

Equivariant Function spaces and Stable Homotopy Theory I

Autor(en): **Becker, J.C. / Schultz, R.E.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **49 (1974)**

PDF erstellt am: **28.05.2024**

Persistenter Link: <https://doi.org/10.5169/seals-37976>

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Equivariant Function Spaces and Stable Homotopy Theory I

J. C. BECKER¹⁾ and R. E. SCHULTZ²⁾

Let $F(S^n)$ denote the space of self-maps of the n -sphere with the compact-open topology and the identity as its basepoint. Results of Dold and Lashof [10] and Stasheff [27] show the importance of $F(S^n)$ in the classification of fiber spaces with fiber (homotopically equivalent to) S^n , and because of this the topological properties of $F(S^n)$ yield (or should yield, at least) considerable information about the topology of manifolds. Actually, for purposes of studying manifolds it is preferable to replace the spaces $F(S^n)$ by a so-called *stable version*. To construct this, we embed $F(S^n)$ in $F(S^{n+1})$ via the unreduced suspension functor and set

$$F = \text{inj} \lim_k F(S^k).$$

(In the literature, this space is usually called G ; however, we shall soon find it convenient to let G designate a compact Lie group).

If we are given an action of a compact Lie group G on S^n , we shall let $F_G(S^n)$ denote the subspace (submonoid, in fact) of all self-maps of S^n that are *equivariant* with respect to the given actions of G ; we shall restrict our attention to group actions given by free orthogonal representations (see §3). In this paper we shall study the homotopy properties of these spaces $F_G(S^n)$ and their corresponding stable versions. Perhaps the most interesting consequence of our work is a relationship between the stable versions of the spaces $F_G(S^n)$ and stable homotopy theory that generalizes the fundamentally important natural isomorphism

$$\theta X: [X, F] \simeq \{X, S^0\}$$

essentially due to G. Whitehead [32], where $[,]$ and $\{ , \}$ denote homotopy classes of ordinary maps and S -maps respectively and X is a CW complex.

Just as the spaces $F(S^n)$ and F and the isomorphism θX are applicable to the topology of manifolds, the spaces $F_G(S^n)$, their stable analogs, and the results of this paper are applicable to the study of manifolds with G -actions. Applications of our results along these lines appear in [35] and [36].

We wish to thank Mark Mahowald for suggestions which contributed substantially to the formulation of our results.

¹⁾ Partially supported by NSF Grant GP-34197

²⁾ Partially supported by NSF Grant GP-19530

1. Introduction

We shall describe some of our results more precisely in this section. Let G be a compact Lie group and W a free G -module (see §3). Let $S(W)$ denote the underlying unit sphere of W . If V is a submodule of W , we denote by $F(V | W)$ the space of G -equivariant maps $S(V^*) \rightarrow S(W)$, where V^* is the orthogonal complement of V in W . If W' is another free G -module, then $S(W \oplus W')$ is equivariantly homeomorphic to the join of $S(W)$ and $S(W')$; furthermore, the orthogonal complement of V in $W \oplus W'$ is $V^* \oplus W'$. Hence the join functor induces an inclusion of $F(V | W)$ in $F(V | W \oplus W')$. We define

$$F(V) = \text{inj} \lim_k F(V | kW),$$

where kW denotes the k -fold sum of W with itself and V is included in the first factor. If V is the trivial G -module $\{0\}$ we write F_G in place of $F(\{0\})$.

Our main result (Theorem (6.6)) gives a description of $F(V)$ as a space constructed from the classifying space of G in a natural way. For example, F_G is describable as follows: let B_G be a classifying space for G with total space E_G , let \mathfrak{G} be the Lie algebra of G and G act on \mathfrak{G} via the adjoint representation; the balanced product of E_G and \mathfrak{G} is a vector bundle over BG which we shall call ζ and whose Thom space we shall call B_G^ζ . Then F_G is homotopy equivalent to $Q(B_G^\zeta)$, where $Q(Y)$ is defined for pointed spaces Y by

$$Q(Y) = \text{inj} \lim_k \Omega^k S^k Y.$$

The homotopy equivalence is best understood using its alternate stable homotopy theoretic interpretation. Namely, under the canonical natural isomorphism

$$\theta X: [X, Q(Y)] \cong \{X, Y\}$$

it takes the form of a natural isomorphism

$$\varphi X: [X, F_G] \cong \{X, BG^\zeta\}.$$

If G is the trivial group, then φX is essentially the same as the previously mentioned θX .

There are many generalizations of the spaces F_G , and it is natural to ask whether they too are describable as $Q(Y)$ for suitable choices of Y . We mention two results in this direction:

(i) If G is finite and acts orthogonally on its real group algebra via the regular representation, the homotopy type of F_G is essentially given by results of Graeme Segal [25, Prop. 2 and Corollary to Prop. 7]. Using the techniques of [24] one can derive special cases of Segal's results from some of our results and vice versa.

(ii) Suppose G is finite and acts freely and topologically on S^n ; results of R. Lee [17] and T. Petrie [22] show that some finite groups admit such actions (smooth actions, in fact) but not linear ones. In this case one can still define F_G and prove analogs of our results. Details will appear in Part II of this paper.

Sections 2 through 4 contain preliminary material on ex-spaces, vector bundles, and the transfer map for fiber bundles. Our main results are stated in Sections 5 and 6; some of the more technical arguments are postponed to Sections 7, 8, and 9. Finally, we consider the following problem: If H is a closed subgroup of G , there is an inclusion of F_G in F_H because every G -equivariant map is automatically H -equivariant; determine the image of $\pi_*(F_G)$ in $\pi_*(F_H)$. The last three sections (10–12) contain some quantitative results on this problem.

2. Sectioned Bundles

Let B denote a locally finite CW-complex. In the terminology of James [14], an *ex-space* of B is an object $\xi = (E_\xi, B, p_\xi, \Delta_\xi)$ consisting of maps $p_\xi: E_\xi \rightarrow B$ and $\Delta_\xi: B \rightarrow E_\xi$ such that $p_\xi \Delta_\xi$ is the identity. If ξ and ξ' are ex-spaces, we denote by $[\xi, \xi']$ the set of homotopy classes of fiber and cross section preserving maps $E_\xi \rightarrow E_{\xi'}$. Ex-spaces may be regarded as generalizations of pointed spaces and many of the standard constructions for pointed spaces, such as reduced join, wedge, etc., carry over to ex-spaces. This is usually done by performing the construction ‘fiberwise’. For detailed accounts see [14], [15], [4].

An ex-space ξ will be called a *sectioned bundle* if it has the following local product structure. There is a pointed space F , with base point (say) x_0 , a cover $\{U\}$ of B by open sets, and homeomorphisms $\psi_U: U \times F \rightarrow p_\xi^{-1}(U)$ such that the following diagrams are commutative.

$$\begin{array}{ccc} U \times F & \xrightarrow{\psi_U} & p_\xi^{-1}(U) \\ \searrow p & & \swarrow p_\xi \\ & U & \end{array} \quad \begin{array}{ccc} U \times F & \xrightarrow{\psi_U} & p_\xi^{-1}(U) \\ \nwarrow \Delta & & \nearrow \Delta_\xi \\ & U & \end{array}$$

Here p is the projection and Δ is the cross section $b \rightarrow (b, x_0)$. We will also assume that F is a finite complex and $(E_\xi, \Delta_\xi(B), p_\xi)$ has the homotopy extension property [4; section 2].

The fiberwise reduced join of ξ and α will be denoted by $\xi \wedge \alpha$. There is a suspension map

$$\sigma: [\xi, \xi'] \rightarrow [\xi \wedge \alpha, \xi' \wedge \alpha] \tag{2.1}$$

defined by $f \rightarrow f \wedge 1$, and the following suspension theorem is proved in [15] (see also [14]).

(2.2) THEOREM. *Suppose that α is a sphere bundle and the fiber of ξ' is $(n-1)$ -connected. Then σ is injective if E_ξ is $(2n-1)$ -connected and surjective if E_ξ is $2n$ -connected.*

If Y and \hat{Y} are homeomorphic pointed spaces let $H(Y, \hat{Y})$ denote the space of base point preserving homeomorphisms from Y to \hat{Y} . If $\xi = (E, B, p, \Delta)$ and $\hat{\xi} = (\hat{E}, \hat{B}, \hat{p}, \hat{\Delta})$ are sectioned bundles with the same fiber F , let

$$H(E, \hat{E}) = \bigcup_{(b, \hat{b}) \in B \times \hat{B}} H(p^{-1}(b), \hat{p}^{-1}(\hat{b}))$$

and let $q: H(E, \hat{E}) \rightarrow B \times \hat{B}$ denote the obvious projection. For each pair of coordinate maps

$$\psi_U: U \times F \rightarrow p^{-1}(U), \quad \psi_V: V \times \hat{F} \rightarrow \hat{p}^{-1}(V)$$

we obtain

$$\psi_{U \times V}: (U \times V) \times H(F, \hat{F}) \rightarrow q^{-1}(U \times V)$$

by $(b, \hat{b}, \varphi) \rightarrow \psi_b \varphi \psi_{\hat{b}}^{-1}$. Let $H(E, \hat{E})$ have the smallest topology such that each $\psi_{U \times V}$ is continuous. Then, with this topology, it is easy to check that $(H(E, \hat{E}), B \times \hat{B}, q)$ is a fiber bundle which we denote by $H(\xi, \hat{\xi})$. Now the following bundle covering homotopy property is an immediate consequence of the covering homotopy property for $H(\xi, \hat{\xi})$.

(2.3) THEOREM. *Let $H: B \times I \rightarrow \hat{B}$ and $k: E \rightarrow \hat{E}$ be such that $\hat{p}k = H_0$, k is cross section preserving, and k is a homeomorphism on each fiber. Then there is $K: E \times I \rightarrow \hat{E}$ such that $pK = H$, $K_0 = k$, K_t is cross section preserving, and K_t is a homeomorphism on each fiber.*

We conclude this section with some notation and remarks. If X is a pointed space with base point x_0 , let \hat{X} denote the sectioned bundle $(B \times X, B, p, \Delta)$ where $p(b, x) = b$ and $\Delta(b) = (b, x_0)$. If α is a vector bundle over B , define $\bar{\alpha}$ to be the sectioned bundle obtained by taking the fiberwise one point compactification of E_α and letting $\Delta_{\bar{\alpha}}$ be the cross section at infinity. Observe that $\overline{\alpha \oplus \beta}$ is canonically equivalent to $\bar{\alpha} \wedge \bar{\beta}$.

There is a functor T from sectioned bundles to pointed spaces defined by $T(\xi) = E_\xi / \Delta_\xi(B)$. If α is a vector bundle, $T(\bar{\alpha})$ is simply the Thom space of α which we will alternately denote by $T(\alpha)$ or B^α . More generally, if $A \subset B$ let

$$(B, A)^\alpha = E_{\bar{\alpha}} / \Delta_{\bar{\alpha}}(B) \cup p_{\bar{\alpha}}^{-1}(A).$$

If X is a pointed space we have $T(X \wedge \xi) = X \wedge T(\xi)$. Note also that projection onto the second factor induces a bijection

$$[\xi, \hat{X}] \rightarrow [T(\xi), X]. \tag{2.4}$$

3. Vector Bundles

Suppose that M is a compact differentiable manifold without boundary and G is a compact Lie group acting freely and differentiably on M . By a result of Gleason [11] the orbit map $p: M \rightarrow M/G$ has the structure of a principal G -bundle (in fact, a smooth bundle, compare [34]). The tangent bundles of M and M/G are related as follows. Let $\text{Ad}(G)$ denote the G -module determined by the adjoint representation of G . The vector bundle with fiber $\text{Ad}(G)$ associated with $p: M \rightarrow M/G$ will be denoted by ζ . One then has an identification

$$\tau(M)/G \simeq \zeta \oplus \tau(M/G), \quad (3.1)$$

and this identification is natural with respect to smooth G -maps [28].

A G -module V will always be assumed to be real, finite dimensional, and equipped with a G -invariant metric. The unit sphere of V will be denoted by $S(V)$ and the quotient space $S(V)/G$ by $M(V)$. We say that V is *free* if G acts freely on $S(V)$. In this case $M(V)$ is a smooth manifold and $p: S(V) \rightarrow M(V)$ is a principal G -bundle.

Suppose now that W is a free G -module and $V \subset W$ is a submodule. Let U denote the orthogonal complement of V in W and let η denote the balanced product vector bundle

$$\eta = (S(U) \times V/G, M(U), p) \quad (3.2)$$

Let ξ be the sectioned bundle

$$\xi = (S(U) \times S(W)/G, M(U), p, \Delta) \quad (3.3)$$

where $\Delta[u] = [u, u]$. We have an identification

$$\xi \simeq \overline{\eta \oplus \tau(S(U))/G} \quad (3.4)$$

given by

$$[u, v \oplus u'] \rightarrow \left[u, \frac{v}{1 - u \cdot u'} \right] \oplus \left[u, \frac{u' - (u \cdot u')u}{1 - u \cdot u'} \right]$$

Combining this with (3.1) we have

$$\xi \simeq \overline{\eta \oplus \zeta \oplus \tau}, \quad (3.5)$$

where τ is the tangent bundle of $M(U)$.

We will also need a description of the Thom space of $\eta \oplus \zeta$ along the lines of [2, Proposition (4.3)]. The map

$$S(U) \times (V \oplus \text{Ad}(G)) \rightarrow S(W) \times \text{Ad}(G)$$

by

$$(u, v, y) \rightarrow (\sqrt{1 - (|v|^2/(1 + |v|^2))}u \oplus (1/(1 + |v|^2))y, y)$$

is equivariant and its quotient extends to an identification

$$M(U)^{\eta \oplus \zeta} \simeq (M(W), M(V))^{\zeta}. \quad (3.6)$$

4. The Transfer

In this section we will give a brief description of the transfer or ‘umkehr’ map associated with a differentiable fiber bundle. Our account follows that of Boardman [6]. By a manifold we mean a compact differentiable manifold without boundary. Let N be a manifold and M a submanifold of N with normal bundle ω . Choose an embedding $E_\omega \subset N$ of E_ω as a tubular neighborhood of M . Let α be a sectioned bundle over N and consider the maps

$$\begin{array}{ccc} E_\alpha & \begin{array}{c} \downarrow p_\alpha \\ E_\omega \end{array} & \begin{array}{c} \xrightarrow{k_t} \\ E_\alpha \end{array} & \begin{array}{c} \downarrow p_\alpha \\ E_\omega \end{array} \\ & & j_t & \\ & & E_\omega & \end{array}, \quad 0 \leq t \leq 1. \quad (4.1)$$

where j_t is the canonical homotopy given by $j_t(x) = (1-t)x$, and k_t is a sectioned bundle morphism covering j_t such that k_0 is the identity, k_t is the identity on $E_\alpha|_M$ (where $M \subset E_\omega$ is the 0-section), and k_t is a homeomorphism on each fiber. Such a homotopy exists by (2.3). Define

$$h_\alpha: \alpha|_{E_\omega} \rightarrow p_\omega^*(\alpha|_M) \quad (4.2)$$

by $h_\alpha(a) = (p_\alpha(a), k_1(a))$ and let

$$\tilde{h}_\alpha: \alpha|_{E_\omega} \rightarrow \alpha|_M \quad (4.3)$$

denote the map k_1 . The Pontrjagin-Thom map

$$c: T(\alpha) \rightarrow T(\bar{\omega} \wedge \alpha|_M) \quad (4.4)$$

is then given by

$$c(a_x) = \begin{cases} x \wedge \tilde{h}_\alpha(a_x), & x \in E_\omega \\ \infty, & \text{if } x \notin E_\omega. \end{cases}$$

It follows by a standard argument that the homotopy class of c does not depend on the particular choice of covering homotopy.

Let $p: M \rightarrow N$ be a differentiable fiber bundle. Choose an embedding $\hat{p}: M \rightarrow N \times \mathbb{R}^s$ homotopic to p and let ω denote the normal bundle. If α is a sectioned bundle over N there is the product bundle $\alpha \times 0$ over $N \times \mathbb{R}^s$ and $\alpha \times 0|_M \simeq p^*(\alpha)$. Since $T(\alpha \times 0) = T(\alpha) \times \mathbb{R}^s/\mathbb{R}^s$, the Pontrjagin-Thom map has the form

$$c: T(\alpha) \times \mathbb{R}^s/\mathbb{R}^s \rightarrow T(p^*(\alpha) \oplus \omega).$$

Representing S^s as the one point compactification of R^s , c may be extended to a map

$$t: T(\alpha) \wedge S^s \rightarrow T(p^*(\alpha) \oplus \omega). \quad (4.5)$$

In particular, if G is a compact Lie group acting freely on a manifold M and H is a closed subgroup, we have the fiber bundle $p: M/H \rightarrow M/G$. Let ζ_G (respectively, ζ_H) denote the bundle over M/G (respectively, M/H) having fiber $\text{Ad}(G)$ (respectively, $\text{Ad}(H)$). Now

$$\tau(M/H) \oplus \omega \simeq p^*(\tau(M/G) \oplus R^s).$$

Adding $\zeta_H \oplus p^*(\zeta_G)$ to both sides and using (3.1) we have

$$\tau(M)/H \oplus p^*(\zeta_G) \oplus \omega \simeq \tau(M)/H \oplus \zeta_H \oplus R^s.$$

For sufficiently large s we may cancel $\tau(M)/H$ obtaining an equivalence

$$p^*(\zeta_G) \oplus \omega \simeq \zeta_H \oplus R^s. \quad (4.6)$$

Thus, the map t of (4.5) yields

$$t: T(\alpha \wedge \zeta_G) \wedge S^s \rightarrow T(p^*(\alpha) \wedge \zeta_H) \wedge S^s. \quad (4.7)$$

The stable homotopy class of this map does not depend on the particular choice of embedding because of the following: (a) isotopic embeddings determine homotopic maps. (b) the effect of replacing $\hat{p}: M/G \rightarrow M/H \times R^s$ by $i\hat{p}: M/G \rightarrow M/H \times R^{s+1}$, where i is the usual inclusion, is to replace t by its suspension. (c) for sufficiently large s , any two embeddings homotopic to p are isotopic.

We shall call t in (4.7) the *transfer* associated with the bundle $p: M/H \rightarrow M/G$. It is easily seen that t is functorial with respect to smooth G -maps. Moreover, if H has finite index in G (so that p is a finite covering map) t agrees with the transfer defined and axiomatized by Roush [23]. A proof of this fact will be given in the appendix.

Consider now the situation of the previous section. If V is a G -module write $V = V_G$ and let V_H denote its underlying H -module. Suppose that $V_G \oplus U_G = W_G$. Let η_G and η_H be as in (3.2). We have the fiber bundle $p: M(U_H) \rightarrow M(U_G)$ and since $p^*(\eta_G) = \eta_H$ we obtain a transfer map

$$T(\eta_G \oplus \zeta_G) \wedge S^s \rightarrow T(\eta_H \oplus \zeta_H) \wedge S^s.$$

Making the identification (3.6) we have

$$t: (M(W_G), M(V_G))^{\zeta_G} \wedge S^s \rightarrow (M(W_H), M(V_H))^{\zeta_H} \wedge S^s. \quad (4.8)$$

5. The Spaces $F(V \mid W)$

If α and β are sectioned bundles, let $\mathcal{M}(\alpha, \beta)$ denote the space of fiber and cross

section preserving maps $E_\alpha \rightarrow E_\beta$, with the compact-open topology. Recall that if Y is a pointed space

$$Q(Y) = \text{inj} \lim_k \mathcal{M}(S^k, Y \wedge S^k).$$

Let V and W be free G -modules such that $V \subset W$ and $V \neq W$. Let V^* denote the orthogonal complement of V in W . We define $F(V | W)$ to be the pointed space of G -equivariant maps $S(V^*) \rightarrow S(W)$, the inclusion map being the base point. Our objective is to construct a map

$$\lambda: F(V | W) \rightarrow Q((M(W), M(V))^{\zeta}). \quad (5.1)$$

Let

$$\xi = (S(V^*) \times S(W)/G, M(V^*), p, \Delta) \quad (5.2)$$

where p and Δ are induced by the projection and diagonal respectively. From (3.4) we have an identification

$$\xi \simeq \overline{\eta \oplus \zeta \oplus \tau}, \quad (5.3)$$

where τ is the tangent bundle of $M(V^*)$ and $\eta = (S(V^*) \times V/G, M(V^*), p)$. The function

$$\theta: F(V | W) \rightarrow \mathcal{M}(\dot{S}^0, \xi) \quad (5.4)$$

defined by sending $f: S(V^*) \rightarrow S(W)$ to $f': S^0 \times M(V^*) \rightarrow S(V^*) \times S(W)/G$, where $f'(0, [y]) = [y, f(y)]$ and $f'(\infty, [y]) = [y, y]$ is easily seen to be a homeomorphism of function spaces.³⁾ Making the identification (3.4), θ becomes

$$\theta: F(V | W) \rightarrow \mathcal{M}(\dot{S}^0, \overline{\eta \oplus \zeta \oplus \tau}). \quad (5.5)$$

Choose an embedding $M(V^*) \subset R^s$ and let ν denote the normal bundle. Let $\psi: \tau \oplus \nu \rightarrow R^s$ denote the associated trivialization and $c: S^s \rightarrow T(\nu)$ the Pontrjagin-Thom map. The map λ is to be the following composition.

$$\begin{aligned} F(V | W) &\xrightarrow{\theta} \mathcal{M}(\dot{S}^0, \overline{\eta \oplus \zeta \oplus \tau}) \xrightarrow{\sigma} \mathcal{M}(\bar{\nu}, \overline{\eta \oplus \zeta \oplus \tau \oplus \nu}) \\ &\xrightarrow{\mathcal{M}(1 \oplus \psi)} \mathcal{M}(\bar{\nu}, \overline{\eta \oplus \zeta \oplus R^s}) \xrightarrow{T} \mathcal{M}(T(\nu), T(\eta \oplus \zeta) \wedge S^s) \\ &\xrightarrow{\mathcal{M}(c)} \mathcal{M}(S^s, T(\eta \oplus \zeta) \wedge S^s) \rightarrow Q(T(\eta \oplus \zeta)) \\ &\longrightarrow Q((M(W), M(V))^{\zeta}). \end{aligned} \quad (5.6)$$

Here σ is suspension and the last map is given by the identification (3.6). It is easy to check that the homotopy class of λ does not depend on the choice of embedding.

³⁾ We use S^n to denote the one point compactification of R^n . The sphere of unit vectors in R^{n+1} will be denoted by $S(R^{n+1})$.

(5.7) THEOREM. λ is an n -equivalence where $n = \dim(V) + \dim(W) + \dim(G) - 2$.

Proof. It follows from the suspension theorem (2.2) that σ is an n -equivalence. It remains to show that $\mathcal{M}(c)T$ is an n -equivalence for large s . Let $\alpha = \eta \oplus \zeta \oplus R^s$. Choose a complementary bundle β and let $\varphi: \beta \oplus \alpha \rightarrow R^t$ be a trivialization. We then have a duality map

$$\mu: S^{s+t} \rightarrow T(v \oplus \beta) \wedge T(\alpha)$$

given by the composite

$$\begin{aligned} S^{s+t} &\xrightarrow{c \wedge 1} T(v) \wedge S^t \xrightarrow{T(1 \oplus \varphi^{-1})} T(v \oplus \beta \oplus \alpha) \\ &\xrightarrow{\Delta} T(v \oplus \beta) \wedge T(\alpha), \end{aligned}$$

where Δ is the diagonal map. Let X be a finite complex such that $\dim(X) \leq n$. The associated correspondence

$$D_\mu: [X \wedge T(v \oplus \beta), S^t] \rightarrow [X \wedge S^{s+t}, T(\alpha) \wedge S^t]$$

defined by sending $f: X \wedge T(v \oplus \beta) \rightarrow S^t$ to the map

$$X \wedge S^{s+t} \xrightarrow{1 \wedge \mu} X \wedge T(v \oplus \beta) \wedge T(\alpha) \xrightarrow{f \wedge 1} S^t \wedge T(\alpha) \rightarrow T(\alpha) \wedge S^t$$

is bijective, provided we are in the stable range. Let us take t to be large enough so that this is the case.

We have the following commutative diagram.

$$\begin{array}{ccccc} [\dot{X} \wedge \bar{v}, \bar{\alpha}] & \xrightarrow{T} & [X \wedge T(v), T(\alpha)] & \xrightarrow{(1 \wedge c)^\#} & [X \wedge S^s, T(\alpha)] \\ \downarrow \sigma & & & & \downarrow \sigma \\ [\dot{X} \wedge \overline{v \oplus R^t}, \overline{\alpha \oplus R^t}] & \xrightarrow{T} & [X \wedge T(v) \wedge S^t, T(\alpha) \wedge S^t] & \xrightarrow{(1 \wedge c \wedge 1)^\#} & [X \wedge S^{s+t}, T(\alpha) \wedge S^t] \\ \uparrow 1 \wedge (1 \oplus \varphi^{-1})^\# & & & & \uparrow D_\mu \\ [\dot{X} \wedge \overline{v \oplus \beta \oplus \alpha}, \overline{\alpha \oplus R^t}] & & & & \\ \uparrow & & & & \\ [\dot{X} \wedge \overline{v \oplus \beta \oplus \alpha}, \overline{R^t \oplus \alpha}] & & & & \\ \uparrow \sigma & & & & \\ [\dot{X} \wedge \overline{v \oplus \beta}, \overline{R^t}] & \xrightarrow{T} & [X \wedge T(v \oplus \beta), S^t] & & \end{array}$$

For sufficiently large s the suspension maps in the above diagram are bijective and therefore $(1 \wedge c)^\# T$ is bijective as desired.

We will now consider the functorial properties of the map λ . We identify the unreduced join $S(V) * S(W)$ with $S(V \oplus W)$ by the map $[v, w, t] \rightarrow tv \oplus \sqrt{1-t^2}w$. If

$V \subset U \subset W$ there is an inclusion map

$$j: F(V \mid U) \rightarrow F(V \mid W) \quad (5.8)$$

induced by the join operation as follows. Let V^* denote the orthogonal complement of V in U and U^* the orthogonal complement of U in W . Then j is defined by sending $f: S(V^*) \rightarrow S(U)$ to $f * 1: S(V^*) * S(U^*) \rightarrow S(U) * S(U^*)$. Let

$$i: (M(U), M(V))^\zeta \rightarrow (M(W), M(V))^\zeta \quad (5.9)$$

denote the inclusion, and let X denote a finite complex.

(5.10) *The following diagram is commutative.*

$$\begin{array}{ccc} [X, F(V \mid U)] & \xrightarrow{\lambda^*} & [X, Q((M(U), M(V))^\zeta)] \\ \downarrow j^* & & \downarrow Q(i)^* \\ [X, F(V \mid W)] & \xrightarrow{\lambda^*} & [X, Q((M(W), M(V))^\zeta)]. \end{array}$$

Let

$$r: F(V \mid W) \rightarrow F(U \mid W) \quad (5.11)$$

denote the map defined by restricting $f: S(V^*) \rightarrow S(W)$ to $S(U^*)$, and let

$$c: (M(W), M(V))^\zeta \rightarrow (M(W), M(U))^\zeta \quad (5.12)$$

be the collapsing map.

(5.13) *The following diagram is commutative*

$$\begin{array}{ccc} [X, F(V \mid W)] & \xrightarrow{\lambda^*} & [X, Q((M(W), M(V))^\zeta)] \\ \downarrow r^* & & \downarrow Q(c)^* \\ [X, F(U \mid W)] & \xrightarrow{\lambda^*} & [X, Q((M(W), M(U))^\zeta)]. \end{array}$$

Finally, if H is a closed subgroup of G there is the natural forgetful map

$$\varphi: F(V_G \mid W_G) \rightarrow F(V_H \mid W_H), \quad (5.14)$$

and for sufficiently large s , there is a transfer map

$$t: (M(W_G), M(V_G))^{\zeta_G} \wedge S^s \rightarrow (M(W_H), M(V_H))^{\zeta_H} \wedge S^s \quad (5.15)$$

as in (4.8).

(5.16) *The following diagram is commutative*

$$\begin{array}{ccc} [X, F(V_G | W_G)] & \xrightarrow{\lambda_{\#}} & [X, Q((M(W_G), M(V_G))^{\zeta_G})] \\ \downarrow \varphi_{\#} & & \downarrow Q(t)_{\#} \\ [X, F(V_H | W_H)] & \xrightarrow{\lambda_{\#}} & [X, Q((M(W_H), M(V_H))^{\zeta_H})]. \end{array}$$

Proofs for (5.10), (5.13) and (5.16) are given in section 8.

6. The Spaces $F(V)$.

Given a free G -module V , choose a free G -module W such that $V \subset W$ and $V \neq W$. Let kW denote the k -fold direct sum of W and define

$$F(V) = \text{inj} \lim_k F(V | kW) \quad (6.1)$$

and

$$B(V)^{\zeta} = \text{inj} \lim_k (M(kW), M(V))^{\zeta} \quad (6.2)$$

If X is a pointed finite CW-complex the map

$$\lambda_{\#} : [X; F(V | kW)] \rightarrow [X, Q((M(kW), M(V))^{\zeta})]$$

is, by (5.10), compatible with the above inclusions. Hence we obtain

$$\lambda(V) : [X; F(V)] \rightarrow [X; Q(B(V)^{\zeta})] \quad (6.3)$$

as the injective limit of the $\lambda_{\#}$. As a result of theorem (4.5) we have

(6.4) **THEOREM.** *$\lambda(V)$ is a natural equivalence of homotopy functors on the category of finite CW-complexes.*

We next show that $F(V)$ has the homotopy type of a CW-complex. To do this it is sufficient to show that the spaces $F(V | W)$ have the homotopy type of a CW-complex. Since $F(V | W)$ is homeomorphic to the space of cross sections to the bundle $S(V^*) \times S(W)/G \rightarrow M(V^*)$, the result for $F(V | W)$ is a consequence of the following.

(6.5) **LEMMA.** *Let $p: E \rightarrow B$ be a Hurewicz fibration with fiber F . Suppose that B is compact and both B and F have the homotopy type of a CW-complex. Then the space of cross sections to p has the homotopy type of a CW-complex.*

Proof. Let \mathcal{CH} denote the category of spaces having the homotopy type of a CW-complex. First, suppose that $p: E \rightarrow B$ is a fibration such that E and B are in \mathcal{CH} . We will show that the fiber F is in \mathcal{CH} . If we replace the inclusion $i: F \rightarrow E$ by

a fibration $i': F' \rightarrow E$ in the usual way, the fiber over e has the homotopy type of $\Omega(B, p(e))$ [21]. By a result of Milnor [19], $\Omega(B, p(e))$ is in \mathcal{CH} . Hence by a theorem of Stasheff [27], F' is in \mathcal{CH} . Therefore F is in \mathcal{CH} .

Now let $p: E \rightarrow B$ be as in the statement of the lemma. By the exponential law, $p': E^B \rightarrow B^B$ is also a Hurewicz fibration and since both E^B and B^B are in \mathcal{CH} [19], the fiber over the identity is in \mathcal{CH} . This is just the space of cross sections to p .

As a consequence of (6.4), we have proved the following:

(6.6) THEOREM. *The space $F(V)$ is homotopy equivalent to $Q(B(V)^\zeta)$.*

Since the homotopy type of $B(V)^\zeta$ clearly does not depend on the choice of ambient G -module W , Theorem (6.6) has an obvious consequence.

(6.7) COROLLARY. *The homotopy type of $F(V)$ depends only on the representation V .*

There are two functorial properties of the transformation $\lambda(V)$. Firstly, if V is a submodule of U we obtain from (5.13) the following commutative diagram

$$\begin{array}{ccc} [X; F(V)] & \xrightarrow{\lambda(V)} & [X; Q(B(V)^\zeta)] \\ \downarrow r_\# & & \downarrow Q(c)_\# \\ [X; F(U)] & \xrightarrow{\lambda(U)} & [X; Q(B(U)^\zeta)] \end{array} \quad (6.8)$$

Secondly, if H is a closed subgroup of G , we have a *transfer*

$$t_\#: [X; Q(B(V_H)^{\zeta_H})] \rightarrow [X; Q(B(V_G)^{\zeta_G})] \quad (6.9)$$

defined to be the injective limit of the maps $Q(t)_\#$, where $Q(t)$ is the map appearing in (5.16). Then by (5.16) we have a commutative diagram

$$\begin{array}{ccc} [X; F(V_G)] & \xrightarrow{\lambda(V_G)} & [X; Q(B(V_G)^{\zeta_G})] \\ \downarrow \varphi_\# & & \downarrow t_\# \\ [X; F(V_H)] & \xrightarrow{\lambda(V_H)} & [X; Q(B(V_H)^{\zeta_H})] \end{array} \quad (6.10)$$

Actually, by the methods of [6], one can construct in a natural way, a map $t: Q(B(V_G)^{\zeta_G}) \rightarrow Q(B(V_H)^{\zeta_H})$ which realizes the transfer $t_\#$. Since we will not need such a map, we do not carry out the construction here.

If V is the trivial G -module $\{0\}$ we shall write F_G in place of $F(V)$ and B_G^ζ in place of $B(V)^\zeta$. Thus, F_G is the injective limit of the space of G -equivariant self maps of $S(kW)$ and B_G^ζ is the Thom space of the bundle with fiber $\text{Ad}(G)$ associated to the universal principal G -bundle.

We shall now examine some special cases of the preceding results. First, if G is the trivial group we write F in place of F_G . In this case $B_G^\zeta = S^{\infty+}$ may be identified

with S^0 by collapsing S^∞ to a point. Let us write $Q^{(0)}(S^0)$ (respectively, $Q^{(1)}(S^0)$) to denote $Q(S^0)$ with the constant map (respectively, the identity map) as base point. We will relate

$$\lambda: [X; F] \rightarrow [X; Q^{(0)}(S^0)] \quad (6.11)$$

to a more familiar map. Let

$$T: [X; Q^{(1)}(S^0)] \rightarrow [X; Q^{(0)}(S^0)] \quad (6.12)$$

be defined as follows. First let $T': \Omega^k(S^k) \rightarrow \Omega^k(S^k)$ send f to the composite

$$S^k \xrightarrow{h} S^k \vee S^k \xrightarrow{1 \vee Rf} S^k \vee S^k \xrightarrow{g} S^k,$$

where h is the pinching map, R is the reflection $(x_1, x_2, \dots, x_k) \rightarrow (-x_1, x_2, \dots, x_k)$, and g is the folding map. (With respect to loop addition T' sends f to $1 - f$). Let $H: S^k \times I \rightarrow S^k$ denote the canonical homotopy from $T'(1)$ to the constant map. Then T is to be the injective limit of

$$[X; \Omega^k(S^k)] \xrightarrow{T'^\#} [X; \Omega^k(S^k)] \xrightarrow{H^\#} [X; \Omega^k(S^k)]$$

There is also a natural inclusion $\iota: F \rightarrow Q^{(1)}(S^0)$ defined by sending $f: S(R^k) \rightarrow S(R^k)$ to its radial extension $\hat{f}: S^k \rightarrow S^k$ given by $\hat{f}(tv) = tf(v)$, $t \geq 0$, $|v| \geq 1$.

(6.13) THEOREM. *The triangle*

$$\begin{array}{ccc} [X, F] & \xrightarrow{\lambda} & [X; Q^{(0)}(S^0)] \\ & \searrow \iota_\# & \nearrow T \\ & [X; Q^{(1)}(S^0)] & \end{array}$$

is commutative.

A proof of (6.13) will be given in Section 9.

Now let K denote one of the fields R , C or H , the real, complex, or quaternionic numbers and let d denote the dimension of K over R . Let $G = S^{d-1}$ and let V denote the standard representation of G on R^d given by scalar multiplication. Then the space $F(kV)$, which we shall now denote by L_k , is the injective limit over n of the spaces L_k^n , where L_k^n is the space of S^{d-1} -equivariant maps $S^{d(n-k)-1} \rightarrow S^{dn-1}$.

Let S^{d-1} act on $S^{dn-1} \times S^{d-1}$ by $(x, y) \rightarrow (gx, gyg^{-1})$, $g \in S^{d-1}$. The *quasi-projective* space \tilde{P}_n defined by James [12] is the space obtained from $S^{dn-1} \times S^{d-1} / S^{d-1}$ by identifying the section $S^{dn-1} \times \{1\} / S^{d-1}$ to a point (see [2; section 5]). It is easy to see that \tilde{P}_n is the Thom space of the bundle with fiber $\text{Ad}(S^{d-1})$ associated with the bundle $S^{dn-1} \rightarrow P_n$, where P_n is the projective space S^{dn-1} / S^{d-1} [2]. Let $\tilde{P}_\infty = \text{inj lim}_n \tilde{P}_n$ and let \tilde{P}_0 be the base point of \tilde{P}_∞ . Then with these changes in notation we obtain

from (6.6) a homotopy equivalence

$$L_k \simeq Q(\tilde{P}_\infty / \tilde{P}_k). \quad (6.14)$$

In particular,

$$F_{S^{d-1}} \simeq Q(\tilde{P}_\infty). \quad (6.15)$$

Note that $R\tilde{P}_\infty = RP^{\infty+}$ and $C\tilde{P}^\infty = (CP^{\infty+}) \wedge S^1$.

7. Morphisms of Sectioned Bundles

In this section we take up some properties of the mapping set $[\alpha, \beta]$ which will be needed to establish the functorial properties of the transformation λ .

Suppose that N is a manifold and $M \subset N$ is a submanifold with normal bundle ω . Let $E_\omega \subset N$ as a tubular neighborhood. Then if α is a sectioned bundle over N we have

$$h_\alpha: \alpha|_{E_\omega} \rightarrow p_\omega^*(\alpha|_M), \quad \tilde{h}_\alpha: \alpha|_{E_\omega} \rightarrow \alpha|_M$$

as in (4.2) and (4.3). Let β denote another sectioned bundle over N and define

$$e: [\bar{\omega} \wedge \alpha|_M, \beta|_M] \rightarrow [\alpha, \beta] \quad (7.1)$$

by

$$e(f)(a_x) = \begin{cases} h_\beta^{-1}(x, f(x \wedge \tilde{h}_\alpha(a_x))), & x \notin E_\omega \\ \Delta_\beta(x), & x \in E_\omega. \end{cases}$$

The map e is easily seen to be natural with respect to suspension. That is, if γ is another sectioned bundle over N , the following diagram is commutative.

$$\begin{array}{ccc} [\bar{\omega} \wedge \alpha|_M, \beta|_M] & \xrightarrow{\sigma} & [\bar{\omega} \wedge (\alpha \wedge \gamma)|_M, (\beta \wedge \gamma)|_M] \\ \downarrow e & & \downarrow e \\ [\alpha, \beta] & \xrightarrow{\sigma} & [\alpha \wedge \gamma, \beta \wedge \gamma]. \end{array} \quad (7.2)$$

The relation between e and the Pontrjagin-Thom map $c: T(\alpha) \rightarrow T(\bar{\omega} \wedge \alpha|_M)$ is given by the following commutative diagram.

$$\begin{array}{ccc} [\bar{\omega} \wedge \alpha|_M, \beta|_M] & \xrightarrow{T} & [T(\bar{\omega} \wedge \alpha|_M), T(\beta|_M)] \\ \downarrow e & & \downarrow c^\# \\ & & [T(\alpha), T(\beta|_M)] \\ & & \downarrow i_\# \\ [\alpha, \beta] & \xrightarrow{T} & [T(\alpha), T(\beta)]. \end{array} \quad (7.3)$$

Here $i: T(\beta \mid M) \rightarrow T(\beta)$ denotes the inclusion. To prove (7.3) we have

$$T e(f)(a_x) = \begin{cases} h_\beta^{-1}(x, f(x \wedge \tilde{h}_\alpha(a_x))), & x \in E_\omega \\ \infty, & x \notin E_\omega, \end{cases}$$

and

$$i: T(f) c(a_x) = \begin{cases} f(x \wedge \tilde{h}_\alpha(a_x)), & x \in E_\omega \\ \infty, & x \notin E_\omega. \end{cases}$$

A connecting homotopy H is given by

$$H(a_x, t) = \begin{cases} h_\beta^{-1}(tx, f(x \wedge \tilde{h}_\alpha(a_x))), & x \in E_\omega \\ \infty, & x \notin E_\omega. \end{cases}$$

Now consider the restriction map

$$r: [\alpha, \beta] \rightarrow [\alpha \mid M, \beta \mid M]. \quad (7.4)$$

Note that for $f: \alpha \rightarrow \beta$ we have $\tilde{h}_\beta f \simeq f \mid M \tilde{h}_\alpha$ since both are the end of a homotopy from $\alpha \mid E_\omega$ to $\beta \mid E_\omega$ which begins at f and covers the homotopy j_t of (4.1). From this observation and a straightforward calculation we obtain the following commutative diagram.

$$\begin{array}{ccc} [\alpha, \beta] & \xrightarrow{T} & [T(\alpha), T(\beta)] \\ \downarrow r & & \downarrow c^\# \\ [\alpha \mid M, \beta \mid M] & & [T(\alpha), T(\beta \mid M \wedge \bar{\omega})] \\ \downarrow \sigma & & \downarrow c^\# \\ [\alpha \mid M \wedge \bar{\omega}, \beta \mid M \wedge \bar{\omega}] & \xrightarrow{T} & [T(\alpha \mid M \wedge \bar{\omega}), T(\beta \mid M \wedge \bar{\omega})]. \end{array} \quad (7.5)$$

Suppose now that $p: M \rightarrow N$ is a map and α, β are sectioned bundles over N . There is then the induced map

$$p^*: [\alpha, \beta] \rightarrow [p^*(\alpha), p^*(\beta)] \quad (7.6)$$

defined by $p^*(f)(m, a) = (m, f(a))$, $m \in M$, $a \in E_\alpha$. Suppose further that $p: M \rightarrow N$ is a differentiable fiber bundle. Let $\hat{p}: M \rightarrow N \times R^s$ be an embedding homotopic to p , let ω denote the normal bundle, and let $\pi: N \times R^s \rightarrow N$ denote the projection. Let us also choose \hat{p} so that $\pi \hat{p} = p$. Then $p^*(\alpha) = \pi^*(\alpha) \mid M$ and under this identification the map p^* of (7.6) corresponds to

$$[\alpha, \beta] \xrightarrow{\pi^*} [\pi^*(\alpha), \pi^*(\beta)] \xrightarrow{r} [p^*(\alpha), p^*(\beta)].$$

where r is the restriction map. Hence by the commutativity of (7.5) and the definition

of the transfer t , we have the following commutative diagram

$$\begin{array}{ccc}
 [\alpha, \beta] \xrightarrow{T} [T(\alpha), T(\beta)] \xrightarrow{\sigma} [T(\alpha) \wedge S^s, T(\beta) \wedge S^s] & & \\
 \downarrow p^* & & \downarrow t^* \\
 [p^*(\alpha), p^*(\beta)] & & [T(\alpha) \wedge S^s, T(p^*(\beta) \wedge \bar{\omega})] \\
 \downarrow \sigma & & \downarrow t^* \\
 [p^*(\alpha) \wedge \bar{\omega}, p^*(\beta) \wedge \bar{\omega}] \xrightarrow{T} [T(p^*(\alpha) \wedge \bar{\omega}), T(p^*(\beta) \wedge \bar{\omega})] & &
 \end{array} \quad (7.7)$$

8. The Functorial Properties of λ

We will first establish property (5.10). Let U , V , and W be free G -modules such that $V \subset U \subset W$. Let V^* denote the orthogonal complement of V in W and V^{**} the orthogonal complement of V in U . We then have $M(V^{**}) \subset M(V^*)$. We let η, ζ, τ denote the bundles over $M(V^*)$ which appear in the definition of $\lambda(V|W)$ and η_0, ζ_0, τ_0 those over $M(V^{**})$ which appear in the definition of $\lambda(V|U)$. Let ω denote the normal bundle of $M(V^{**})$ in $M(V^*)$.

Let X be a finite complex. Since the restriction of τ to $M(V^{**})$ is $\tau_0 \oplus \omega$, we have

$$[\dot{X}, \overline{\eta_0 \oplus \zeta_0 \oplus \tau_0}] \xrightarrow{\sigma} [\dot{X} \wedge \bar{\omega}, \overline{\eta_0 \oplus \zeta_0 \oplus \tau_0 \oplus \omega}] \xrightarrow{e} [\dot{X}, \overline{\eta \oplus \zeta \oplus \tau}].$$

and we denote this composite by \hat{e} . A lengthy but straightforward calculation shows that the following diagram is commutative.

$$\begin{array}{ccc}
 [X, F(V|U)] \xrightarrow{\theta^*} [\dot{X}, \overline{\eta_0 \oplus \zeta_0 \oplus \tau_0}] & & \\
 \downarrow j^* & & \downarrow \hat{e} \\
 [X, F(V|W)] \xrightarrow{\theta^*} [\dot{X}, \overline{\eta \oplus \zeta \oplus \tau}] & &
 \end{array} \quad (8.1)$$

Now let $M(V^*) \subset R^s$ with normal bundle v and let v_0 denote normal bundle of the composite embedding $M(V^{**}) \subset R^s$. Then $v_0 \simeq \omega \oplus (v|_{M(V^{**})})$ so that, by (7.2) and the definition of \hat{e} we obtain a commutative diagram

$$\begin{array}{ccc}
 [\dot{X}, \overline{\eta_0 \oplus \zeta_0 \oplus \tau_0}] \xrightarrow{\sigma} [\dot{X} \wedge \bar{v}_0, \overline{\eta_0 \oplus \zeta_0 \oplus \tau_0 \oplus v_0}] & & \\
 \downarrow \hat{e} & & \downarrow e \\
 [\dot{X}, \overline{\eta \oplus \zeta \oplus \tau}] \xrightarrow{\sigma} [\dot{X} \wedge \bar{v}, \overline{\eta \oplus \zeta \oplus \tau \oplus v}] & &
 \end{array} \quad (8.2)$$

Let $\psi: \tau \oplus v \rightarrow R^s$ and $\psi_0: \tau_0 \oplus v_0 \rightarrow R^s$ denote the trivializations associated with the embeddings. Since ψ_0 is the restriction of ψ we have the commutativity relation.

$$\begin{array}{ccc}
 [\dot{X} \wedge \bar{v}_0, \overline{\eta_0 \oplus \zeta_0 \oplus \tau_0 \oplus v_0}] \xrightarrow{(1 \oplus \psi_0)^*} [\dot{X} \wedge \bar{v}_0, \overline{\eta_0 \oplus \zeta_0 \oplus R^s}] & & \\
 \downarrow e & & \downarrow e \\
 [\dot{X} \wedge \bar{v}, \overline{\eta \oplus \zeta \oplus \tau \oplus v}] \xrightarrow{(1 \oplus \psi)^*} [\dot{X} \wedge \bar{v}, \overline{\eta \oplus \zeta \oplus R^s}] & &
 \end{array} \quad (8.3)$$

Now, by (7.3) we have a commutative diagram

$$\begin{array}{ccc}
 [\dot{X} \wedge \bar{v}_0, \overline{\eta_0 \oplus \zeta_0 \oplus R^s}] & \xrightarrow{T} & [X \wedge T(v_0), T(\eta_0 \oplus \zeta_0) \wedge S^s] \\
 \downarrow e & & \downarrow c^\# \\
 & & [X \wedge T(v), T(\eta_0 \oplus \zeta_0) \wedge S^s] \\
 & & \downarrow i^\# \\
 [\dot{X} \wedge \bar{v}, \overline{\eta \oplus \zeta \oplus R^s}] & \xrightarrow{T} & [X \wedge T(v), T(\eta \oplus \zeta) \wedge S^s].
 \end{array} \tag{8.4}$$

Property (5.10) now follows easily from the commutativity of the diagrams (8.1) through (8.4), together with the relation

$$\begin{array}{ccc}
 & T(v) & \\
 c \nearrow & & \downarrow c \\
 S^s & & T(v_0). \\
 c \searrow & &
 \end{array} \tag{8.5}$$

We turn now to the proof of (5.16). Let V_G and W_G be free G -modules such that $V_G \subset W_G$ and let V_H and W_H denote their underlying H -modules. Let $p: M(V_G^*) \rightarrow M(V_H^*)$ denote the projection and choose an embedding $\hat{p}: M(V_H^*) \rightarrow M(V_G^*) \times R^{s_1}$ such that $\pi \hat{p} = p$, where $\pi: M(V_G^*) \times R^{s_1} \rightarrow M(V_G^*)$ is the projection. Let ω denote the normal bundle to this embedding. The bundles over $M(V_G^*)$ which appear in the definition of λ will be denoted by a subscript G and those over $M(V_H^*)$ by a subscript H . We then have $p^*(\eta_G) = \eta_H$ and $p^*(\zeta_G \oplus \tau_G) = \zeta_H \oplus \tau_H$.

We have the following commutative diagram

$$\begin{array}{ccc}
 [X, F(V_G | W_G)] & \xrightarrow{\theta} & [\dot{X}, \overline{\eta_G \oplus \zeta_G \oplus \tau_G}] \\
 \downarrow \varphi & & \downarrow p^* \\
 [X, F(V_H | W_H)] & \xrightarrow{\theta} & [\dot{X}, \overline{\eta_H \oplus \zeta_H \oplus \tau_H}]
 \end{array} \tag{8.6}$$

Now choose an embedding $M(V_G^*) \subset R^{s_1}$ with normal bundle v_G and let v_H denote the normal bundle of the composite embedding

$$M(V_H^*) \xrightarrow{\hat{p}} M(V_G^*) \times R^{s_1} \rightarrow R^{s_1 + s_2}.$$

We then have the relation

$$v_H \simeq p^*(v_G) \oplus \omega, \tag{8.7}$$

and from (4.6),

$$\zeta_H \oplus R^{s_1} \simeq p^*(\zeta_G) \oplus \omega. \tag{8.8}$$

Let $\psi_G: \tau_G \oplus \nu_G \rightarrow R^{s_2}$ and $\psi_H: \tau_H \subset \nu_H \rightarrow R^{s_1+s_2}$ denote the trivializations associated with the embeddings. Making use of the identifications (8.7) and (8.8) we have the following commutative diagrams

$$\begin{array}{ccc}
 [\dot{X}, \overline{\eta_G \oplus \zeta_G \oplus \tau_G}] & \xrightarrow{(1 \oplus \psi_G) \# \sigma} & [\dot{X} \wedge \bar{\nu}_G, \overline{\eta_G \oplus \zeta_G \oplus R^{s_2}}] \\
 \downarrow p^* & & \downarrow p^* \\
 & & [\dot{X} \wedge p^*(\bar{\nu}_G), \overline{\eta_H \oplus p^*(\zeta_G) \oplus R^{s_2}}] \\
 & & \downarrow \sigma \\
 [\dot{X}, \overline{\eta_H \oplus \zeta_H \oplus \tau_H}] & \xrightarrow{(1 \oplus \psi_H) \# \sigma} & [\dot{X} \wedge \bar{\nu}_H, \overline{\eta_H \oplus \zeta_H \oplus R^{s_1+s_2}}]
 \end{array} \tag{8.9}$$

Next, by the commutativity of (7.8) we have (see (4.7))

$$\begin{array}{ccc}
 [\dot{X} \wedge \bar{\nu}_G, \eta_G \oplus \zeta_G \oplus R^{s_2}] & \xrightarrow{T} & [X \wedge T(\nu_G), T(\eta_G \oplus \zeta_G) \wedge S^{s_2}] \\
 \downarrow p^* & & \downarrow \sigma \\
 [\dot{X} \wedge p^*(\bar{\nu}_G), \eta_H \oplus p^*(\zeta_G) \oplus R^{s_2}] & & [X \wedge T(\nu_G) \wedge S^{s_1}, T(\eta_G \oplus \zeta_G) \wedge S^{s_1+s_2}] \\
 \downarrow \sigma & & \downarrow t \# \\
 [\dot{X} \wedge \nu_H, \eta_H \oplus \zeta_H \oplus R^{s_1+s_2}] & \xrightarrow{T} & [X \wedge T(\nu_H), T(\eta_H \oplus \zeta_H) \wedge S^{s_1+s_2}] \\
 & & \uparrow (1 \wedge t) \#
 \end{array} \tag{8.10}$$

Finally, from the relation

$$\begin{array}{ccc}
 & T(\nu_G) \wedge S^{s_1} & \\
 c \swarrow & \downarrow t & \searrow c \\
 S^{s_1+s_2} & & T(\nu_H)
 \end{array}$$

we obtain the following commutative diagram.

$$\begin{array}{ccc}
 [X \wedge T(\nu_G), T(\eta_G \oplus \zeta_G) \wedge S^{s_2}] & \xrightarrow{c \#} & [X \wedge S^{s_2}, T(\eta_G \oplus \zeta_G^G) \wedge S^{s_2}] \\
 \downarrow t \# \sigma & & \downarrow t \# \sigma \\
 [X \wedge T(\nu_G) \wedge S^{s_1}, T(\eta_H \oplus \zeta_H) \wedge S^{s_1+s_2}] & & \\
 \uparrow (1 \wedge t) \# & \searrow (1 \wedge c) \# & \\
 [X \wedge T(\nu_H), T(\eta_H \oplus \zeta_H) \wedge S^{s_1+s_2}] & \xrightarrow{(1 \wedge t) \#} & [X \wedge S^{s_1+s_2}, T(\eta_H \oplus \zeta_H) \wedge S^{s_1+s_2}]
 \end{array} \tag{8.11}$$

Property (5.16) now follows from the commutativity of the diagrams (8.6) through (8.11).

Property (5.13) requires a similar analysis but we will leave the details to the reader. The key relation needed here is given in (7.5).

9. Proof of (6.13)

Let X be a finite complex such that $\dim(X) < n-1$ and let p_2 denote the projection $X \times S^n \rightarrow S^n$. The proof of (6.13) is based on the following commutative diagram

$$\begin{array}{ccccc} [X \times S(R^n), S(R^n)] & \xrightarrow{s} & [X \times S^n, S^n] & \xrightarrow{T'} & [X \times S^n, S^n] \\ \downarrow & & \downarrow & & \downarrow \\ [X, F] & \xrightarrow{\iota} & [X, Q^{(1)}(S^0)] & \xrightarrow{T} & [X, Q^{(0)}(S^0)] \end{array}$$

Here s is defined by $s(f)(x, tv) = tf(x, v)$, $t \geq 0$, $|v| = 1$. The vertical maps are given by the obvious exponential correspondence and T' is the map $T'(u) = [p_2] - u$. Since we are in the stable range $[X \times S^n, S^n]$ has a natural abelian group structure.

Let $f: X \times S(R^n) \rightarrow S(R^n)$ represent an element of $[X, F]$ and let

$$\lambda(f): X \times S^n \rightarrow S^n \quad (9.2)$$

represent its image under the equivalence $\lambda: [X, F] \rightarrow [X, Q^{(0)}(S^0)]$. From the commutativity of the above diagram it is sufficient to show that

$$[\lambda(f)] = [p_2] - [s(f)]. \quad (9.3)$$

To do this we will give an explicit description of $\lambda(f)$. The standard embedding $S(R^n) \subset R^n$ has a trivial normal bundle and a tubular neighborhood map $S(R^n) \times R \rightarrow R^n$ is given by $(v, t) \rightarrow e^t v$. Hence, the associated Pontrjagin-Thom map

$$c: S^n \rightarrow S^1 \times S(R^n)/S(R^n)$$

is given by $c(v) = (\log|v|, v/|v|)$. (It will be understood throughout this section that a point for which a formula is not defined is to be mapped to the base point.)

Let $\psi: \gamma \oplus \dot{R} \rightarrow R^n$ denote the standard trivialization $\psi((v, w) \oplus t) = tv + w$. If v is a non-zero vector let $\hat{v} = v/|v|$. Using this data to construct λ , we have

$$\lambda(f)(x, v) = \frac{f(x, \hat{v}) - (\hat{v} \cdot f(x, \hat{v})) \hat{v}}{1 - \hat{v} \cdot f(x, \hat{v})} + \log|v| \hat{v}.$$

Let

$$h: S^n \times S^n \rightarrow S^n \quad (9.4)$$

be defined by

$$h(v, w) = \frac{\hat{w} - (\hat{v} \cdot \hat{w}) \hat{v}}{1 - \hat{v} \cdot \hat{w}} + \log|v| \hat{v}.$$

Let $a \in \pi_n(S^n)$ denote a generator and let $a_1, a_2 \in \pi_n(S^n \times S^n)$ denote the image of a under inclusion onto the first and second factor respectively.

(9.5) LEMMA. *Suppose that n is odd. Then $h_*(a_1) = a$ and $h_*(a_2) = -a$.*

Proof. Since h maps the diagonal to the base point we have $h_*(a_1 + a_2) = 0$. Now let $d: S^n \rightarrow S^n \times S^n$ send v to $(v, -v)$ and consider the composite $hd: S^n \rightarrow S^n$. Its adjoint $(hd)': S(R^n) \rightarrow \Omega(S^n)$ is given by

$$(hd)'(v)(t) = \begin{cases} \log(t)v, & t > 0 \\ -\log(t)v, & t < 0. \end{cases}$$

Let $i: S(R^n) \rightarrow \Omega(S^n)$ denote the adjoint of the identity. Evidently, $(hd)'$ represents $[i] - [iA]$, where $A: S(R^n) \rightarrow S(R^n)$ is the antipodal map. If n is odd $[iA] = -[i]$ and $(hd)'$ represents $2[i]$. Therefore hd has degree 2. Since $d_*(a) = a_1 - a_2$ we have $h_*(a_1 - a_2) = 2a$. The lemma follows now from this and the relation $h_*(a_1 + a_2) = 0$.

We suppose now that n is odd. The map $\lambda(f)$ admits a factorization

$$X \times S^n \xrightarrow{\tilde{f}} S^n \times S^n \xrightarrow{h} S^n$$

where $\tilde{f}(x, v) = (v, s(f)(x, v))$. Because of the dimensional restriction on X we may deform \tilde{f} into $S^n \vee S^n$. That is, there exists a homotopy commutative diagram of the form

$$\begin{array}{ccc} X \times S^n & \xrightarrow{\tilde{f}} & S^n \times S^n \xrightarrow{h} S^n \\ & \searrow \tilde{f} & \uparrow \\ & & S^n \vee S^n \end{array}$$

It now follows from the lemma and an elementary diagram chase that $h\tilde{f} = \lambda(f)$ represents $[p_2] - [s(f)]$.

10. The Image of $\pi_*(F_G)$ in $\pi_*(F)$, $G = Z_p$.

The stable homotopy theoretic interpretation of the forgetful homomorphism from F_G to F_H yields considerable information on the image of $\pi_*(F_G)$ in $\pi_*(F_H)$. There is a natural division into two cases depending on whether G is finite or infinite; we defer the infinite case to the next two sections.

We begin with an easy observation.

(10.1) PROPOSITION. *Suppose G is finite and admits a free linear representation. Then the induced homomorphism*

$$\pi_*(F_G) \otimes Z[|G|^{-1}] \rightarrow \pi_*(F) \otimes Z[|G|^{-1}]$$

is an isomorphism.

Proof. According to (6.10), the above map is equivalent to the transfer homomorphism

$$\tau_*: S_*(B_G^+) \rightarrow S_*(S^{\infty+})$$

tensoring with $Z[|G|^{-1}]$. However, if $p: S^\infty \rightarrow B_G$ is projection, the composite $(p^+)_* \circ \tau_*$ is an isomorphism when tensored with $Z[|G|^{-1}]$ (see [23]). By a spectral sequence argument, $(p^+)_*$ is an isomorphism when tensored with $Z[|G|^{-1}]$. Hence the same is true of τ_* .

As one might expect, considerably stronger results hold for suitable choices of G . We limit our discussion to the following

(10.2) THEOREM. *Let $G = Z_p$, where p is a prime. Then the forgetful map from $\pi_*(F_G)$ to $\pi_*(F)$ is surjective in positive dimensions.*

Proof. By (10.1) the image of the forgetful map contains all torsion in $\pi_*(F)$ of order prime to p . Since $\pi_*(F)$ is finite in positive dimensions, it suffices to prove that the p -primary component of $\pi_*(F_G)$ maps onto the p -primary component of $\pi_*(F)$ in positive dimensions. We shall establish this using results of D. S. Kahn and S. B. Priddy [16]; the cases $p=2$ and $p \neq 2$ require separate treatment.

Case 1. $p=2$. In this case $B_G = RP^\infty$. Embed RP^∞ in the infinite special orthogonal group via the reflection construction; since SO is contained in F_G (linear maps are Z_2 -equivariant) and F_{Z_2} is homotopy equivalent to $Q(RP^{\infty+})$, this yields a map from RP^∞ to $Q(RP^{\infty+})$. The results of [18] imply the existence of a unique map

$$h: Q(RP^\infty) \rightarrow Q(RP^{\infty+}) \quad (10.3)$$

which is a map of infinite loop spaces and makes the following diagram commute:

$$\begin{array}{ccccc} \pi_*(RP^\infty) & \longrightarrow & \pi_*(Q(RP^\infty)) & & \\ \downarrow p^* & & \downarrow h^* & & \\ \pi_*(SO) & \rightarrow \pi_*(F_{Z_2}) \xrightarrow{\lambda_2^*} & \pi_*(Q(RP^{\infty+})) & & \\ \searrow J^* & \downarrow & \downarrow t^* & & \\ & \pi_*(F) \xrightarrow{\lambda_1^*} & \pi_*(Q(S^0)) & & \end{array} \quad (10.4)$$

It is well-known that $\lambda_1 J p$ induces an isomorphism of fundamental groups. Thus by [16, Theorem 4.1] its adjoint induces a surjection of 2-primary components in positive-dimensional homotopy. But this adjoint induces $t_* h_*$ in homotopy by standard adjoint functor formulas, and hence t_* must also induce a surjection of 2-primary components in positive-dimensional homotopy.

Case 2. $p \neq 2$. Suppose $f: X \rightarrow QY$ is continuous where X and Y are pointed CW-complexes. Then there is an essentially unique factorization of f through Y as an S -map (i.e., in the category of CW-spectra). Hence for any cohomology theory h^* there is a canonical induced homomorphism

$$f^*: h^*(Y) \rightarrow h^*(X) \quad (10.6)$$

making the following diagram commutative

$$\begin{array}{ccc} h^*(Q(Y)) & \xrightarrow{f^*} & h^*(X) \\ \downarrow i^* & \nearrow f^* & \\ h^*(Y) & & \end{array}$$

Furthermore the correspondence $f \rightarrow f^*$ is functorial. Let $L = B_{Z_p}$ and let $t: Q(L^+) \rightarrow Q(S^0)$ denote some map which realizes the transformation

$$t_{\#}: [\ ; Q(L^+)] \rightarrow [\ ; Q(S^0)].$$

For any such choice of t we have the following commutative diagram (where H^* denotes singular cohomology with Z_p coefficients).

$$\begin{array}{ccccc} & H^*(F) & \xleftarrow{\lambda^*} & H^*(Q(S^0)) & \\ & \downarrow & & \downarrow t^* & \\ H^*(U) & \leftarrow H^*(F_{Z_p}) & \xleftarrow{\lambda^*} & H^*(Q(L^+)) & \\ & \searrow \lambda_{\#} & & \downarrow i^* & \\ & & & H^*(L^+) & \end{array} \quad (10.7)$$

Let $\sigma(q_i) \in H^{2i(p-1)-1}(F)$ represent the loop-suspension of the i -th Wu class

$$q_i \in H^{2i(p-1)}(BF) \quad (10.8)$$

and let $r_i = \lambda^{*-1}(\sigma(q_i))$. By the results of Kahn and Priddy [16, Remark 4.3] together with a lemma of Tsuchiya [30, Lemma 6.3], in order to show that the adjoint of the composite

$$L^+ \xrightarrow{i} Q(L^+) \xrightarrow{t} Q(S^0)$$

induces an epimorphism of p -primary components in stable homotopy (in positive dimensions) it is sufficient to show that the images of the r_i in $H^{2i(p-1)-1}(L^+)$ are non zero. From the diagram (10.7) this will follow by showing that the classes $\sigma(q_i)$ map non-trivially into $H^*(U)$. Now the image of $\sigma(q_i)$ in $H^*(U)$ is the loop-suspension of the i -th Wu class in $H^*(BU)$ which is a non zero multiple of the Chern class of dimension $(p-1)i$ modulo decomposables (see [33] or [30, p. 120]). Hence it is non zero in $H^*(U)$.

11. The Image of $\pi_*(F_G)$ in $\pi_*(F)$, G Infinite

In contrast to the above results for $G = Z_p$ the image of $\pi_k(F_G)$ in $\pi_k(F)$ is always a proper subgroup if G is infinite and $k \equiv \pm 1 \pmod{8}$ with the exception of $k = 1$ if

$G \neq S^3$ (since F_{S^3} is 2-connected by (6.6), clearly the generator of $\pi_1(F) = Z_2$ does not come from $\pi_*(F_{S^3})$). The proof has two basic ingredients – an investigation of the image of $\pi_*(U)$ in $\pi_*(F)$ and a computation of the Adams e -invariant of elements in $\pi_*(F)$ which come from torsion in $\pi_*(F_{S^1})$.

In [8] Browder essentially proved that $\pi_*(U) \rightarrow \pi_*(F_{S^1})$ is monic. Using his methods one can prove a much stronger result.

(11.1) THEOREM. *The map from $\pi_*(U)$ to $\pi_*(F_{S^1})$ is an injection onto a direct summand, and the complementary summand of the latter group is finite.*

We shall need the notion of G -equivariant fiber bundle as defined by Tom Dieck [29]; however, all of our equivariant bundles will be over trivial G -spaces, and hence the formulation of equivariant local triviality is easily understandable. In particular, if $\text{Top}(X, \varphi)$ is the group of G -equivariant homeomorphisms of the G -space X with action $\varphi: G \times X \rightarrow X$, then equivariant (X, φ) bundles over a trivial base are classified by maps from the base into $B \text{Top}(X, \varphi)$.

The Dold-Lashof classification of ordinary fiber bundles up to fiber homotopy type [10, Theorem 7.5, p. 303] generalizes to equivariant fiber bundles over trivial G -spaces with only minor changes.

(11.2) PROPOSITION. *Let (X, φ) be as above, and let $F(X, \varphi)$ be its space of equivariant self-maps. Two equivariant fiber bundles over a CW complex with fiber (X, φ) are equivariantly fiber homotopy equivalent if and only if the composites of their classifying maps with the induced function*

$$B \text{Top}(X, \varphi) \rightarrow BF(X, \varphi).$$

are homotopic.

The following result generalizes the main step in Browder's argument. It is apparently well-known but (to our knowledge) unpublished.

(11.3) LEMMA. (i) *Let ξ be an n -dimensional complex vector bundle over a finite complex, and assume that its unit sphere bundle is equivariantly fiber homotopically trivial (with the obvious free S^1 action). Then the complex K -theoretic Chern classes of ξ are trivial.* (ii) *Let ξ be an n -dimensional quaternionic vector bundle over a finite complex, and assume that the unit sphere bundle of ξ is equivariantly fiber homotopically trivial (with the obvious free S^3 action). Then the real K -theoretic symplectic Pontrjagin classes of ξ are trivial.*

The characteristic classes mentioned above are defined in [9].

Proof. (i) Let $S(\xi)$ be the associated S^{2n-1} bundle of ξ and let $P(\xi)$ be the associated CP^{n-1} bundle. Then $S(\xi) \rightarrow P(\xi)$ is a principal S^1 bundle projection we shall call the *canonical line bundle* of ξ . An equivariant fiber homotopy equivalence

from $S(\xi)$ to $B \times S^{2n-1}$ induces a fiber homotopy equivalence from $P(\xi)$ to $B \times CP^{n-1}$ under which the canonical line bundle over $B \times CP^{n-1}$ (namely, $\text{id} \times p: B \times S^{2n-1} \rightarrow B \times (P^{n-1})$) pulls back to the canonical line bundle on ξ . Since K -theoretic Chern classes satisfy an analog of the Grothendieck relation for ordinary Chern classes (compare [9, pp. 45–48] or [3, pp. 84, 109], Browder's argument [8, p. 33] works for complex K -theory as well as singular cohomology.

(ii) This follows from a virtually identical argument with canonical quaternionic line bundles replacing complex line bundles and KO -theoretic symplectic Pontrjagin classes [9, pp. 45–48, 52–58] replacing K -theoretic Chern classes.

(11.4) COROLLARY. *If ξ satisfies the hypotheses of Proposition 8.3, it is stably trivial.*

Proof. The results of [9, Section 9] show that the first K -theoretic Chern or symplectic Pontrjagin class of ξ is its stable equivalence class in $K^2(X) \cong \tilde{K}(X)$ or $KO^4(X) \cong \widetilde{KSp}(X)$.

Proof of Theorem (11.1). Since U and F_{S^1} are both arcwise connected, the result is trivial for π_0 . We shall first prove the result for π_1 and use the low-dimensional cases in providing the higher-dimensional ones.

Let $F(CP^{n-1})$ be the space of self maps of CP^{n-1} . Regarding C^n as an S^1 module we have the space $F_{S^1}(C^n)$. A result of James [13] states that the 'passage to orbit space' homomorphism

$$F_{S^1}(C^n) \rightarrow F(CP^{n-1}) \quad (11.5)$$

is a fibration whose fiber is homeomorphic to the space of functions from CP^{n-1} to S^1 . It is easy to show that the latter is a $K(Z, 1)$ and the inclusion of S^1 as the set of diagonal matrices is an explicit homotopy equivalence. Thus we have the following commutative diagram whose rows represent fibrations and whose left-hand vertical map is a homotopy equivalence;

$$\begin{array}{ccccc} S^1 & \rightarrow & U_n & \longrightarrow & PSU_n \\ \downarrow \cong & & \downarrow & & \downarrow \\ X & \rightarrow & F_{S^1}(C^n) & \rightarrow & F(CP^{n-1}) \end{array} \quad (11.6)$$

as usual, PSU_n denotes the projective group. Consider the induced mappings of fundamental groups; in the first row one obtains the short exact sequence $0 \rightarrow Z \rightarrow \rightarrow Z \rightarrow Z_n \rightarrow 0$. By (11.4), the induced map from $\pi_1(U_n) = Z$ to $\pi_1(F_{S^1}(C^n))$ is monic. Thus the induced map from $\pi_1(X)$ to $\pi_1(F_{S^1}(C^n))$ is also monic; notice that $\pi_1(F_{S^1}(C^n)) = Z$ holds if $n \geq 2$ by Theorem (5.7). An application of [26, Theorem 4.11, p. 452] shows that $\pi_1(F(CP^{n-1})) \cong Z_n$, and it follows that the bottom row of the above diagram also yields the short exact sequence $0 \rightarrow Z \rightarrow Z \rightarrow Z_n \rightarrow 0$ in funda-

mental groups. But this forces the map from $\pi_1(U_n)$ to $\pi_1(F_{S^1}(C^n))$ to be an isomorphism. Since $\pi_1(U_n) \cong \pi_1(U)$ and $\pi_1(F_{S^1}(C^n)) \cong \pi_1(F_{S^1})$ if n is large, the proof of the theorem in dimension 1 is complete.

Consider the following extended fibration sequence

$$U_n \rightarrow F_{S^1}(C^n) \xrightarrow{g} Y_n \xrightarrow{f} BU_n \xrightarrow{h} BF_{S^1}(C^n). \quad (11.7)$$

By the results of the previous paragraphs, Y_n is 1-connected. Thus Lemma (6.5) and results of Stasheff [27] and Milnor [19] imply Y_n has the homotopy type of a CW complex with finitely many cells in each dimension.

Let W_n be the $2n$ -skeleton of such a complex homotopically equivalent to Y_n , and let $j: W_n \rightarrow Y_n$ be the 'inclusion' map. Then hfj is homotopically trivial, so that the composite of f_i with the canonical map from BU_n to BU is homotopically trivial by Corollary (11.4). Since (BU, BU_n) is $(2n+1)$ -connected and $\dim W_n \leq 2n$, it follows that ffj is homotopically trivial. Since f is a fibration, this means that j factors through g up to homotopy. Since g is a fibration, this means that the induced fibration

$$U_n \rightarrow j^*F_{S^1}(C^n) \rightarrow W_n$$

has a cross section. Therefore

$$\pi_*(j^*F_{S^1}(C^n)) \cong \pi_*(W_n) \oplus \pi_*(U_n).$$

However, the pair $(F_{S^1}(C^n), j^*F_{S^1}(C^n))$ is $2n$ -connected, and hence it is immediate that $\pi_i(U_n) \rightarrow \pi_i(F_{S^1}(C^n))$ is an injection onto a direct summand if $i < 2n$. Since (U, U_n) is $2n$ -connected and $(F_{S^1}, F_{S^1}(C^n))$ is $(2n-2)$ -connected by 5.5 and 6.6, an obvious diagram chase shows that $\pi_*(U) \rightarrow \pi_*(F_{S^1})$ is also an injection onto a direct summand. The finiteness of the complementary summand follows because rank $\pi_i(F_{S^1})$ is 1 if i is odd and 0 if i is even, the same as the corresponding rank of $\pi_i(U)$.

(11.8) *Addendum to 11.1.* A completely analogous argument shows that $\pi_*(Sp) \rightarrow \pi_*(F_{S^3})$ is an injection onto a direct summand with finite complementary summand; we shall omit the details.

(11.9) **THEOREM.** *Let n be odd, and let $u \in \pi_n(F_{S^1})$ have finite order. Then the image of u in $\pi_n(F)$ has trivial complex e -invariant.*

See [1, §3] for the definition and properties of the complex Adams e -invariant.

Proof. Let $T: S^{2m+1}(CP^{r+}) \rightarrow S^{2m}(S^{2r+1+})$ be the transfer, where $r \gg n$ and $2m \gg r$. Let $u': S^{2m+n} \rightarrow S^{2m+1}(CP^{r+})$ correspond to u . The image v of u in $\pi_n(F)$ corresponds to cTu' , where $c: S^{2m}(S^{2r+1+}) \rightarrow S^{2m}$ collapses the $S^{2m+2r+1}$ wedge factor.

To show $e_c(\text{image } u) = 0$, it suffices to prove that $\tilde{K}(C(v)) \cong \tilde{K}(S^{2m}) \oplus \tilde{K}(S^{2m+n+1})$

as modules over the Adams ψ operations (compare [1, §6]). Consider the following diagram

$$\begin{array}{ccccccc}
 S^{2m+n} & \xrightarrow{v} & S^{2m} & \rightarrow & C(v) & \rightarrow & S^{2m+n+1} \longrightarrow S^{2m+1} \\
 \downarrow u' & & \downarrow = & & \downarrow & & \downarrow Su' \\
 S^{2m+1}(CP^{r+}) & \xrightarrow{cT} & S^{2m} & \rightarrow & Y & \rightarrow & S^{2m+2}(CP^{r+}) \rightarrow S^{2m+1}
 \end{array} \quad (11.9)$$

Apply \tilde{K} to this diagram; since $\tilde{K}(X)=0$ if X is a finite complex with cells of only odd dimensions, we have the following commutative diagram:

$$\begin{array}{ccccccc}
 0 \leftarrow \tilde{K}(S^{2m}) \leftarrow \tilde{K}(Y) \leftarrow \tilde{K}(S^{2m+2}CP^{r+}) \leftarrow 0 \\
 \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
 0 \leftarrow \tilde{K}(S^{2m}) \leftarrow \tilde{K}(C(v)) \leftarrow \tilde{K}(S^{2m+n+1}) \leftarrow 0
 \end{array} \quad (11.10)$$

Let α generate $\tilde{K}(S^{2m})=Z$, let $\xi' \in \tilde{K}(Y)$ map to α , let $\xi \in \tilde{K}(C(v))$ denote the image of ξ' .

It suffices to show that $\psi^k(\xi)=k^m\xi$. By naturality,

$$\psi^k(\xi) = k^m\xi + \tau, \quad (11.11)$$

where $\tau \in \text{Image}(u')^*$. But the order of $(u')^*$ is finite since the order of u' is; since $\tilde{K}(S^{2m+n+1})=Z$, this means $(u')^*$ must vanish. Therefore $\tilde{K}(C(v))$ splits as a ψ -module.

Theorems (11.1) and (11.9) readily yield the following result:

(11.12) THEOREM. (i) Let $\mu_k (k \geq 1)$ denote the Adams-Barratt element in $\pi_{8k+1}(F)$. Then μ_k is not in the image of $\pi_{8k+1}(F_{S^1})$.

(ii) Let $\sigma_k (k \geq 1)$ denote the generator of the image of J in dimension $8k-1$. Then σ_k is not in the image of $\pi_{8k-1}(F_{S^1})$.

(iii) In the notation of (ii), twice σ_k is not in the image of $\pi_{8k-1}(F_{S^3})$.

Proof. The results of Adams show that μ_k and $2\sigma_k$ have nontrivial e -invariant [1, pp. 68 and 44–45]. Thus they can only come from elements of $\pi_*(F_{S^3})$ or $\pi_*(F_{S^1})$ having infinite order. An easy application of Theorem (11.1) and its addendum shows that if they come from $\pi_*(F_{S^3})$ or $\pi_*(F_{S^1})$, they also come from $\pi_*(Sp)$ or $\pi_*(U)$ respectively. Since μ_k is not in the image of J , conclusion (i) follows. On the other hand, the Bott periodicity theorems imply that $\pi_{8k-1}(G)=Z$ if $G=0, U$, or Sp and the canonical maps

$$\begin{array}{l}
 \pi_{8k-1}(U) \rightarrow \pi_{8k-1}(0) \\
 \pi_{8k-1}(Sp) \rightarrow \pi_{8k-1}(0)
 \end{array}$$

are multiplication by 2 and 4 respectively (for example, see [7]). This shows that σ_k and $2\sigma_k$ do not come from $\pi_{8k-1}(F_{S^1})$ and $\pi_{8k-1}(F_{S^3})$ respectively, proving (ii) and (iii).

12. The Image of $\pi_*(F_{S^3})$ in $\pi_*(F_{S^1})$

The pathologies discussed in Section 11 are definitely 2-primary in nature. For example, if p is odd the generators of the p -primary components of the image of J always come from $\pi_*(F_{S^3})$; in fact, they come from $\pi_*(Sp)$ because the canonical map from $\pi_*(Sp)$ to $\pi_*(0)$ is an isomorphism mod (graded) finite 2-groups. Thus one is led to ask whether the induced map from $\pi_*(F_{S^3}) \otimes Z[\frac{1}{2}]$ to $\pi_*(F) \otimes Z[\frac{1}{2}]$ is surjective in positive dimensions. Although we cannot prove this, we can prove that the images of $\pi_*(F_{S^3}) \otimes Z[\frac{1}{2}]$ and $\pi_*(F_{S^1}) \otimes Z[\frac{1}{2}]$ in $\pi_*(F) \otimes Z[\frac{1}{2}]$ are the same.

By Theorem (5.15) the above statement is equivalent to saying that the images of the transfer homomorphisms

$$\begin{aligned} S_*((H\tilde{P}^\infty)^\infty) \otimes Z[\tfrac{1}{2}] &\rightarrow S_*(S^0) \otimes Z[\tfrac{1}{2}] \\ S_*(S(CP^{\infty+})) \otimes Z[\tfrac{1}{2}] &\rightarrow S_*(S^0) \otimes Z[\tfrac{1}{2}] \end{aligned}$$

are equal. We shall deduce this using the following result.

(12.1) THEOREM. *Let k be the involution of CP^∞ given by conjugation. Then the transfer from $S_*(H\tilde{P}^\infty) \otimes Z[\frac{1}{2}]$ to $S_*(S(CP^\infty)) \otimes Z[\frac{1}{2}]$ is surjective, and its image is the subgroup left fixed by $S(k^+)_*$.*

Assuming this, we state and prove the fact mentioned above.

(12.2) THEOREM. *The images of $S_*(H\tilde{P}^\infty) \otimes Z[\frac{1}{2}]$ and $S_*(S(CP^{\infty+})) \otimes Z[\frac{1}{2}]$ in $S_*(S^0) \otimes Z[\frac{1}{2}]$ are equal.*

Proof. Let S^∞ be the total space of the universal S^1 bundle over CP^∞ . Then k lifts to an involution l of S^∞ , and by the naturality of the transfer we have the following commutative diagram:

$$\begin{array}{ccccc} S(CP^{\infty+}) & \rightarrow & S^{\infty+} & \simeq & S^0 \\ \downarrow S(k^+) & & \downarrow l^+ & & \downarrow id \\ S(CP^{\infty+}) & \rightarrow & S^{\infty+} & \simeq & S^0 \end{array}$$

It follows that if $y \in S_*(S(CP^{\infty+}))$, then y and $S(k^+)_* y$ have the same image in $S_*(S^0)$. Clearly this remains true after tensoring with $Z[\frac{1}{2}]$.

Consider the element $\frac{1}{2}(y + S(k^+)_* y)$ in $S_*(S(CP^{\infty+})) \otimes Z[\frac{1}{2}]$. By the discussion of the preceding paragraph its image in $S_*(S^0) \otimes Z[\frac{1}{2}]$ is the same as the image of y . On the other hand, it is clearly left fixed by $S(k^+)_*$, so that it lies in the image of $S_*(H\tilde{P}^\infty) \otimes Z[\frac{1}{2}]$ by Theorem (12.1).

Let N be the normalizer of S^1 in S^3 ; then the transfer from $H\tilde{P}^\infty$ to $S(CP^{\infty+})$ factors through BN^ζ . The proof of Theorem (12.1) has two parts – an examination of the image of $S_*(H\tilde{P}^\infty) \otimes Z[\frac{1}{2}]$ in $S_*(BN^\zeta) \otimes Z[\frac{1}{2}]$ and an examination of the image of $S_*(BN^\zeta) \otimes Z[\frac{1}{2}]$ in $S_*((CP^{\infty+}) \otimes Z[\frac{1}{2}])$.

(12.3) PROPOSITION. *The induced homomorphism from $S_*(H\tilde{P}^\infty) \otimes Z[\frac{1}{2}]$ to $S_*(BN^\zeta) \otimes Z[\frac{1}{2}]$ is an isomorphism.*

Proof. Let $k \geq 0$ be given, and let n be large with respect to k . It suffices to prove that

$$t_* S_k((HP^{n-1})^{\zeta(S^3)}) \rightarrow S_k((S^{4n-1}/N)^{\zeta(N)})$$

is an isomorphism when tensored with $Z[\frac{1}{2}]$.

The Atiyah-Hirzebruch spectral sequence for stable homotopy theory yields a spectral sequence map converging to the homomorphism under consideration. On the E_2 level it takes the form

$$t_*: H_p((HP^{n-1})^\zeta; S_q) \otimes Z[\frac{1}{2}] \rightarrow H_p((S^{4n-1}/N)^\zeta; S_q) \otimes Z[\frac{1}{2}].$$

The homology groups of X^ζ are isomorphic to unreduced cohomology groups of X (where $X = HP^{n-1}$ or S^{4n-1}/N) by the Thom isomorphism and Poincaré duality. Techniques of Boardman [6, §6] show that under these isomorphisms t_* corresponds to the cohomology map induced by the projection

$$p: S^{4n-1}/N \rightarrow HP^{n-1}.$$

Therefore it suffices to know that p^* is an isomorphism in $Z[\frac{1}{2}]$ -module coefficients. This follows from the Serre spectral sequence; for p is an orientable fiber bundle projection whose fiber is RP^2 , a $Z[\frac{1}{2}]$ -acyclic space.

We shall need a slight generalization of a familiar result on the transfer in singular cohomology.

(12.4) PROPOSITION. *Suppose $p: X \rightarrow Y$ is a regular n -sheeted covering (Y is a CW complex) and G is the full group of covering transformations. Let ξ be a k -plane bundle over Y whose pullback to X is trivial, and let $p^\xi: S^k X^+ \rightarrow Y^\xi$ denote the induced map of Thom spaces.*

(i) *If $t: Y^\xi \rightarrow S^k X^+$ is the transfer, then $p^\xi t$ is an isomorphism in any homology theory taking values in the category of $Z[1/n]$ -modules.*

(ii) *Let h_* be a homology theory taking values in the category of $Z[1/n]$ -modules. Then t_* is injective and its image is the stationary set of $h_*(S^k X^+)$ under the action of G induced by covering transformations.*

The proof of the first part is an exercise in the techniques of [6, §6] and [23]. The proof of the second part is an elementary algebraic exercise based on the canonical isomorphism from $h_*(S^k X^+)/G$ to $h_*(Y^\xi)$ induced by p^ξ .

The following result and Proposition (12.3) imply Theorem (12.1).

(12.5) PROPOSITION. *The transfer map from $S_*(BN^\zeta) \otimes Z[\frac{1}{2}]$ to $S_*(S(CP^\infty)^+) \otimes Z[\frac{1}{2}]$ is injective and its image is the subgroup left fixed under $S(k)_*$.*

Proof. If ζ is the line bundle over BN given by the adjoint representation, then the pullback of ζ to CP^∞ is trivial. On the other hand, CP^∞ is a double covering of BN , and an elementary argument shows that the covering involution of CP^∞ is homotopic to k . Thus the proposition follows from Proposition (12.4).

APPENDIX

13. The Transfer

Let $p:M \rightarrow N$ be a finite covering space where M and N are compact smooth manifolds without boundary. In section 4, we described a well known method of associating with a sectioned bundle α over N an S -map.

$$t: T(\alpha) \wedge S^s \rightarrow T(p^*(\alpha)) \wedge S^s.$$

For the purposes of this section we refer to t as the ‘umkehr’ map. On the other hand, there are general constructions of Roush [23] and of Kahn and Priddy [16] which associate with a finite covering pair a wrong way map called the ‘transfer’. In particular, for the covering pair $(E_{p^*(\alpha)}, M) \rightarrow (E_\alpha, N)$ there is a transfer

$$\tau: T(\alpha) \wedge S^s \rightarrow T(p^*(\alpha)) \wedge S^s.$$

The object of this appendix is to give a direct proof that the umkehr map agrees with the transfer. In this direction Roush has shown that their induced homomorphisms agree for any (co) homology theory h for which N is h -orientable (taking $\alpha=0$).

We begin by describing the transfer for finite coverings. Let \mathcal{C} denote the subcategory of the stable homotopy category of CW-spectra [6,31] having pointed CW-complexes as objects. Let G be a finite group and H a subgroup. Let \mathcal{P}_G denote the category whose objects are CW-pairs (X, A) with a free and cellular action of G on X which leaves A invariant. The morphisms in \mathcal{P}_G are to be equivariant maps of pairs. We will call (X, A) a free G -pair. There is the forgetful functor $\mathcal{R}: \mathcal{P}_G \rightarrow \mathcal{P}_H$ obtained by restricting the action of G to H . There is also the quotient functor $\mathcal{Q}_G: \mathcal{P}_G \rightarrow \mathcal{C}$ defined by sending (X, A) to $X/A/G$. As usual, we write X^+ for $X/\Phi = X \cup \{+\}$ and, in general, $+$ will denote the base point of a pointed space. If $f: (X, A) \rightarrow (X', A')$ is a G -map, we also let f denote the quotient map $f: X/A/G \rightarrow X'/A'/G$.

There is a ‘suspension’ functor $\mathcal{P}_G \rightarrow \mathcal{P}_G$ defined by sending (X, A) to the pair $(X, A) \times (S^1, +)$ with G acting on the first factor. Note that the quotient of $(X, A) \times (S^1, +)$ is equal to $X/A/G \wedge S^1$.

Suppose that $\Delta: X/G \rightarrow X/H$ is a cross section to the covering $p: X/H \rightarrow X/G$.

There is then a retraction $q: X/A/H \rightarrow X/A/G$ by

$$q(y) = \begin{cases} p(y), & \text{if } y = \Delta(p(y)). \\ +, & \text{otherwise.} \end{cases}$$

(13.1) DEFINITION. An $H-G$ transfer is a natural transformation $\tau: \mathcal{Q}_G \rightarrow \mathcal{Q}_H \mathcal{R}$ having the following properties:

- (a) $\tau(X, A) \times (S^1, +) = \tau(X, A) \wedge 1$.
- (b) If $\Delta: X/G \rightarrow X/H$ is a cross section.

the composite

$$X/A/G \xrightarrow{\tau} X/A/H \xrightarrow{q} X/A/G$$

is the identity.

Although our formulation of the transfer is slightly different than Roush's his results are easily translated. Hence we have

(13.2) THEOREM. (Roush [23]). *There exists a unique $H-G$ transfer.*

The construction of τ that follows is equivalent to that of Roush and also of Kahn and Priddy. If Y is a pointed space let $P(Y)$ denote the space of functions $\sigma: G/H \rightarrow Y$, where G/H denotes the set of left cosets of H in G . Let G act on $P(Y)$ by $g\sigma(wH) = \sigma(g^{-1}wH)$, $g, w \in G$. We have an equivariant embedding

$$(G/H)^+ \wedge Y \rightarrow P(Y)$$

by $wH \wedge y \rightarrow \sigma$, where $\sigma(wH) = y$ and $\sigma(w'H) = +$ if $w'H \neq wH$. Topologically, the pair $(P(Y), (G/H)^+ \wedge Y)$ is simply the n -fold product of Y modulo the n -fold wedge, where n is the index of H in G . Hence it is a $(2s-1)$ -connected pair if Y is $(s-1)$ -connected.

Now we may write

$$P(Q(Y)) = \text{inj} \lim_k \Omega^i(P(Y \wedge S^k))$$

and

$$Q((G/H)^+ \wedge Y) = \text{inj} \lim_p \Omega^k((G/H)^+ \wedge (Y \wedge S^k)).$$

Moreover, the embedding (13.3) is compatible with the injective limit maps and so we obtain

$$i: Y((G/H)^+ \wedge Y) \rightarrow P(Q(Y)). \quad (13.4)$$

By the remarks of the preceding paragraph, the relative homotopy groups of the pair $(P(Q(Y)), Q((G/H)^+ \wedge Y))$ are trivial.

Now let (X, A) be a free G -pair and set $Y = X/A/H$. Define

$$\varphi: (X, A) \rightarrow (P(Y), +) \quad (13.5)$$

by $\varphi(x)(wH) = [w^{-1}x]$. Then φ is a G -map. We will also let φ denote the map $(X, A) \rightarrow (P(Q(Y)), +)$ obtained by composing with the canonical inclusion $P(Y) \subset P(Q(Y))$. Consider the diagram

$$\begin{array}{ccc} (X, A) & \xrightarrow{\varphi} & (P(Q(Y)), +) \\ & \searrow \varphi' & \uparrow i \\ & & (Q((G/H)^+ \wedge Y), +) \xrightarrow{Q(\lambda)} Q(Y), \end{array}$$

where λ is the ‘folding map’ $(G/H)^+ \wedge Y \rightarrow Y$ defined by $\lambda(wH \wedge y) = y$. There are no obstructions to equivariantly deforming φ relative to A into $Q((G/H)^+ \wedge Y)$. The end of such a homotopy is denoted by φ' in the diagram. Upon taking quotients $Q(\lambda)$ yields a map

$$\tau': X/A/G \rightarrow Q(Y) = Q(X/A/H). \quad (13.7)$$

Now the transfer τ is the map in the stable homotopy category which is the adjoint of τ' . It is easy to check that τ is well defined and meets the requirements of definition (13.1).

To obtain a transfer on the category of n -fold coverings let $G = \mathcal{S}_n$, the symmetric group on n letters, and let $H = \mathcal{S}_{n-1}$. If $p: (E, E') \rightarrow (B, B')$ is an n -fold covering pair let X denote the total space of the associated principal G -bundle. Precisely, X is the space of maps $\sigma: \{1, \dots, n\} \rightarrow E$ such that σ is fiber preserving and one-one. Let A be the subspace of maps whose image lies in E' . If G acts on X by $\sigma \rightarrow \sigma\psi^{-1}$, $\psi \in G$, we have a free G -pair (X, A) and the assignment which sends the covering pair to (X, A) is clearly functorial. Moreover $p: (X/H, A/H) \rightarrow (X/G, A/G)$ is naturally equivalent to the original covering pair. The identifications $X/H \rightarrow E$ and $X/G \rightarrow B$ are given by $\sigma \rightarrow \sigma(n)$ and $\sigma \rightarrow p\sigma(n)$ respectively. Hence the H - G transfer yields a transfer for n -fold coverings.

Now let $p: M \rightarrow N$ be a finite covering of index n where M and N are smooth manifolds. By the preceding remarks, we may write it in the form $p: X/H \rightarrow X/G$ where $G = \mathcal{S}_n$, $H = \mathcal{S}_{n-1}$, and X is a smooth manifold. To define the umkehr map we will construct a particular embedding

$$\hat{p}: X/H \rightarrow X/G \times R^s \quad (13.8)$$

Let V denote the G -module consisting of R^n plus the action of $G = \mathcal{S}_n$ on R^n through permutations. There is an embedding $X/H \rightarrow X \times V/G$ by $[x]_H \rightarrow [x, e_n]_G$. Now for the vector bundle $\pi: X \times V/G \rightarrow X/G$ there is, for large, s , a map

$$\sigma: X \times V/G \rightarrow R^s \quad (13.9)$$

which is a monomorphism on each fiber. Let \hat{p} be the composite embedding

$$X/H \rightarrow X \times V/G \xrightarrow{(\pi, \sigma)} X/G \times R^s.$$

Explicitly, $\hat{p}([x]) = ([x], \sigma([x, e_n]))$. The embedding \hat{p} has trivial normal bundle and for ε sufficiently small we have a tubular neighborhood map

$$\hat{p}: X/H \times R^s \rightarrow X/G \times R^s \quad (13.10)$$

by $\hat{p}([x], v) = ([x], \varrho(x, v))$, where

$$\varrho: X \times R^s \rightarrow R^s \quad (13.11)$$

is defined by $\varrho(x, v) = \sigma([x, e_m]) + \varepsilon v/1 + |v|$.

Let β be a sectioned bundle over X/G and α its pullback over X . Then $\beta = \alpha/G$ and $p^*(\beta) = \alpha/H$. Using the above tubular neighborhood embedding, the umkehr map

$$t: T(\alpha/G) \wedge S^s \rightarrow T(\alpha/H) \wedge S^s \quad (13.12)$$

is given by

$$t([a] \wedge v) = \begin{cases} [g^{-1}a] \wedge v', & \text{if } v = \varrho(g^{-1}p_\alpha(a), v') \\ +, & \text{otherwise} \end{cases}$$

On the other hand there is the transfer

$$\tau: T(\alpha/G) \wedge S^s \rightarrow T(\alpha/H) \wedge S^s \quad (13.13)$$

associated with the free G -pair (E_α, X) .

We will show now that $t = \tau$. To this end let $Y = T(\alpha/H)$ and define

$$\theta: (E_\alpha, X) \times (S^s, +) \rightarrow (G/H)^+ \wedge Y \wedge S^s$$

by

$$\theta(a, v) = \begin{cases} gH \wedge [g^{-1}a] \wedge v', & v = \varrho(g^{-1}p_\alpha(a), v') \\ +, & \text{otherwise} \end{cases}$$

Consider the following diagram

$$\begin{array}{ccc} (E_\alpha, X) & \xrightarrow{\varphi} & (P(Y \wedge S^s), +) \\ & \searrow \theta & \uparrow i \\ & & ((G/H)^+ \wedge Y \wedge S^s, +) \xrightarrow{\lambda} (Y \wedge S^s, +) \end{array}$$

Since the umkehr map t is the quotient of $\lambda\theta$, we will have $\pi = t$ provided $i\theta$ is equivariantly homotopic to φ . The required homotopy F_t is given by

$$F_t(a, v)(gH) = \begin{cases} [g^{-1}a] \wedge v', & \\ \text{if } v = t\varrho(g^{-1}p_\alpha(a), v') + (1-t)v', & \\ +, & \text{otherwise.} \end{cases}$$

REFERENCES

- [1] ADAMS, J. F. *On the groups $J(X)$ – IV*, Topology 5 (1966), 21–80.
- [2] ATIYAH, M. F., *Thom complexes*, Proc. London Math. Soc. 11 (1961), 291–310.
- [3] —, *K-theory*, Benjamin, New York, 1967.
- [4] BECKER, J. C., *On the existence of A_k -structures on stable vector bundles*, Topology 9 (1970), 367–384.
- [5] —, *Vector fields on quotient manifolds of spheres*, Indiana Univ. Math. Jour. 22 (1973), 859–871.
- [6] BOARDMAN, J. M., *Stable homotopy theory*, Chapter V – *Duality and Thom Spectra*, mimeographed. University of Warwick, 1966.
- [7] BOTT, R., *The stable homotopy of the classical groups*, Ann. of Math. 70 (1959), 313–337.
- [8] BROWDER, W., *Surgery and the theory of differentiable transformation groups*, Proc. of the Conf. of Transformation Groups (New Orleans, 1967), 1–46. Springer-Verlag, New York, 1968.
- [9] CONNER, P. and FLOYD, E., *The relation of cobordism to K-theories*, Lecture Notes in Mathematics 28, Springer-Verlag, New York, 1966.
- [10] DOLD, A. and LASHOF, R., *Quasifibrations and fibre homotopy equivalence of bundles*, Ill. J. Math. 3 (1959), 285–305.
- [11] GLEASON, A., *Spaces with a compact lie group of transformations*, Proc. Amer. Math. Soc. 1 (1950), 35–43.
- [12] JAMES, I. M. *Spaces associated with Stiefel manifolds*, Proc. London Math. Soc. 9 (1959), 115–140.
- [13] —, *The space of bundle maps*, Topology 2 (1963), 45–59.
- [14] —, *Bundles with special structure – I*, Ann. Math. 89 (1969), 359–390.
- [15] —, *Ex-homotopy theory – I*, Ill. Jour. Math. 15 (1971), 324–337.
- [16] KAHN, D. S. and PRIDDY, S. B., *Applications of the transfer to stable homotopy theory*, to appear.
- [17] LEE, R., *The spherical space form problem*, Topology 12 (1973).
- [18] MAY, J. P., *The geometry of iterated loop spaces*, Lecture Notes in Mathematics 271, Springer-Verlag, New York, 1972.
- [19] MILNOR, J. *On spaces having the homotopy type of a CW-complex*, Trans. Amer. Math. Soc. 90 (1959), 272–280.
- [20] —, *Lectures on characteristic classes*, mimeographed, Princeton University, 1957.
- [21] MOSHER, R. and TANGORA, M., *Cohomology operations and applications in homotopy theory*, Harper and Row, New York, 1968.
- [22] PETRIE, T., *Free metacyclic group actions on spheres*, Ann. of Math. 94 (1971), 108–124.
- [23] ROUSH, F. W., *Transfer in generalized cohomology theories*, Thesis, Princeton University, 1971.
- [24] SCHULTZ, R. E., *Homotopy decompositions of equivariant function spaces – I*, Math. Zeitschrift 131 (1973), 49–75; *ibid.* II, 132 (1973), 69–90.
- [25] SEGAL, G., *Equivariant stable homotopy theory*, Actes, Congrès internat. math. (Nice, 1970) T. 2, 59–63. Gauthier-Villars, Paris, 1971.
- [26] SPANIER, E. H., *Algebraic topology*, McGraw-Hill, New York, 1966.
- [27] STASHEFF, J., *A classification theorem for fibre spaces*, Topology 2 (1963), 239–246.
- [28] SZCZARBA, R., *On tangent bundles of fibre spaces and quotient spaces*, Amer. J. Math. 86 (1964), 685–697.
- [29] TOM DIECK, TAMMO, *Faserbündel mit Gruppenoperation*, Arch. Math. (Basel) 20 (1969), 136–143.
- [30] TSUCHIYA, A., *Characteristic classes for spherical fiber spaces*, Nagoya Math. J. 43 (1971), 1–39.
- [31] VOGT, R., *Boardman's stable homotopy category*, mimeographed, Aarhus University, 1970.
- [32] WHITEHEAD, G. W., *On products in homotopy groups*, Ann. of Math. 47 (1946), 460–475.
- [33] WU WEN-TSÜN, *On Pontrjagin classes – II* (Chinese), Acta Math. Sinica 4 (1954), 171–199. English translation, Amer. Math. Soc. Translations (2) 92 (1970), 63–92.
- [34] YANG, C. T., *The triangulability of the orbit space of a differentiable transformation group*, Bull., Amer. Math. Soc. 69 (1963) 405–408.

- [35] SCHULTZ, R., *Homotopy sphere pairs admitting semifree differentiable actions*, Amer. J. Math., to appear.
- [36] ———, *Homotopy-theoretic restrictions for smooth actions on exotic spheres*, to appear.

Received February 13/August 15, 1973.

*Purdue University
Division of Math. Sci.
Lafayette, Ind.*