Amalgamated Free Products of Groups and Homological Duality

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1. Introduction

1.1. Among the contexts where amalgamated free products of groups occur are presentations of groups and fundamental groups of spaces: A group freely presented by generators and relations can often be considered as an amalgamated free product $G=G_1 *_S G_2$ of subgroups G_1, G_2 and S which are better known than G. The fundamental group $\pi_1(X)$ of a union $X=X_1 \cup_Y X_2$ of spaces X_1, X_2 with identified subspace Y (all path-connected), where $\pi_1(Y) \to \pi_1(X)$ is injective, is an amalgamated free product $\pi_1(X)) *_{\pi_1(Y)} \pi_1(X_2)$.

In both these instances simple examples are available where amalgamation of duality groups (i.e., groups with homological duality generalizing Poincaré duality, cf. [1]) again yields duality groups; or where a group known to be a duality group – for example because it admits a closed manifold as an Eilenberg-Mac Lane space – can be decomposed into an amalgamated free product. A simple illustration of this is given by the closed orientable surface of genus 2 considered as a union of two tori with a disc removed; a similar decomposition is available for closed "sufficiently large" 3-manifolds (cf. [7]). Section 4 contains a list and detailed description of examples of these and other types.

1.2. The purpose of this paper is to show that under suitable conditions amalgamation of duality groups leads to duality groups. These conditions are essentially on the dimensions of the groups G_1, G_2 and S. We first prove that *for $G=G_1 *_S G_2$, with $S \neq G_1$ and G_2, to be a duality group of dimension n the following condition on the respective cohomology dimensions* cd *is necessary*:

$$n-1 \leqslant \operatorname{cd} S \leqslant \operatorname{cd} G_j \leqslant n, \quad j=1,2. \tag{1.1}$$

In particular, if G is a duality group of dimension >1 then $\operatorname{cd} S>0$. The above result thus contains, and explains in a more precise way, the known fact ([1], Corollary 1.5) that a duality group of dimension >1 cannot be a non-trivial free product. The lower bound for the cohomology dimension of S is also useful in applications, e.g. to torsion-free arithmetic groups (known, by the work of Borel-Serre [9], to be duality groups).

Conversely, if G_1, G_2 and S are duality groups of dimensions fulfilling the inequalities (1.1), then G is a duality group in the case $\operatorname{cd} G_1=\operatorname{cd} G_2=n$, $\operatorname{cd} S=n-1$, and then $\operatorname{cd} G=n$; and also in the case $\operatorname{cd} G_1=\operatorname{cd} G_2=\operatorname{cd} S=\operatorname{cd} G=n-1$. In the other remaining dimension cases ($\operatorname{cd} G_1 = \operatorname{cd} S = n-1$, $\operatorname{cd} G_2 = n$; and $\operatorname{cd} G_1 = \operatorname{cd} G_2 = \operatorname{cd} S = n-1$, but $\operatorname{cd} G = n$) additional conditions on certain restriction homomorphisms must be fulfilled. For the precise statements see Theorems 3.2, 3.3 and 3.5. The additional conditions always hold if S has finite index in G_1, or in G_1 and G_2, respectively.

The proofs of these statements are based on the *Mayer-Vietoris sequence* for amalgamated free products. We recall that sequence briefly in Section 2, with a short sketch of a proof. Moreover, general properties of duality groups established in other papers ([1], [2], [3], [4]) are heavily used. All proofs become simpler if the groups involved are assumed to admit finite projective resolutions[1]); or equivalently, for finitely presented groups, to admit Eilenberg-Mac Lane complexes dominated by finite complexes (this remark is useful for applications, but our procedure is entirely algebraic). This view-point is adopted in Section 3.

1.3. In Section 5 we prove the same statements without finiteness assumptions. The main tool here, aside from the Mayer-Vietoris sequence, is a property of groups, fulfilled by all duality groups, which is examined in a broader context in [4]: namely, to have finite cohomology dimension and to admit an "elementary duality" property in the top dimension. We call such groups "of type (FD_*)". Groups admitting finite projective resolutions belong to that class; and so do, more generally, groups of finite cohomology dimension admitting a projective resolution which is finitely generated in the top dimension[1]). The arguments used in Section 5 deal essentially with amalgamated free products of groups of type (FD_*). For these a few further dimension relations can be obtained.

2. The Mayer-Vietoris Sequence

2.1. Given two monomorphisms of groups $\iota_1 : S \to G_1$, $\iota_2 : S \to G_2$ one denotes by $G_1 *_S G_2$ the generalized free product of G_1 and G_2 with amalgamated subgroups $\iota_1(S)$ and $\iota_2(S)$, in short the "amalgamated free product". $G = G_1 *_S G_2$ is defined as factor group $(G_1 * G_2)/N$, where N is the normal subgroup of $G_1 * G_2$ generated by all $\iota_1(s)\,\iota_2(s)^{-1}$, $s \in S$. The natural maps $\kappa_j : G_j \to G$, $j = 1, 2$, are monomorphisms; one often identifies G_j with $\kappa_j(G_j)$ and S with $\kappa_1\iota_1(S) = \kappa_2\iota_2(S)$, and one then has $G_1 \cap G_2 = S$. The diagram

$$\begin{array}{ccc} S & \overset{\iota_1}{\rightarrowtail} & G_1 \\ {\scriptstyle \iota_2}\downarrow & & \downarrow{\scriptstyle \kappa_2} \\ G_2 & \underset{\kappa_1}{\rightarrowtail} & G \end{array}$$

is a push-out diagram in the category of groups.

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[1]) See "Note added in proof" at the end of this paper.

One writes $\mathbf{Z}(G/S)$ for the (right) G-module freely generated, as an Abelian group, by the cosets Sx of G modulo S with G-action by (right) translations; and similarly for $\mathbf{Z}(G/G_j)$, $j=1, 2$.

PROPOSITION 2.1. (Swan [6]). *With* $G=G_1 *_S G_2$ *there is associated a short exact sequence of (right) G-modules*

$$\mathbf{Z}(G/S) \overset{\alpha}{\rightarrowtail} \mathbf{Z}(G/G_1) \oplus \mathbf{Z}(G/G_2) \overset{\beta}{\twoheadrightarrow} \mathbf{Z} \tag{2.1}$$

where $\alpha(Sx)=(G_1x, -G_2x)$ *and* $\beta(G_1x, 0)=\beta(0, G_2x)=1$, $x \in G$.

2.2. For an amalgamated free product $G=G_1 *_S G_2$ there are *Mayer-Vietoris sequences* relating the (co) homology groups of G to those of G_1, G_2 and S. Although these sequences are well-known (cf. [6], [7], [8]), we will give a simple proof showing how to deduce them almost immediately from (2.1); moreover we get a description of the connecting homomorphisms which we will use in our applications.

PROPOSITION 2.2. *For an amalgamated free product* $G=G_1 *_S G_2$, *a left G-module A and a right G-module B one has long exact sequences* $(k \in \mathbf{Z})$

$$\cdots \to H^k(G; A) \xrightarrow{(\mathrm{res}^*,\, \mathrm{res}^*)} H^k(G_1; A) \oplus H^k(G_2; A) \xrightarrow{(\mathrm{res}^*,\, -\mathrm{res}^*)} H^k(S; A) \xrightarrow{\delta} H^{k+1}(G; A) \to \cdots \tag{2.2}$$

and

$$\cdots \to H_k(S; B) \xrightarrow{(\mathrm{cor}_*,\, -\mathrm{cor}_*)} H_k(G_1; B) \oplus H_k(G_2; B) \xrightarrow{(\mathrm{cor}_*,\, \mathrm{cor}_*)} A_k(G; B) \xrightarrow{\delta} H_{k-1}(G; B) \to \cdots \tag{2.3}$$

The maps res and cor$_*$ are induced by the respective subgroup inclusions.*

Proof. Since (2.1) is a sequence of free Abelian groups, we have exact sequences of left (right) G-modules by diagonal action

$$\mathrm{Hom}(\mathbf{Z}, A) \rightarrowtail \mathrm{Hom}(\mathbf{Z}(G/G_1), A) \oplus \mathrm{Hom}(\mathbf{Z}(G/G_2), A) \twoheadrightarrow \mathrm{Hom}(\mathbf{Z}(G/S), A)$$

and

$$B \otimes \mathbf{Z}(G/S) \rightarrowtail (B \otimes \mathbf{Z}(G/G_1)) \oplus (B \otimes Z(G/G_2)) \twoheadrightarrow B \otimes \mathbf{Z}.$$

Now, for any subgroup $H \subset G$, the maps

$$\xi: \mathrm{Hom}(\mathbf{Z}(G/H), A) \to \mathrm{Hom}_H(\mathbf{Z}G, A)$$
$$\eta: B \otimes \mathbf{Z}(G/H) \to B \otimes_H \mathbf{Z}G$$

given by $\xi(f)(x)=xf(Hx)$ and $\eta(b\oplus Hx)=bx^{-1}\otimes x$, $x\in G$, $f\in\mathrm{Hom}(\mathbf{Z}(G/H), A)$, $b\in B$, are G-module isomorphisms. We thus get exact sequences of left (right) G-modules

$$A\rightarrowtail \mathrm{Hom}_{G_1}(\mathbf{Z}G, A)\oplus\mathrm{Hom}_{G_2}(\mathbf{Z}G, A)\twoheadrightarrow\mathrm{Hom}_S(\mathbf{Z}G, A) \tag{2.4}$$

and

$$B\otimes_S\mathbf{Z}G\rightarrowtail(B\otimes_{G_1}\mathbf{Z}G)\oplus(B\otimes_{G_2}\mathbf{Z}G)\twoheadrightarrow B. \tag{2.5}$$

Since, for any subgroup $H\subset G$, one has $H^k(G;\mathrm{Hom}_H(\mathbf{Z}G, A))\cong H^k(H; A)$ and $H_k(G; B\otimes_H\mathbf{Z}G)\cong H_k(H; B)$, the long coefficient sequences corresponding to (2.4) and (2.5) respectively are precisely the desired Mayer-Vietoris sequences. The homomorphisms δ and ∂ can easily be described as connecting homomorphisms in the coefficient sequences.

2.3. As an application we discuss conditions for an amalgamated free product to be of type (FP), or $(\overline{FP})$. A group G is said to be of type (FP), if the trivial G-module $\mathbf{Z}$ admits a finite projective resolution over $\mathbf{Z}G$; of type $(\overline{FP})$ if it admits a finitely generated free resolution. $(\overline{FP})$ together with finite cohomology dimension $\mathrm{cd}\,G$ is equivalent to (FP). The results of this section could be obtained by explicit use of resolutions, but we prefer here a procedure based on the homology Mayer-Vietoris sequence and on the criteria for (FP) and $(\overline{FP})$ given in [2], Proposition 3.2.

THEOREM 2.3. *Let $G=G_1*_SG_2$ be an amalgamated free product.*

(i) *If G_1 and G_2 are of type $(\overline{FP})$, then G is of type $(\overline{FP})$ if and only if S is.*

(ii) *If G and S are of type $(\overline{FP})$, then so are G_1 and G_2.*

Moreover, the same statements hold for type (FP).

Proof. Let G_1 and G_2 be of type $(\overline{FP})$. We consider the sequence (2.3) with $B=\prod\mathbf{Z}G$, an arbitrary direct product of copies of $\mathbf{Z}G$. By [2], Proposition 3.2, since $\mathbf{Z}G$ is G_j-free, we have $H_k(G_j; B)=0$ for $k\geqslant 1, j=1, 2$. Thus (2.3) yields

$$H_k(G;\prod\mathbf{Z}G)\cong H_{k-1}(S;\prod\mathbf{Z}G),\qquad k\geqslant 2.$$

Moreover, by Proposition 2.1, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0\to H_1(G;\prod\mathbf{Z}G) & \to & (\prod\mathbf{Z}G)\otimes_S\mathbf{Z} & \to & (\prod\mathbf{Z}G)\otimes_{G_1}\mathbf{Z}\oplus(\prod\mathbf{Z}G)\otimes_{G_2}\mathbf{Z} & \twoheadrightarrow & (\prod\mathbf{Z}G)\otimes_G\mathbf{Z} \\ & & \downarrow\lambda & & \downarrow\mu & & \downarrow\nu \\ 0 & \to & \prod\mathbf{Z}(G/S) & \to & \prod\mathbf{Z}(G/G_1)\oplus\prod\mathbf{Z}(G/G_2) & \twoheadrightarrow & \prod\mathbf{Z} \end{array}$$

The maps λ, μ and ν are epimorphisms. G_1 and G_2, being of type (FP), are finitely generated, and so is $G=G_1*_SG_2$; hence μ and ν are isomorphisms.

Now let S be of type $(\overline{FP})$. Then $H_k(S; \prod \mathbf{Z}G) \cong \prod H_k(S; \mathbf{Z}G)$ for all $k \in \mathbf{Z}$. For $k=0$, this tells that λ is an isomorphism, and thus $H_1(G; \prod \mathbf{Z}G)=0$. For $k \geqslant 1$, we have $H_{k+1}(G; \prod \mathbf{Z}G) \cong H_k(S; \prod \mathbf{Z}G) = \prod H_k(S; \mathbf{Z}G)=0$, since $\mathbf{Z}G$ is S-free. Thus $H_k(G; \prod \mathbf{Z}G)=0$ for all $k \geqslant 1$ and all direct products $\prod$; by Proposition 3.2 of [2] this implies that G is of type $(\overline{FP})$.

Conversely, assume that G is of type $(\overline{FP})$. It then follows that $H_k(S; \prod \mathbf{Z}G)=0$ for $k \geqslant 1$. We may assume that the index $|G:S|$ is ∞ (since for $|G:S|<\infty$ any finitely generated resolution over $\mathbf{Z}G$ is also finitely generated over $\mathbf{Z}S$). Then one has a short exact sequence of S-modules $\mathbf{Z}S \rightarrowtail \mathbf{Z}G \twoheadrightarrow \mathbf{Z}G$, and hence a short exact sequence $\prod \mathbf{Z}S \rightarrowtail \prod \mathbf{Z}G \twoheadrightarrow \prod \mathbf{Z}G$. The corresponding coefficient sequence in homology yields $H_k(S; \prod \mathbf{Z}S)=0$ for all $k \geqslant 1$. Moreover, the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to & (\prod \mathbf{Z}S) \otimes_S \mathbf{Z} & \to & (\prod \mathbf{Z}G) \otimes_S \mathbf{Z} & \to & (\prod \mathbf{Z}G) \otimes_S \mathbf{Z} & \to 0 \\
& \varrho \downarrow & & \lambda \downarrow & & \lambda \downarrow & \\
0 \to & \prod \mathbf{Z} & \to & \prod \mathbf{Z}(G/S) & \to & \prod \mathbf{Z}(G/S) & \to 0
\end{array}$$

shows that ϱ is an isomorphism; hence $\mathbf{Z}$ is finitely presented over $\mathbf{Z}S$, or equivalently, S is finitely generated. By the $(\overline{FP})$-criterion, Prop. 3.2 of [2], it follows that S is of type $(\overline{FP})$. We thus have proved (i).

To prove (ii), one considers as before the sequence (2.3) with $B=\prod \mathbf{Z}G$. Assuming G and S to be of type $(\overline{FP})$, the criterion yields $H_k(G_j; \prod \mathbf{Z}G)=0$ for $j=1, 2$ and $k \geqslant 1$. By arguments analogous to those above one then easily checks that the conditions of the criterion are fulfilled, i.e., G_1 and G_2 are of type $(\overline{FP})$.

As to the statements (i) and (ii) for type (FP), all that remains is to check that the respective groups have finite cohomology dimension. In the case (ii), $\operatorname{cd} G < \infty$ of course implies $\operatorname{cd} G_j < \infty$, $j=1, 2$. In the case (i) one assumes $\operatorname{cd} G_j < \infty$, $j=1, 2$; if $\operatorname{cd} S < \infty$, the sequence (2.2) yields $\operatorname{cd} G < \infty$, while the converse implication is again obvious.

3. Amalgamated free Products of Duality Groups: Type (FP)

3.1. We recall (cf. [1]) that G is a *duality group* of dimension n if there is a dualizing right G-module C and a fundamental class $e \in H_n(G; C)$ such that the cap-product $e \cap -$ induces isomorphisms

$$H^k(G; A) \xrightarrow{\cong} H_{n-k}(G; C \otimes A)$$

for every left G-module A and all $k \in \mathbf{Z}$ ($C \otimes A$ is a right G-module by diagonal action). If $C = \mathbf{Z}$ as an Abelian group, G is called Poincaré duality group. In the present section we discuss conditions for an amalgamated free product $G = G_1 *_S G_2$ to be a duality group. We restrict ourselves to groups of type (FP). As we will show in section 5, all results are in fact valid without that restriction. However, the proofs and the technique used for groups of type (FP) is simpler so that a separate treatment may be justified[1]). The main tool here are the Theorems 4.4 and 4.5 of [1] which give criteria for duality without explicitly involving the cap-product: A group G of type (FP) is a duality group of dimension n if and only if $H^k(G; \mathbf{Z}G)=0$ for $k \neq n$ and torsion-free for $k=n$ (and then $C=H^n(G; \mathbf{Z}G)$); or if and only if $H^k(G; A)=0$ for $k \neq n$ and all induced G-modules $A=L \otimes \mathbf{Z}G$.

3.2. We thus assume G_1, G_2 and S, and hence also G, to be of type (FP). If G is a duality group of dimension n, *lower bounds* for $\operatorname{cd} S$ (and hence for $\operatorname{cd} G_1$ and $\operatorname{cd} G_2$) can be obtained as follows.

We suppose that $\operatorname{cd} S < n-1$. Then (2.2) with $A=\mathbf{Z}G$ yields

$$C=H^n(G; \mathbf{Z}G) \cong H^n(G_1; \mathbf{Z}G) \oplus H^n(G_2; \mathbf{Z}G).$$

Since G_j is of type (FP), $j=1,2$, the cohomology functors $H^n(G_j; -)$ commute with direct sums. Since $C \cong \bigoplus_{G/G_1} H^n(G_1; \mathbf{Z}G_1) \oplus H^n(G_2; \mathbf{Z}G)$, where the sum $\bigoplus_{G/G_1}$ is over the cosets of G modulo G_1, as right G_1-modules, we have

$$\begin{aligned} H_n(G_1; C) &\cong \bigoplus_{G/G_1} H_n(G_1; H^n(G_1; \mathbf{Z}G_1)) \oplus H_n(G_1; H^n(G_2; \mathbf{Z}G)) \\ &\cong \bigoplus_{G/G_1} \operatorname{Hom}_{G_1}(H^n(G_1; \mathbf{Z}G_1), H^n(G_1; \mathbf{Z}G_1)) \oplus H_n(G_1; H^n(G_2; \mathbf{Z}G)) \end{aligned}$$

by [4], Theorem 2.4 (see also [3]). Therefore $H_n(G_1; C)=0$ implies $H^n(G_1; \mathbf{Z}G_1)=0$, and similarly for G_2. But at least one of the groups $H^n(G_1; \mathbf{Z}G_1)$, $H^n(G_2; \mathbf{Z}G_2)$ must be $\neq 0$; thus, e.g., $H_n(G_1; C) \neq 0$. On the other hand $H_n(G_1; C) \cong H_n(G; C \otimes_{G_1} \mathbf{Z}G) \cong H_n(G; C \otimes \mathbf{Z}(G/G_1))$ is isomorphic, by duality, to $H^0(G; \mathbf{Z}(G/G_1)) = (\mathbf{Z}(G/G_1))^G$. Under the action of G, $\mathbf{Z}(G/G_1)$ has no fixed element unless the index $|G:G_1|$ is finite. But in $G=G_1 *_S G_2$ the index $|G:G_1|$ is finite only if $G=G_1$, $S=G_2$. We thus have proved

THEOREM 3.1. *Let $G=G_1 *_S G_2$ be a non-trivial amalgamated free product* (i.e., $S \neq G_j$, $j=1,2$) *and let G_1, G_2 and S be of type (FP). If G is a duality group of dimension n, then*

$$n-1 \leqslant \operatorname{cd} S \leqslant \operatorname{cd} G_j \leqslant n, \quad j=1,2. \tag{3.1}$$

3.3. We now give *sufficient* conditions for $G=G_1 *_S G_2$, all groups of type (FP), to be a duality group of dimension n. We will see in particular that all combinations of $\operatorname{cd} S$, $\operatorname{cd} G_1$, $\operatorname{cd} G_2$ which comply with the necessary conditions (3.1) actually occur. Explicit examples will be given in a separate section (§4).

THEOREM 3.2. *Let $G=G_1 *_S G_2$, G_1, G_2 and S of type (FP). If G_1 and G_2 are duality groups of dimension n and S is a duality group of dimension $n-1$, then G is a duality group of dimension n.*

Proof. The Mayer-Vietoris sequence (2.2) with $A=\mathbf{Z}G$ immediately yields $H^k(G; \mathbf{Z}G)=0$ for $k \neq n$, and a short exact sequence

$$0 \to H^{n-1}(S; \mathbf{Z}G) \to H^n(G; \mathbf{Z}G) \to H^n(G_1; \mathbf{Z}G) \oplus H^n(G_2; \mathbf{Z}G) \to 0. \tag{3.2}$$

By duality we have $H^{n-1}(S; \mathbf{Z}G) \cong H^{n-1}(S; \mathbf{Z}S) \otimes_S \mathbf{Z}G$ and $H^n(G_j; \mathbf{Z}G) \cong H^n(G_j; \mathbf{Z}G_j) \otimes_{G_j} \mathbf{Z}G$, $j=1, 2$. These groups are torsion-free over $\mathbf{Z}$. It follows that $H^n(G; \mathbf{Z}G)$ is torsion-free. By [1], Theorem 4.5, G is a duality group of dimension n.

THEOREM 3.3. *Let $G=G_1 *_S G_2$, G_1, G_2 and S of type (FP). If G_2 is a duality group of dimension n, and G_1 and S are duality groups of dimension $n-1$ such that the restriction* $\mathrm{res}^*: H^{n-1}(G_1; A) \to H^{n-1}(S; A)$ *is a monomorphism for all induced G_1-modules A, then G is a duality group of dimension n.*

Remark 3.4. The (necessary) assumption that $\mathrm{res}^*: H^{n-1}(G_1; A) \to H^{n-1}(S; A)$ be a monomorphism for all induced G_1-modules A is fulfilled, in particular, if S has *finite index* in G_1. We will show this when discussing examples (§4); cases where S has infinite index and where the condition holds will also be exhibited.

Proof of Theorem 3.3. Let A be an induced G-module (and hence an induced G_1-, G_2- and S-module). By [1], Prop. 1.4, $H^k(G_1; A)=H^k(S; A)=0$ for $k \neq n-1$, and $H^k(G_2; A)=0$ for $k \neq n$. The sequence (2.2) then yields $H^k(G; A)=0$ for $k \neq n-1$, n and an exact sequence

$$0 \to H^{n-1}(G; A) \to H^{n-1}(G_1; A) \xrightarrow{\mathrm{res}^*} H^{n-1}(S; A) \to H^n(G; A) \to H^n(G_2; A) \to 0.$$

By assumption res* is a monomorphism, hence $H^{n-1}(G; A)=0$. By [1], Theorem 4.4 it follows that G is a duality group, of dimension n.

THEOREM 3.5. *Let $G=G_1 *_S G_2$, G_1, G_2 and S of type (FP), and let G_1, G_2 and S be duality groups of dimension $n-1$.*

(i) *If* $\mathrm{cd}\, G \leqslant n-1$, *then G is a duality group of dimension $n-1$.*

(ii) *If for all induced G-modules A the restrictions* $\mathrm{res}^*: H^{n-1}(G_j; A) \to H^{n-1}(S; A)$ *are monomorphisms, $j=1, 2$, and* $\mathrm{res}^* H^{n-1}(G_1; A) \cap \mathrm{res}^* H^{n-1}(G_2; A)=0$, *then G is a duality group of dimension n.*

Remark 3.6. The assumption (ii) – which is necessary for G to be a duality group of dimension n – is again fulfilled if S has *finite index* in G_1 and G_2, but also in other cases (see examples, §4).

Proof of Theorem 3.5. The sequence (2.2) for induced G-modules, together with [2], Theorem 4.4, yields the result. The assumption (ii) simply tells that the map $(\mathrm{res}^*, -\mathrm{res}^*): H^{n-1}(G_1; A) \oplus H^{n-1}(G_2; A) \to H^{n-1}(S; A)$ is a monomorphism.

4. Examples. Topological Aspects

4.1. In this section we give examples, of algebraic and of topological nature, illustrating the various dimension cases which occur in Section 3.

The *algebraic* examples are explicit applications of Theorems 3.2, 3.3 and 3.5, to groups known to be (low-dimensional) duality groups. They partly concern cases of finite index subgroups S; these cases require some additional algebraic ad hoc arguments (Lemma 4.1 and 4.2 below). It should be mentioned that these actually belong to a more general, rather subtle, context dealt with in detail elsewhere (see [4]). They do not use full duality but only finite cohomology dimension.

The *topological* examples combine topological and algebraic arguments, and partly apply the theorems of Section 3, partly illustrate them. They are based on some remarks on Eilenberg-MacLane complexes of duality groups, and on their unions with identified subcomplexes, and of course on the van Kampen theorem.

4.2. *Algebraic Preliminaries: Subgroups of Finite Index*

LEMMA 4.1. *Let G be a group of type (FP), with* $\operatorname{cd} G=n$, *and $S\subset G$ a subgroup of finite index. Then the restriction* $\operatorname{res}^*: H^n(G; A)\to H^n(S; A)$ *is a monomorphism for every induced G-module $A=L\otimes \mathbf{Z}G$.*

Proof. Since $|G:S|$ is finite, we can identify $C=H^n(G; \mathbf{Z}G)$ with $H^n(S; \mathbf{Z}S)$, cf. [1], §3. By Theorem 4.2 of [1] we have isomorphisms $H^n(G; A)\cong C\otimes_G A$ and $H^n(S; A)\cong C\otimes_S A$. Under these isomorphisms, as shown in [4], the restriction map res^* corresponds to the *transfer* $\operatorname{res}: C\otimes_G A\to C\otimes_S A$, given for arbitrary G-modules A by $\operatorname{res}(c\otimes a)=\sum_i cr_i^{-1}\otimes r_i a$, $c\in C$, $a\in A$, $\{Sr_i\}$ being the right cosets of G modulo S.

For $A=L\otimes\mathbf{Z}G$ one has an isomorphism $\kappa: C\otimes_S(L\otimes\mathbf{Z}G)\cong (C_0\otimes L)\otimes\mathbf{Z}(G/S)$; it is given by

$$\kappa(c\oplus u\oplus x)=cx\oplus u\oplus Sx, \qquad c\in C,\, u\in L,\, x\in G.$$

C_0 denotes the Abelian group underlying C. In particular, $C\otimes_G(L\otimes\mathbf{Z}G)\cong C_0\otimes L$. As a map $C_0\otimes L\to(C_0\otimes L)\otimes\mathbf{Z}(G/S)$ the transfer is given by

$$\begin{aligned}\kappa\operatorname{res}\kappa^{-1}(c\otimes u)&=\kappa\operatorname{res}(c\otimes u\otimes e)\\&=\kappa\left(\sum_i cr_i^{-1}\otimes u\otimes r_i\right)\\&=\sum_i c\otimes u\otimes Sr_i\end{aligned}$$

This is obviously a monomorphism.

LEMMA 4.2. *Let $G=G_1 *_S G_2$, where G_1, G_2 are of type (FP),* $\operatorname{cd} G_1\leqslant n$ *and* $\operatorname{cd} G_2\leqslant n$, *and where S has finite index in both G_1 and G_2. Then the restriction images* *in* $H^n(S; A)$, *for any induced G-module* A, *have intersection* 0:

$$\operatorname{res}^* H^n(G_1; A) \cap \operatorname{res}^* H^n(G_2; A) = 0.$$

Proof. As before we identify $H^n(G_1; \mathbf{Z}G_1) = H^n(G_2; \mathbf{Z}G_2) = H^n(S; \mathbf{Z}S)$ and denote this module by C. The restrictions can be replaced by the transfers $\operatorname{res}: C \otimes_{G_j} A \to C \otimes_S A$, $j=1, 2$. For $A = L \otimes \mathbf{Z}G$ the transfer

$$\operatorname{res}: (C_0 \otimes L) \otimes \mathbf{Z}(G/G_j) \to (C_0 \otimes L) \otimes \mathbf{Z}(G/S)$$

is given by

$$\operatorname{res}(c \otimes u \otimes G_j x) = \sum_{r \in \Gamma_j} c \otimes u \otimes Srx$$

$c \in C_0$, $u \in L$, $x \in G$; Γ_j denotes a set of representatives (including e) of G_j modulo S, $j=1, 2$.

We first consider the transfer map $\operatorname{res}: \mathbf{Z}(G/G_j) \to \mathbf{Z}(G/S)$ given by $\operatorname{res}(G_j x) = \sum_{r \in \Gamma_j} Srx$. We recall that the words of the form $w = g_2 g_1' g_2' g_1'' \cdots$ with letters $g_j, g_j', g_j'' \ldots \in \Gamma_j$, $j=1, 2$, all $\neq e$, and with initial letter from Γ_2, represent the right *cosets* $\neq G_1$ of G modulo G_1. Since cancellation is not possible, the length $\lambda(w)$ of such a word is defined in an obvious way. An element $t_1 \in \mathbf{Z}(G/G_1)$ is a finite sum $t_1 = \sum m G_1 g_2 g_1' g_2' g_1'' \cdots$ with integral coefficients. Its image $\operatorname{res}(t_1) \in \mathbf{Z}(G/S)$ is of the form

$$\operatorname{res}(t_1) = \sum m S g_2 g_1' g_2' \cdots + \sum_{\substack{g_1 \neq e \\ g_1 \in \Gamma_1}} \sum m S g_1 g_2 g_1' \cdots$$

We have divided the sum into two parts according to whether the first letter to the right of mS is in Γ_1 or in Γ_2. Let $\bar{\lambda}(t_1)$ be the maximum length of words occurring in t_1, and let $g_1 w$ be a term in t_1 with $\lambda(w) = \bar{\lambda}(t_1)$. Then there is a term $Sg_1 w$ in the second part of $\operatorname{res}(t_1)$, with $\lambda(g_1 w) = \bar{\lambda}(t_1) + 1$.

If we now assume $\operatorname{res}(t_1) = \operatorname{res}(t_2)$ for some $t_2 \in \mathbf{Z}(G/G_2)$, the term $Sg_1 w$ must occur in the "first part" of $\operatorname{res}(t_2)$, i.e., $g_1 w$ must occur in t_2, and thus $\bar{\lambda}(t_2) \geqslant \bar{\lambda}(t_1) + 1$. But the situation is entirely symmetric in t_1 and t_2, so that $\bar{\lambda}(t_1) \geqslant \bar{\lambda}(t_1) + 2$. Hence if $\operatorname{res}(t_1) = \operatorname{res}(t_2)$, there are *no words* of maximum length in t_1, i.e., $t_1 = 0 = t_2$. Thus we have proved that

$$\operatorname{res} \mathbf{Z}(G/G_1) \cap \operatorname{res} \mathbf{Z}(G/G_2) = 0.$$

Since the restriction maps themselves are monomorphisms, we have a short exact sequence of the form

$$\mathbf{Z}(G/G_1) \oplus \mathbf{Z}(G/G_2) \xrightarrow{(\text{res}, -\text{res})} \mathbf{Z}(G/S) \twoheadrightarrow K \tag{4.1}$$

The cokernel K is torsion-free: Tensoring over $\mathbf{Z}$ with $\mathbf{Z}_p = \mathbf{Z}/(p)$, for a prime p, gives rise to the exact sequence

$$0 \to \mathrm{Tor}(\mathbf{Z}_p, K) \to \mathbf{Z}_p(G/G_1) \oplus \mathbf{Z}_p(G/G_2) \xrightarrow{(\mathrm{res},\, -\mathrm{res})} \mathbf{Z}_p(G/S) \to \mathbf{Z}_p \otimes K \to 0.$$

But the above arguments on $\mathbf{Z}(G/G_j)$ and $\mathbf{Z}(G/S)$ are valid for $\mathbf{Z}_p$-group rings as well (the crucial point was that there is no cancellation of terms in $\mathrm{res}(t_1)$ etc.). Therefore (res, $-$res) is again a monomorphism, and $\mathrm{Tor}(\mathbf{Z}_p, K)$ is 0 for all primes p, i.e., K is torsion-free. If we tensor (2.2) over $\mathbf{Z}$ with $C_0 \otimes L$, we conclude that $(\mathrm{res}, -\mathrm{res})\colon (C_0 \otimes L) \otimes \mathbf{Z}(G/G_1) \oplus (C_0 \otimes L) \otimes \mathbf{Z}(G/G_2) \to (C_0 \otimes L) \otimes \mathbf{Z}(G/S)$ is a monomorphism, whence

$$\mathrm{res}(C_0 \otimes L) \otimes \mathbf{Z}(G/G_1) \cap \mathrm{res}(C_0 \otimes L) \otimes \mathbf{Z}(G/G_2) = 0.$$

This proves Lemma 4.2.

4.3. *Topological Preliminaries. Eilenberg-MacLane Spaces*

If X is a CW-complex, with subcomplexes $X_1 \subset X$ and $X_2 \subset X$ such that $X_1 \cap X_2 = Y$ is not empty, we will write $X = X_1 \cup_Y X_2$. Equivalently, we consider two CW-complexes X_1, X_2 containing non-empty subcomplexes $Y_1 \subset X_1$, $Y_2 \subset X_2$ which are isomorphic. Then, by identifying Y_1 with Y_2 through a given isomorphism and writing $Y = Y_1 = Y_2$, we obtain a complex $X = X_1 \cup_Y X_2$.

We will assume X_1 and X_2 and whence X to be *connected*; Y need in general not be connected. We only consider examples where the fundamental group $\pi_1(Y^{(\nu)})$ maps *monomorphically* into $\pi_1(X_1)$ and $\pi_1(X_2)$, for each component $Y^{(\nu)}$ of Y. – If Y is connected, the van Kampen theorem tells that $\pi_1(X)$ is the amalgamated free product $\pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$.

In the situation described above, let $p\colon \tilde{X} \to X$ be the universal cover of X, and write $p^{-1}(X_j) = \bar{X}_j$, $j = 1, 2$, $p^{-1}(Y^{(\nu)}) = \bar{Y}^{(\nu)}$. Then $\bar{X}_j$ is a certain number of copies of the universal cover $\tilde{X}_j$, $j = 1, 2$, and $\bar{Y}^{(\nu)}$ is a certain number of copies of $\tilde{Y}^{(\nu)}$, for all components $Y^{(\nu)}$ of Y. From $\tilde{X} = \bar{X}_1 \cup_{\bar{Y}} \bar{X}_2$ and the topological Mayer-Vietoris sequence with integral coefficients

$$\cdots \to H_k(\bar{Y}) \to H_k(\bar{X}_1) \oplus H_k(\bar{X}_2) \to H_k(\tilde{X}) \to H_{k-1}(\bar{Y}) \to \cdots$$

we deduce immediately:

(i) If $\tilde{X}_1$, $\tilde{X}_2$ and $\tilde{Y}$ have trivial homology, then the same holds for $\tilde{X}$.

(*ii*) If $\tilde{X}$ and $\tilde{Y}$ have trivial homology, then the same holds for $\tilde{X}_1$ and $\tilde{X}_2$.

By "trivial homology" we mean $H_k = 0$ for $k \geqslant 1$ (no statement about H_0). Note that obviously $H_1(\tilde{X}) = H_1(\tilde{X}_1) = H_1(\tilde{X}_2) = H_1(\tilde{Y}) = 0$).

For a connected complex X, the universal cover $\tilde{X}$ having trivial homology is equivalent to X being aspherical; i.e., to X being an Eilenberg-Mac Lane complex $K(G, 1)$ for its fundamental group $G=\pi_1(X)$. Thus for $X=X_1 \cup_Y X_2$ as before, but with Y connected, (i) tells that if X_1, X_2 and Y are Eilenberg-Mac Lane complexes, so is X; namely $X=K(G, 1)$ where $G=G_1 *_S G_2$, $G_j=\pi_1(X_j)$ for $j=1, 2$, $S=\pi_1(Y)$. And conversely, by (ii), if $X=K(G, 1)$, $Y=K(S, 1)$ then X_1, X_2 are Eilenberg-Mac Lane complexes. The results of Section 3 then have topological interpretations: one replaces (Poincaré) duality groups by (Poincaré) duality Eilenberg-Mac Lane complexes, cf. [4], Section 6, and amalgamated free products by unions of spaces with identified subspaces.

We recall here two topological criteria for duality: (a) If $X=K(G, 1)$ is a closed manifold (i.e., compact without boundary) then G is a Poincaré duality group. (b) if $X=K(G, 1)$ is an m-dimensional compact orientable manifold-with-boundary such that $H_k(\partial\tilde{X})=0$ for all $k\neq q$ (H_0 being reduced) and $H_q(\partial\tilde{X})$ torsion-free, then G is a duality group of dimension $n=m-q-1$ with $C=H_q(\partial\tilde{X})$.

4.4. *Topological Preliminaries: 3-Dimensional Manifolds*

We recall some facts concerning "sufficiently large" 3-manifolds (cf. Waldhausen [7] for terminology and results). Let M denote, throughout this and the following sections, a triangulable *compact connected orientable* 3-*manifold*, and let $G=\pi_1(M)$.

PROPOSITION 4.3. *If M is irreducible and ∂M incompressible in M, then M is an Eilenberg-Mac Lane complex $K(G, 1)$ and G is a duality group of dimension* 2.

Proof. M is aspherical, see e.g. [7], Lemma 1.1.5. The boundary ∂M consists of orientable surfaces of genus >0, and for each component the fundamental group imbeds monomorphically into $\pi_1(M)$. Thus $\partial\tilde{M}$ consists of universal covers of the surfaces occurring in ∂M, i.e., of copies of $\mathbf{R}^2$. The above conditions for duality are therefore fulfilled, with $q=0$, $n=m-q-1=2$. The dualizing module $C=H_0(\partial\tilde{M})$ is $\mathbf{Z}$-free.

Examples of manifolds M which satisfy the assumptions of Proposition 4.3 are the closed complements of non-trivial knots in the 3-sphere. Then ∂M is a torus, and incompressible in M.

Let now M be irreducible and closed ($\partial M=\emptyset$), and assume that M contains an incompressible separating surface Y. Then $M=M_1 \cup_Y M_2$, where M_1, M_2 are compact 3-manifolds with $\partial M_1=\partial M_2=Y$ fulfilling the assumptions of Proposition 4.3. We have $\pi_1(M)=G=G_1 *_S G_2$ with $\pi_1(M_j)=G_j$, $j=1, 2$, $\pi_1(Y)=S$. The groups G_1, G_2 are duality groups of dimension 2, S is a Poincaré duality group of dimension 2, and G is a Poincaré duality group of dimension 3. We thus have an example illustrating (not using) Theorem 3.5. Such examples can be obtained by taking for M_1 and M_2

3-manifolds-with-boundary as in Proposition 4.3, with ∂M_1 and ∂M_2 being surfaces of the same genus, and by identifying ∂M_1 with ∂M_2 through a homeomorphism (e.g., M_1 and M_2 are knot-complements, ∂M_1 and ∂M_2 tori).

In these examples, the necessary condition (ii) of Theorem 3.5 must be fulfilled; in particular, $\mathrm{res}^*: H^2(G_1; A) \to H^2(S; A)$ is a monomorphism for all induced G_1-modules A. This remark yields the following result:

COROLLARY 4.4. *If M is irreducible and ∂M incompressible in M, with $\pi_1(M) = G$, $\pi_1(\partial M) = S$, then the restriction* $\mathrm{res}^*: H^2(G; A) \to H^2(S; A)$ *is a monomorphism for all induced G-modules A.*

Remark 4.5. A similar situation arises if we take two tori X_1, X_2 with an open disc removed, and identify the two boundary circles $\partial X_1 = \partial X_2 = Y$. Then $X = X_1 \cup_Y X_2$ is the closed surface of genus 2. Since X and Y are Eilenberg-Mac Lane spaces $K(G, 1)$, $K(S, 1)$, so are $X_j = K(G_j, 1)$, $j = 1, 2$; G_1 is free on two generators a, b, G_2 on c, d, and S is cyclic generated by $[a, b] = [c, d]$. The group G is presented by $\langle a, b, c, d \mid [a, b]\,[d, c] = e \rangle$.

G_1, G_2 are duality groups of dimension 1, S is a Poincaré duality group of dimension 1, and G is a Poincaré duality group of dimension 2. We thus have an example illustrating (not using) Theorem 3.5, case (ii). As before we get a side-result:

COROLLARY 4.5. *Let G be free on two generators a, b, and S cyclic generated by $[a, b]$. Then the restriction* $\mathrm{res}^*: H^1(G; A) \to H^1(S; A)$ *is a monomorphism for all induced G-modules A.*

4.5. *Examples*

We will apply Theorems 3.2, 3.3 and 3.5 to explicitly given amalgamated free products of groups $G = G_1 *_S G_2$. We write $n_j = \mathrm{cd}\, G_j$, $m = \mathrm{cd}\, S$ and $n = \mathrm{cd}\, G$, and use the symbol $[n_1, n_2, m; n]$ to indicate the dimensions occurring in an example. We recall that Theorem 3.2 refers to the case $[n, n, n-1; n]$, Theorem 3.3 to $[n-1, n, n-1; n]$, Theorem 3.5, case (i) to $[n-1, n-1, n-1; n-1]$ and case (ii) to $[n-1, n-1, n-1; n]$.

EXAMPLE 1 $[2, 2, 1; 2]$. Let G be presented by $\langle a, b, c \mid [a, b] = [a, c] = e \rangle$. This group can be obtained as $G = G_1 *_S G_2$ with $G_1 = \langle a, b \mid [a, b] = e \rangle$, $G_2 = \langle c, d \mid [c, d] = e \rangle$, S infinite cyclic generated by $a \in G_1$ or $d \in G_2$ respectively. G_1 and G_2 are Poincaré duality groups of dimension 2, S of dimension 1. By Theorem 3.2 G is a duality group of dimension 2.

Corresponding to the decomposition $G = G_1 *_S G_2$ one may take for $K(G, 1)$ the space obtained from two tori X_1, X_2 by identifying circles which are generators of $\pi_1(X_1)$ and $\pi_1(X_2)$ respectively (e.g., one takes two tori in $\mathbf{R}^3$ having the same vertical axes of rotation and puts one on top of the other). It follows that this space is a duality complex of formal dimension 2.

EXAMPLE 2 [1, 2, 1; 2]. Let G be presented by $\langle a, b \mid [a^2, b]=e\rangle$. We may write $G=G_1 *_S G_2$, $G_1=\langle a\rangle$, $G_2=\langle b, c \mid [b, c]=e\rangle$, $S=(a^2)=(c)$. Then G_1, G_2, S are Poincaré duality groups of dimension 1, 2, 1 respectively. Since S has finite index in G_1, the restriction condition of Theorem 3.3 is fulfilled (Proposition 4.1). It follows that G is a duality group of dimension 2.

An alternative proof of this fact is obtained as an application of [1], Theorem 5.2.

EXAMPLE 3 [1, 2, 1; 2]. We take $G_1=\langle a, b\rangle$, $G_2=\langle c, d \mid [c, d]=e\rangle$ and $S=([a, b])=(c)$. Then $G=G_1 *_S G_2=\langle a, b, c, d \mid [a, b]=c,\ [c, d]=e\rangle$. Since all finitely generated free groups are duality groups of dimension 1 so is G_1; G_2 and S are Poincaré duality groups of dimensions 2 and 1. The restriction condition for $\mathrm{res}^*: H^1(G_1; A)\to H^1(S; A)$ is fulfilled by Corollary 4.5 (though here the index $|G_1 : S|$ is not finite). By Theorem 3.3, G is a duality group of dimension 2.

A topological description similar to that of Example 1 is easily obtained.

EXAMPLE 4 [2, 3, 2; 3]. Let G_1 be the fundamental group $\pi_1(X_1)$ of the complement of a non-trivial knot in the 3-sphere, $S=\pi_1(\partial X_1)$ the fundamental group of the boundary torus, and $G_2=\pi_1(X_2)$ the fundamental group of the 3-torus X_2. We identify S with the fundamental group of a 2-torus $Y\subset X_2$. Then $G=G_1 *_S G_2=\pi_1(X)$, where X is the union of X_1 and X_2 with ∂X_1 identified with Y (of course, algebraic descriptions of G are available).

G_1 is a duality group of dimension 2, G_2 a Poincaré duality group of dimension 3, S of dimension 2; the restriction condition of Theorem 3.3 is fulfilled by Corollary 4.4. Thus G is a duality group of dimension 3, i.e., X is a duality complex.

EXAMPLE 5 [1, 1, 1; 1]. We take for G_1 and G_2 free groups on two generators, $G_1=\langle s, b\rangle$, $G_2=\langle c, h\rangle$, and $S=(b)=(d)$. Then $G=G_1 *_S G_2$ is free on 2 generators, and we have a trivial illustration of Theorem 3.5, case (i). – A less trivial example, where the theorem is applied, is the following.

EXAMPLE 6 [2, 2, 2; 2]. We take G_1 to be the group called G in Example 1, X_1 the space called X there, $G_1=\pi_1(X_1)$. Let $G_2=\pi_1(X_2)$ be a second copy of the same group, X_2 of the same space. Let $S=\pi_1(Y)$, where Y is one of the tori in X_1 or X_2 respectively. Then $G=G_1 *_S G_2=\pi_1(X)$ with $X=X_1\cup_Y X_2$; this space simply consists of three tori, with common vertical axis, one on top of the other. Algebraically, $G=\langle a, b, c, d \mid [a, d]=[b, d]=[c, d]=e\rangle$. It is clear geometrically, that $\mathrm{cd}\, G=2$; it can also easily be seen from the fact that G is an extension of a cyclic group by a free group. Thus by Theorem 3.5, case (i), G is a duality group of dimension 2.

EXAMPLE 7 [1, 1, 1; 2]. Let $G_1=\langle a\rangle$, $G_2=\langle b\rangle$, $S=(a^2)=(b^3)$, all Poincaré duality groups of dimension 1. Since S has finite index in both G_1 and G_2, condition (ii) of Theorem 3.5 is fulfilled (Proposition 4.2). Hence $G=G_1 *_S G_2=\langle a, b \mid a^2=b^3\rangle$ is a duality group of dimension 2. (Cf. again [1], Theorem 5.2).

We note here that in Section 4.3 examples are given for the dimension cases [2, 2, 2; 3] and [1, 1, 1; 2] where duality of G occurs for topological reasons. They illustrate (but do not use) Theorem 3.5, case (ii), with $|G_1:S|=|G_2:S|=\infty$.

5. Amalgamated free Products of Duality Groups: Type (FD$_*$)

5.1. In this section we show that the results of Section 3 remain valid without the assumption that the groups involved are of type (FP). This requires some modification of the proofs; we first explain the difference in approach.

We recall that for a duality group G of dimension n one has

$$H^k(G; A)=0 \quad \text{for} \quad k\neq n \quad \text{and all induced } G\text{-modules } A. \tag{5.1}$$

For groups of type (FP) condition (5.1) is also *sufficient* for duality. This was essential in Section 3: we proved that (5.1) is carried over, in the appropriate dimensions, from the given duality groups to the amalgamated free product. This part of the arguments, based upon the Mayer-Vietoris sequence, remains valid in the general case.

Now one can show (see [4]) that there is another class of groups, called groups of type (FD_*), for which (5.1) is sufficient for duality; the definition is given below. While we do not know[1]) whether duality groups must be of type (FP), they are necessarily of type (FD_*), as shown in [4], Theorem 2.4. Therefore, to prove the theorems of Section 3 for arbitrary groups, we can start from the fact that those groups which are duality groups by assumption are of type (FD_*); all that remains then to be proved is that this property is carried over to the amalgamated free product. Hereby we rely on the detailed analysis of type (FD_*) made in [4].

5.2. For the definition of type (FD_*) we need a natural "duality" homomorphism φ_* closely related to the cap-product $(e\cap -)$. We recall that one has for left G-modules M, A the natural homomorphism

$$\varphi: M^*\otimes_G A \to \mathrm{Hom}_G(M, A)$$

given by

$$\varphi(f\otimes a)(m)=f(m)\,a, f\in M^*=\mathrm{Hom}_G(M, ZG), \qquad a\in A, m\in M.$$

For a G-projective resolution $\mathscr{P} \twoheadrightarrow \mathbf{Z}$ we thus have a homomorphism of complexes

$$\varphi: \mathscr{P}^* \otimes_G A \to \mathrm{Hom}_G(\mathscr{P}, A). \tag{5.2}$$

Now assume G to be of finite cohomology dimension $\mathrm{cd}\, G = n$. Then $\mathscr{P}$ is split-exact over $\mathbf{Z}G$ in dimensions $\geqslant n$, and thus φ induces

$$\varphi_*: H^n(G; \mathbf{Z}G) \otimes_G A \to H^n(G; A).$$

If we wish to emphasize the coefficient module A, we write φ_*^A for φ_*.

DEFINITION 5.1. A group G is said of type (FD_*) if it has finite cohomology dimension n and if φ_*^A is an isomorphism for all G-modules A.

Remarks 5.2. Let $\mathrm{cd}\, G = n$. It is shown in [4], Theorem 2.4, that if φ_*^F is an epimorphism foı all free G-modules F, then φ_*^A is an isomorphism for all G-modules A. Moreover, φ_*^A is an isomorphism for all G-modules A if and only if there is an element $e \in H_n(G; C)$, $C = H^n(G; \mathbf{Z}G)$, such that $(e \frown -): H^n(G; A) \to C \otimes_G A$ is an isomorphism for all G-modules A, inverse of φ_*^A. In particular, if *G is a duality group then it is of type* (FD_*). – Let G be a group of finite cohomology dimension, and assume that G admits a G-projective resolution $\mathscr{P} \twoheadrightarrow \mathbf{Z}$ with P_n finitely generated[2]). Then φ_*^A is an isomorphism (cf. [1], Theorem 4.2), i.e., G is of type (FD_*). In particular, all groups of type (FP) are of type (FD_*). Note that the converse is not true: There are groups G with $\mathrm{cd}\, G = n$ and P_n finitely generated, but not of type (FP), see [4], Section 2.5.

We first show that the dimension restriction of Theorem 3.1 holds for type (FD_*).

THEOREM 5.3. *Let $G = G_1 *_S G_2$ be a non-trivial amalgamated free product. If G is a duality group of dimension n, and if G_1, G_2 are of type (FD_*), then*

$$n - 1 \leqslant \mathrm{cd}\, S \leqslant \mathrm{cd}\, G_j \leqslant \mathrm{cd}\, G = n, \quad j = 1, 2.$$

Proof. Since G_1 is of type (FD_*), we have $H^n(G_1; \mathbf{Z}G_1) \otimes_{G_1} \mathbf{Z}G \cong H^n(G_1; \mathbf{Z}G)$; in other words, $H^n(G_1; \mathbf{Z}G)$ is isomorphic to $H^n(G_1; \mathbf{Z}G_1) \otimes \mathbf{Z}(G/G_1) \cong \oplus H^n(G_1; \mathbf{Z}G_1)$, the sum being over the cosets of G modulo G_1. Thus the proof of Theorem 3.1 applies without change.

5.3. Before giving the analogue of Theorems 3.2, 3.3 and 3.5 we show that φ_* occurring in the definition is compatible with the Mayer-Vietoris sequence.

We first have to relate φ_* to subgroups $S \subset G$. Let $\mathscr{P} \twoheadrightarrow \mathbf{Z}$ be a G-projective resolution. There is a map $\varphi(S)$ generalizing φ of (5.2), for a G-module A,

$$\varphi(S): \mathrm{Hom}_S(\mathscr{P}, \mathbf{Z}G) \otimes_G A \to \mathrm{Hom}_S(\mathscr{P}, A)$$

[2]) Type (FD_*) is, in fact, equivalent to that property, cf. "Note added in proof" at the end of the paper.

given by $\varphi(S)(f\otimes a)(p)=f(p)\,a, f\in \mathrm{Hom}_S(\mathscr{P}, \mathbf{Z}G), a\in A, p\in\mathscr{P}$. If $\operatorname{cd} S\leqslant n$, we have again an induced homomorphism

$$\varphi(S)_*: H^n(S; \mathbf{Z}G)\otimes_G A\to H^n(S; A).$$

Of course, if $S=G$ then $\varphi(S)_*=\varphi_*$. One has a commutative diagram

$$\begin{tikzcd}
H^n(S; \mathbf{Z}S)\otimes_S A \arrow[rr, "\varphi^A_*"] \arrow[dr, "\varphi_*{}^{\mathbf{Z}G}\otimes A"'] & & H^n(S; A) \\
& H^n(S; \mathbf{Z}G)\otimes_G A \arrow[ur, "\varphi(S)^A_*"'] &
\end{tikzcd}$$

(note that $H^n(S; \mathbf{Z}S)\otimes_S A=[H^n(S; \mathbf{Z}S)\otimes_S \mathbf{Z}G]\otimes_G A)$. If for the group S the map $\varphi_*^{\mathbf{Z}G}$ is an isomorphism (e.g., if S is of type (FD_*)), we may therefore identify the map φ_*^A for S with $\varphi(S)_*^A: H^n(S; \mathbf{Z}G)\otimes_G A\to H^n(S; A)$.

In the Mayer-Vietoris sequence for $G=G_1 *_S G_2$ we have compatibility of φ_* (for the various groups) with all restriction homomorphisms, by [4], Section 3. Compatibility with the connecting homomorphisms is described in the following lemma.

LEMMA 5.4. *Let $G=G_1 *_S G_2$ be an amalgamated free product with* $\operatorname{cd} S<\operatorname{cd} G\leqslant n$, *and A a left G-module. Then the following diagram is commutative*

$$\begin{tikzcd}
H^{n-1}(S; \mathbf{Z}G)\otimes_G A \arrow[r, "\delta\otimes_G A"] \arrow[d, "\varphi(S)^A_*"'] & H^n(G; \mathbf{Z}G)\otimes_G A \arrow[d, "\varphi^A_*"] \\
H^{n-1}(S; A) \arrow[r, "\delta"'] & H^n(G; A)
\end{tikzcd}$$

where δ is the connecting homomorphism in the Mayer-Vietoris sequence (2.2).

Proof. The short exact sequence (2.1) yields an exact sequence of left G-modules

$$A\rightarrowtail \mathrm{Hom}_{G_1}(\mathbf{Z}G, A)\oplus \mathrm{Hom}_{G_2}(\mathbf{Z}G, A)\twoheadrightarrow \mathrm{Hom}_S(\mathbf{Z}G, A).$$

Let $\mathscr{P}\twoheadrightarrow\mathbf{Z}$ be a G-projective resolution. For any subgroup $H\subset G$ one has natural isomorphisms $\mathrm{Hom}_G(\mathscr{P}, \mathrm{Hom}_H(\mathbf{Z}G, A))\cong \mathrm{Hom}_H(\mathscr{P}, A)$. As $\varphi(S)$ commutes with restrictions we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_G(\mathscr{P}, \mathbf{Z}G)\otimes_G A \arrow[r, "\lambda"] \arrow[d, "\varphi"'] & \mathrm{Hom}_{G_1}(\mathscr{P}, \mathbf{Z}G)\otimes_G A\oplus \mathrm{Hom}_{G_2}(\mathscr{P}, \mathbf{Z}G)\otimes_G A \arrow[r, two heads] \arrow[d, "\varphi(G_1)\oplus\varphi(G_2)"'] & \mathrm{Hom}_S(\mathscr{P}, \mathbf{Z}G)\otimes_G A \arrow[d, "\varphi(S)"'] \\
\mathrm{Hom}_G(\mathscr{P}, A) \arrow[r] & \mathrm{Hom}_{G_1}(\mathscr{P}, A)\oplus \mathrm{Hom}_{G_2}(\mathscr{P}, A) \arrow[r, two heads] & \mathrm{Hom}_S(\mathscr{P}, A).
\end{tikzcd}$$

If A is G-free then λ is an monomorphism. Passing to cohomology one thus gets the assertion of the lemma for *free A*. For arbitrary G-modules A, take a free module F

with an epimorphism $F \twoheadrightarrow A$; one then has a commutative diagram

$$\begin{tikzcd}
H^{n-1}(S; ZG)\otimes_G F \arrow[rrr, "\delta\otimes_G F"] \arrow[dr, "\mu"] \arrow[ddd, "\varphi(S)_*^F"'] & & & H^n(G; ZG)\otimes_G F \arrow[dl] \arrow[ddd, "\varphi_*^F"] \\
& H^{n-1}(S; ZG)\otimes_G A \arrow[r, "\delta\otimes_G A"] \arrow[d, "\varphi(S)_*^A"'] & H^n(G; ZG)\otimes_G A \arrow[d, "\varphi_*^A"] & \\
& H^{n-1}(S; A) \arrow[r, "\delta"'] & H^n(G; A) & \\
H^{n-1}(S; F) \arrow[rrr, "\delta"'] \arrow[ur] & & & H^n(G; F) \arrow[ul]
\end{tikzcd}$$

The homomorphisms δ and $\delta\otimes_G$- commute with coefficient maps, and so do φ_* and $\varphi(S)_*$. We have already proved that the outer square is commutative. Since μ is an epimorphism, it follows that the inner square is also commutative.

5.4. We now establish the analogue of Theorems 3.2, 3.3 and 3.5 without finiteness restrictions.

THEOREM 5.5 (cf. Theorem 3.2). *Let $G = G_1 *_S G_2$ where G_1 and G_2 are duality groups of dimension n and S is a duality group of dimension $n-1$. Then G is a duality group of dimension n.*

Proof. According to the preliminary remarks in 5.1 we only have to prove that G is of type (FD_*). It is clear that $\operatorname{cd} G = n$, and that G_1, G_2 and S are of type (FD_*).

In the commutative diagram with exact rows, for a free G-module A,

$$\begin{tikzcd}
H^{n-1}(S; ZG)\otimes_G A \arrow[r, tail] \arrow[d, "\varphi(S)_*"'] & H^n(G; ZG)\otimes_G A \arrow[r, two heads] \arrow[d, "\varphi_*"'] & H^n(G_1; ZG)\otimes_G A \oplus H^n(G_2; ZG)\otimes_G A \arrow[d, "\varphi(G_1)_*\oplus\varphi(G_2)_*"'] \\
H^{n-1}(S; A) \arrow[r, tail] & H^n(G; A) \arrow[r, two heads] & H^n(G_1; A)\oplus H^n(G_2; A)
\end{tikzcd}$$

$\varphi(S)_*$, $\varphi(G_1)_*$ and $\varphi(G_2)_*$ are isomorphisms, and so is φ_* by the 5-lemma. Hence G is of type (FD_*), and thus a duality group of dimension n.

THEOREM 5.6 (cf. Theorem 3.3). *Let $G = G_1 *_S G_2$, where G_2 is a duality group of dimension n and G_1 and S are duality groups of dimension $n-1$. If the restriction* $\operatorname{res}: H^{n-1}(G_1; A) \to H^{n-1}(S; A)$ *is a monomorphism for all induced G_1-modules A then G is a duality group of dimension n.*

Proof. Again $\operatorname{cd} G = n$. Let A be a free G-module. Then one has a commutative diagram with exact rows

$$\begin{array}{ccccccc} H^{n-1}(G_1; ZG)\otimes_G A & \rightarrowtail & H^{n-1}(S; ZG)\otimes_G A & \rightarrow & H^n(G; ZG)\otimes_G A & \twoheadrightarrow & H^n(G_2; ZG)\otimes_G A \\ \downarrow{\scriptstyle \varphi(G_1)_*} & & \downarrow{\scriptstyle \varphi(S)_*} & & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi(G_2)_*} \\ H^{n-1}(G_1; A) & \rightarrowtail & H^{n-1}(S; A) & \rightarrow & H^n(G; A) & \twoheadrightarrow & H^n(G_2; A) \end{array}$$

Since $\varphi(G_j)_*$, $j=1,2$ and $\varphi(S)_*$ are isomorphisms, so is φ_*. Hence G is of type (FD_*), and therefore a duality group of dimension n.

THEOREM 5.7 (cf. Theorem 3.5). *Let $G=G_1 *_S G_2$, where G_1, G_2 and S are duality groups of dimension $n-1$.*

(i) *If* $\mathrm{cd}\,G \leqslant n-1$, *then G is a duality group of dimension $n-1$.*

(ii) *If for all induced G-modules A the restrictions* $\mathrm{res}^*: H^{n-1}(G_j; A) \rightarrow H^{n-1}(S; A)$ *are monomorphisms, $j=1,2$, and* $\mathrm{res}^* H^{n-1}(G_1; A) \cap \mathrm{res}^* H^{n-1}(G_2; A)=0$, *then G is a duality group of dimension n.*

Proof. Again $\mathrm{cd}\,G<\infty$, namely $=n-1$ in case (i), $=n$ in case (ii). For a free G-module A one has commutative diagrams, in case (i)

$$\begin{array}{ccccc} H^{n-1}(G; ZG)\otimes_G A & \rightarrowtail & H^{n-1}(G_1; ZG)\otimes_G A \oplus H^{n-1}(G_2; ZG)\otimes_G A & \twoheadrightarrow & H^{n-1}(S; ZG)\otimes_G A \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi(G_1)_*\oplus\varphi(G_2)_*} & & \downarrow{\scriptstyle \varphi(S)_*} \\ H^{n-1}(G; A) & \rightarrowtail & H^{n-1}(G_1; A)\oplus H^{n-1}(G_2; A) & \twoheadrightarrow & H^{n-1}(S; A); \end{array}$$

and in case (ii)

$$\begin{array}{ccccc} H^{n-1}(G_1; ZG)\otimes_G A \oplus H^{n-1}(G_2; ZG)\otimes_G A & \rightarrowtail & H^{n-1}(S; ZG)\otimes_G A & \twoheadrightarrow & H^n(G; ZG)\otimes_G A \\ \downarrow{\scriptstyle \varphi(G_1)_*\oplus\varphi(G_2)_*} & & \downarrow{\scriptstyle \varphi(S)_*} & & \downarrow{\scriptstyle \varphi_*} \\ H^{n-1}(G_1; A)\oplus H^{n-1}(G_2; A) & \rightarrowtail & H^{n-1}(S; A) & \twoheadrightarrow & H^n(G; A), \end{array}$$

with exact rows in both cases. The 5-lemma again shows that φ_* is an isomorphism; i.e., G is of type (FD_*), and hence a duality group of dimension $n-1$ or n respectively.

5.5. It is clear that the method of this section applies more generally to amalgamated free products of groups which are not assumed to be duality groups, but just groups of type (FD_*). One then obtains relations for the various *dimensions* involved, as follows.

PROPOSITION 5.8. *Let $G=G_1 *_S G_2$, where* $\mathrm{cd}\,G_1<n-1$, $\mathrm{cd}\,S<n-2$, $\mathrm{cd}\,G_2=n$, *and where G_2 is of type (FD_*). Then* $\mathrm{cd}\,G=n$ *and G is of type (FD_*).*

PROPOSITION 5.9. *Let $G=G_1 *_S G_2$, where* $\mathrm{cd}\,G_1=n-1$, $\mathrm{cd}\,S=n-1$, $\mathrm{cd}\,G_2=n$, *and where S and G_2 are of type (FD_*). Then* $\mathrm{cd}\,G=n$ *and G is of type (FD_*).*

PROPOSITION 5.10. *Let* $G = G_1 *_S G_2$, *where* $\mathrm{cd}\, G_1 = \mathrm{cd}\, G_2 = \mathrm{cd}\, S = n$, $\mathrm{cd}\, G = n+1$, *and where* S *is of type* (FD_*). *Then* G *is of type* (FD_*).

The proofs are similar to those above and can be left to the reader. Note that these results provide a method for constructing examples of groups which are of type (FD_*) but *not* of type (FP): For those groups which are assumed to be of type (FD_*) one may take type (FP) with the respective cohomology dimensions; for the others, groups which are not of type (FP), e.g., which are not finitely generated but have the appropriate finite cohomology dimensions.

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Note added in proof: It was recently proved by R. Strebel that duality groups are necessarily of type (FP), and that type (FD_*) is equivalent to the existence of a projective resolution of finite length which is finitely generated in the top dimension. – It thus turns out that the treatment in Section 3 is sufficient for our main results; Section 5 is still of interest with regard to the method used, and to groups of type (FD_*) which are not duality groups.

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Received March 14, 1974