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1. Accola [1] gave a criterion for an analytic self-mapping of Riemann surface to be the identity mapping as follows:

THEOREM A. *Let W be a Riemann surface of genus greater than one. Let f be an automorphism of W. Suppose that there are four independent cycles $C_1, \ldots, C_4$, so that $C_1 \times C_3 = C_2 \times C_4 = 1$ and $C_i \times C_j = 0$ if $i+j \equiv 1 \pmod 2$. Suppose $f(C_i) \sim C_i$ (reads $f(C_i)$ is weakly homologous to C_i) for $i = 1, \ldots, 4$. Then f is the identity mapping. Here $C_i \times C_j$ denotes the intersection number of C_i and C_j.*

If any one of the assumptions of this theorem is omitted, then the conclusion fails. In this sense Theorem A is best possible. Despite of this fact it is still open to discuss the possibility of giving weaker conditions. In this paper we shall try to seek for such weaker conditions for an analytic self-mapping to be the identity mapping.

2. At first we shall consider some conditions which imply that an analytic self-mapping reduces to an automorphism. The present author and Kubota [3] obtained such a condition already. We state it in a weak form.

THEOREM B. *Let W be a Riemann surface of positive genus and with non-abelian fundamental group, f an analytic self-mapping of W and C a non-dividing cycle on W. If $f(C) \approx C$ (reads $f(C)$ is homologous to C), then f is an automorphism of finite period.*

In terms of weak homology we have an analogue of Theorem B as follows:

THEOREM 1. *Let W and f be as in Theorem B. Suppose that $f(\gamma)$ is a dividing cycle for every dividing cycle γ. Let C be a non-dividing cycle. If $f(C) \sim C$ then f is an automorphism of finite period.*

Proof. Since $f(\gamma) \sim 0$ for every $\gamma \sim 0$, we have $f^n(C) \sim C$ for every positive integer n, where f^n denotes the n-th iteration of f. Suppose that f is not an automorphism. Then $\{f^n\}$ tends to a point on W or a component of the ideal boundary of W uniformly on every compact subset of W (Cf. [2, 3, 4]). This contradicts the fact that $f^n(C) \sim C$ and that C is a non-dividing cycle (Cf. Proof of Lemma 2 of [3]).

Using Theorem B and Theorem 1 we can paraphrase Theorem A under weaker assumptions. We omit them.

On the other hand as a supplementary result of Theorem B we have

THEOREM 2. *Let W be an open Riemann surface of positive genus. Let f be an analytic self-mapping of W. Suppose there exist two cycles C_1 and C_2, so that $C_1 \times C_2 = 1$ and $f(C_i) \approx C_i$ $(i = 1, 2)$. Then f is the identity mapping.*

This is a counterpart of a result of Marden, Richards and Rodin [4, p. 224].

Proof. Since $f(C_1) \approx C_1$ and C_1 is a non-dividing cycle, f is an automorphism of finite period. Therefore, there is a canonical subregion R of W such that $C_1, C_2 \subset R$ and that f is an automorphism of R. Hence we may assume that W is a compact bordered Riemann surface. Let $\hat{W}$ be the double of W, and let $\hat{f}$ be the extension of f to $\hat{W}$. We can carry C_1 and C_2 to $\hat{W} - W$ in the standard manner. Applying Theorem A to $\hat{W}$ and $\hat{f}$, we conclude that f is the identity mapping.

3. Using a condition on the topological characters of a Riemann surface, we have

THEOREM 3. *Let W be a Riemann surface of genus g $(\geqq 2)$ and with k $(\geqq 0)$ boundary components. Let f be an automorphism of W. Let C_1 and C_2 be cycles on W such that $C_1 \times C_2 = 1$. If $f(C_i) \sim C_i$ $(i = 1, 2)$, then the period of f is a common divisor of $g - 1$ and k. If $k = 0$ or ∞, then the period of f is a divisor of $g - 1$. If $g = \infty$, then the period of f is a divisor of k.*

This theorem yields immediately

COROLLARY. *Under the same hypotheses of Theorem* 3, *if one of three cases* (1) $g = 2$, (2) $k = 1$ *and* (3) $g - 1$ *and* k *are coprime, is assumed, then f is the identity.*

If it is not the case of Corollary we can construct a pair of W and f whose period is the greatest common divisor of $g - 1$ and k.

To prove Theorem 3 we need a lemma due to Accola [1].

LEMMA. *Let W be a compact bordered Riemann surface of genus greater than one. Let f be an automorphism of W so that $f(C_i) \sim C_i$ $(i = 1, 2)$, and $C_1 \times C_2 = 1$. If f has a fixed point or f leaves a boundary component invariant, then f is the identity mapping.*

Proof of Theorem 3. Since f is an automorphism of W, the hypothesis of Theorem 1 is fulfilled. Thus f has a finite period. If g is finite, there is a compact Riemann surface of genus g, and it has an automorphism whose period is the same as the period of f. Such a Riemann surface is constructed as follows: Since g is finite, there is a compact subregion of W whose genus is g, and the restriction of f to it is an automorphism of it. Then there is a compact Riemann surface $\hat{W}$ of genus g which contains that region as its subregion, so that the automorphism of the region can be extended conformally to it in the unique way (Cf. Oikawa [5]).

We also denote the extension by f. Let Φ be the cyclic group generated by f. Denote the order of Φ by n $(\geqq 2)$. For each $j < n$, f^j has no fixed points by virtue of Lemma. Therefore the natural projection of $\hat{W}$ onto the quotient $\hat{W}/\Phi$ has no branch points.

Let g_0 denote the genus of $\hat{W}/\Phi$. We have $n=(g-1)/(g_0-1)$ by the Riemann-Hurwitz relation.

If k is finite and positive, since f has a finite period, there is a canonical subregion R of W, so that the restriction of f to R is an automorphism of R and the number of boundary components of R is k. The restriction of f to R induces a permutation of the boundary components of R. It is known that every permutation can be written as the product of disjoint cycles. In our case, by virtue of Lemma every cycles of the permutation has the same period. Thus the period of f is a divisor of k. This completes the proof of Theorem 3.

Restricting ourselves to the hyperelliptic case, we have the following:

THEOREM 4. *Let W be a hyperelliptic Riemann surface. Let f, C_1 and C_2 be as in Theorem* 3. *Then the period of f is at most two. Particularly, if W is of even genus, then f is the identity.*

Proof. Let W be the hyperelliptic Riemann surface defined by the equation $y^2=P(x)$, where $P(x)$ is a polynomial of degree at least six and each zero of $P(x)$ is simple. There always exists the automorphism of W, so called the sheets exchange, which we denote by s. Let ϱ be the projection of W onto the x-sphere. Tsuji [6] showed that $\varrho\circ f\circ\varrho^{-1}$ is an elliptic linear transformation of the x-sphere.

If f has a fixed point, then it is the identity by Lemma. Suppose that f has no fixed point. Then $s\circ f$ has four fixed point. Since, $s\circ f=f\circ s$, we have $(s\circ f)^2=f^2$. Hence f^2 is the identity.

If W is of even genus, then f has an odd period by Theorem 3. Thus f is the identity.

In Theorem 4, if W is of odd genus, there is a pair of W and f so that f is not the identity.

Let W be the Riemann surface defined by the equation $y^2=x^{2g+2}-1$ where g is an odd number. Let C_1 be a cycle on W which surrounds $g+1$ points $e^{n\pi i/(g+1)}$, $n=0, 1,\ldots, g$, once, respectively. Let C_2 be a cycle on W which surrounds $g+1$ points $e^{n\pi i/(g+1)}$, $n=1, 2,\ldots, g+1$, once, respectively. Let f be the automorphism of W such that $(x, y)\rightarrow(-x, -y)$. Then these satisfy the assumption of Theorem 4, but f is not the identity.

Combining Theorem A and Theorem B, we have

THEOREM 5. *Let W be a Riemann surface of genus greater than one and with not more than four boundary components. Let f be an analytic self-mapping of W. Let $C_1,\ldots, C_4$ be non-dividing cycles, such that $C_1\times C_3=C_2\times C_4=1$ and $C_i\times C_j=0$ if $i+j\equiv 1$ (mod 2). Suppose $f(C_i)\sim C_i$ ($i=1,\ldots, 4$). Then f is the identity mapping.*

Proof. Since there are at most three independent dividing cycles, there is a linear combination of C_1 to C_4 such that

$$f\left(\sum_{i=1}^{4} n_i C_i\right) \approx \sum_{i=1}^{4} n_i C_i.$$

Hence, by Theorem B, f is an automorphism. Thus f is the identity by Theorem A.

4. In Theorem 5 the number four of cycles is best possible. We shall show it by an example.

Let E_1 be a triply connected plane region bounded by $\{|z|=1, 0 \leqq \arg z \leqq \pi, 7\pi/6 \leqq \arg z \leqq 11\pi/6\}$, $\{|z|=7/8, \pi \leqq \arg z \leqq 7\pi/6, 11\pi/6 \leqq \arg z \leqq 2\pi\}$, $\{\arg z=0, \pi, 7\pi/6$ and $11\pi/6, 7/8 \leqq |z| \leqq 1\}$, $\{\arg z=\pi/8, 7\pi/8, 2/3 \leqq |z| \leqq 1\}$, $\{|z|=2/3, \pi/12 \leqq \arg z \leqq \pi/8, 7\pi/8 \leqq \arg z \leqq 11\pi/12\}$ and $\{\arg z=13\pi/12, 23\pi/12, 5/8 \leqq |z| \leqq 3/4\}$. We distinguish from E_1 slits along $\{\arg z=\pi/12, 11\pi/12, 5/8 \leqq |z| \leqq 3/4, 7/8 \leqq |z|<1\}$, $\{\arg z=\pi/6, 11\pi/6, 25/192 \leqq |z| \leqq 3/16, 49/192 \leqq |z| \leqq 1/3\}$ and $\{\arg z=\pi/4, 3/4 \leqq |z|<1\}$.

Let E_2 be a simply connected plane region bounded by $\{|z|=1/2, 1, 0 \leqq \arg z \leqq \pi/6\}$, $\{\arg z=0, \pi/6, 1/2 \leqq |z| \leqq 1\}$, $\{\arg z=\pi/24, 2/3 \leqq |z| \leqq 1\}$ and $\{|z|=2/3, \pi/24 \leqq \arg z \leqq \pi/12\}$. We distinguish from E_2 slits along $\{\arg z=\pi/12, 5/8=|z| \leqq 3/4, 7/8 \leqq |z|<1\}$.

Let E_3 be a simply connected plane region bounded by $\{|z|=1/2, 1, 5\pi/6 \leqq \arg z \leqq \pi\}$, $\{\arg z=5\pi/6, \pi, 1/2 \leqq |z| \leqq 1\}$, $\{\arg z=23\pi/24, 2/3 \leqq |z| \leqq 1\}$ and $\{|z|=2/3, 11\pi/12 \leqq \arg z \leqq 23\pi/24\}$. We distinguish from E_3 slits along $\{\arg z=11\pi/12, 5/8 \leqq |z| \leqq 3/4, 7/8 \leqq |z| \leqq 1\}$.

Let E_4 be a simply connected plane region bounded by $\{|z|=1, \pi/6 \leqq \arg z \leqq \pi/3\}$, $\{\arg z=\pi/3, 1/12 \leqq |z| \leqq 1\}$, $\{|z|=1/12, -\pi/3 \leqq \arg z \leqq \pi/3\}$, $\{\arg z=-\pi/3, 1/12 \leqq |z| \leqq 1/2\}$, $\{|z|=1/2, -\pi/3 \leqq \arg z \leqq \pi/6\}$ and $\{\arg z=\pi/6, 1/2 \leqq |z| \leqq 1\}$. We distinguish from E_4 slits along $\{\arg z=\pi/6, 11\pi/6, 25/192 \leqq |z| \leqq 3/16, 49/192 \leqq |z| \leqq 1/3\}$ and $\{\arg z=\pi/4, 3/4 \leqq |z|<1\}$.

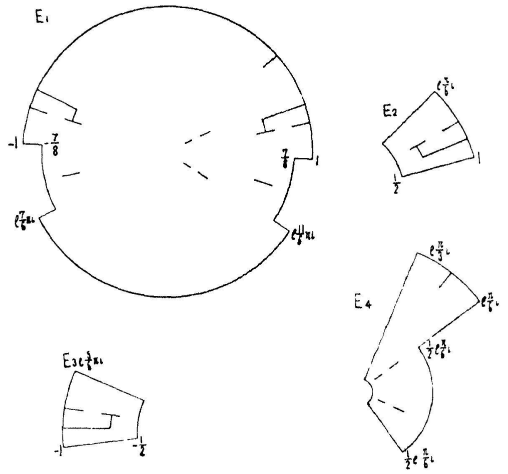
We construct the desired Riemann surface W by joining E_2, E_3 and E_4 to E_1 along their corresponding distinguished slits in the standard manner. Then W has five boundary components.

Let γ_{11} (resp. γ_{21}) be a simple closed curve on W surrounding $3e^{\pi i/12}/4$ and $7e^{\pi i/12}/8$ (resp. $3e^{11\pi i/12}/4$ and $7e^{11\pi i/12}/8$) once, respectively. Let γ_{31} (resp. γ_{41}) be a simple closed curve on W surrounding a slit $\{\arg z=13\pi/12$ (resp. $23\pi/12$), $5/8 \leqq |z| \leqq 3/4\}$ once. Let γ_{12} (resp. γ_{22}) be a simple closed curve on W surrounding $3e^{\pi i/6}/16$ and $49e^{\pi i/6}/192$ (resp. $3e^{11\pi i/6}/16$ and $49e^{11\pi i/6}/192$) once, respectively. Let γ_{32} (resp. γ_{42}) be a simple closed curve on W surrounding $25e^{\pi i/6}/192$ and $3e^{\pi i/6}/16$ (resp. $25e^{11\pi i/6}/192$ and $3e^{11\pi i/6}/16$) once, respectively. The orientations of these curves γ_{ij} are determined appropriately so that the discussion below holds good.

Put

$$C_i=\gamma_{i1}+\gamma_{i2} \quad (i=1,\dots,4).$$

Let ϕ be the natural projection mapping W into z-plane. Then f is determined so



that $\phi \circ f \circ \phi^{-1} = z^2/3$. Such an f is uniquely determined. Thus we have $C_1 \times C_3 = C_2 \times C_4 = 1$, $C_i \times C_j = 0$ if $i+j \equiv 0 \pmod 2$ and $f(C_i) \sim C_i$ $(i=1, \dots, 4)$. But f is not an automorphism.

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