4. *Assume that $f: G \to \mathbf{R}^n$ is a (orientation preserving) homeomorphism, $f \in$ ACL, and that $f$ is differentiable a.e. Then $f$ is a quasiconformal mapping if and only if every point $x \in G$ has a neighbourhood $U$ such that $\varphi: u \to u' = u \circ f^{-1}$ is an isomorphism of* BMO $(U)$ *onto* BMO $(fU)$ *which satisfies*

$$\|u'\|_* \leqslant C_0 \|u\|_*$$

*for all $u \in$* BMO $(U)$ *with a fixed constant $C_0$ independent of $U$.*

The proof for the quasiconformality of a homeomorphism $f: G \to \mathbf{R}^n$ satisfying all the above hypotheses is contained in the proof of Theorem 3. In order to show that a quasiconformal mapping gives rise to local isomorphisms of BMO $(U)$ onto BMO $(U')$, $U' = fU$, the full strength of the Lemmata 2 and 4 has to be used.

For $x \in G$ we choose a neighbourhood $U$ in such a way that

$$4n^{1/2}(1+2k)\,\mathrm{dia}\,U' < \mathrm{dist}(U', \partial G'),$$

where $k$ is the constant of Lemma 4. If $P'$ is a cube with $\mathrm{dia}\,P' \leqslant 4n^{1/2}\,\mathrm{dia}\,U'$ and with $P' \cap U' \neq \emptyset$, then

$$\mathrm{dist}(P', \partial G') \geqslant \mathrm{dist}(U', \partial G') - \mathrm{dia}P' > \mathrm{dia}P'(1+2k) - \mathrm{dia}P' = 2k\,\mathrm{dia}P'.$$

Hence by Lemma 4 there exists a cube $Q \subset G$ with $fQ = Q' \supset P'$, $|Q'| \subseteq k^n n^{n/2} |P'|$ and with

$$\mathrm{dist}(Q', \partial G') > \mathrm{dia}\,Q'. \tag{4.1}$$

For a given function $u \in$ BMO $(U)$ with $u' = u \circ f^{-1}$ we can then proceed as in the proof of Theorem 2. Condition (4.1) ensures the validity of Lemma 2. It follows (see (2.11)) that

$$\fint_{P'} |u'(z) - u'_{P'}|\,dz \leqslant C\,\|u\|_*$$

and this inequality holds for all $P'$ with $\mathrm{dia}P' \leqslant 4n^{1/2}\,\mathrm{dia}\,U'$. Therefore by Lemma 9

$$\|u'\|_* \leqslant 4C\|u\|_* = C_0\|u\|_*.$$

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