

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 50 (1975)

**Artikel:** Commutators of Diffeomorphisms: II.  
**Autor:** Mather, John N.  
**DOI:** <https://doi.org/10.5169/seals-38790>

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## Commutators of Diffeomorphisms: II<sup>1)</sup>

by JOHN N. MATHER

This paper is a sequel to [2], and we will assume the reader is familiar with the terminology and results of [2]. Our main result is the following.

**THEOREM 1.** *Let  $M$  be a smooth  $n$ -manifold. If  $n \geq r \geq 1$ , then  $\text{Diff}(M, r)$  is perfect.*

From Epstein's theorem [1], it then follows that we have:

**COROLLARY.** *Under the same hypothesis,  $\text{Diff}(M, r)$  is simple.*

Combining this with the result in [2], we see that the only missing differentiability class is  $r = n + 1$ . For  $r = n + 1$ , it is still unknown whether Theorem 1 holds.

Theorem 1 is an immediate consequence of the case when  $M = \mathbf{R}^n$ . The proof of Theorem 1 is very closely related to the proof of Theorem 1 in [2]. In a sense, we have turned the proof of the latter upside down.

In the final section, we give a result concerning the connectivity of Haefliger's classifying space as an application of our method. This is analogous to the result concerning the connectivity of Haefliger's classifying space we obtained in [2], but for low differentiability, rather than high differentiability.

### §1. A Refinement of Theorem 1

We will actually prove a refinement of Theorem 1. Let  $\alpha$  be a modulus of continuity, and  $r$  a positive integer.

**THEOREM 2.** *Suppose either of the following holds.*

- a)  $n > r \geq 1$
- b)  $r = n$ ,  $\alpha$  is defined on all of  $[0, \infty)$ , and there exists  $\beta$ , with  $0 < \beta < 1$ , such that  $\alpha(tx) \leq t^\beta \alpha(x)$  for all  $x \geq 0$  and all  $t \geq 1$ .

*Then  $\text{Diff}(M, r, \alpha)$  is perfect.*

Theorem 1 is an immediate consequence, since  $\text{Diff}(M, r) = \bigcup_{\alpha} \text{Diff}(M, r, \alpha)$ , where the union is taken over all moduli of continuity in the case  $r < n$ , and all moduli of continuity satisfying the supplementary condition (b), when  $r = n$ .

Of course, it is enough to prove Theorem 2 in the case  $M = \mathbf{R}^n$ , to obtain the

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<sup>1</sup> This research was partially supported by NSF grant GP 43613X.

general result. We reduce the proof in §2 to the construction of certain mappings  $P_{i,A}$ . These mappings are constructed in subsequent sections.

## §2. Strategy of the Proof

The main technical step is the construction of certain mappings  $P_{i,A}$  of function spaces, and the proof of a number of properties of the  $P_{i,A}$ . The domain of  $P_{i,A}$  is a  $C^1$  neighborhood of the identity in the space of  $C^1$  diffeomorphisms of  $\mathbf{R}^n$  with support in  $\text{int } D_{i,A}$ . The range of  $P_{i,A}$  is the set of  $C^1$  diffeomorphisms of  $\mathbf{R}^n$  with support in  $\text{int } D_{i-1,A}$ .

There is a close parallel between the construction of  $P_{i,A}$  which we will give below and the construction of  $\Psi_{i,A}$  which we already gave in [2]. Note, however, that there is already a difference: we have reversed the domain and range.

Now we list the properties we will show  $P_{i,A}$  to have.

*Properties of  $P_{i,A}$ .*

- 1)  $P_{i,A}(\text{id}) = \text{id}$ ,
- 2) If  $u$  is in  $C^{r,\alpha}$ , then so is  $P_{i,A}(u)$ .
- 3) The restriction of  $P_{i,A}$  to the set of  $C^r$  diffeomorphisms in its domain is continuous with respect to the  $C^r$  topologies on its domain and range.
- 4) If  $u$  is in the domain of  $P_{i,A}$ , then  $u$  is isotopic to the identity through an isotopy with support in  $\text{int } D_{i,A}$  and  $P_{i,A}(u)$  is isotopic to the identity through an isotopy with support in  $\text{int } D_{i-1,A}$ .

From 4), if  $u \in \text{dom } P_{i,A}$  and  $u$  is  $C^{r,\alpha}$ , then  $u, P_{i,A}(u) \in \text{Diff}(\mathbf{R}^n, r, \alpha)$ .

- 5) If  $u \in \text{dom } P_{i,A}$  and  $u$  is  $C^{r,\alpha}$  then  $[u] = [P_{i,A}(u)]$  in the commutator quotient group of  $\text{Diff}(\mathbf{R}^n, r, \alpha)$ .

- 6) There exists  $\delta > 0$ ,  $C > 0$  such that

$$\mu_{r,\alpha}(P_{i,A}(u)) \leq CA^{-1} \mu_{r,\alpha}(u),$$

if  $u$  is of class  $C^{r,\alpha}$ , lies in the domain of  $P_{i,A}$ , and satisfies  $\mu_{r,\alpha}(u) < \delta$ . Moreover,  $C$  is independent of  $A$ .

The estimate for  $P_{i,A}$  given in 6) is in a sense the “inverse” of the estimate for  $\Psi_{i,A}$  given in §3, (6) of [2].

In the rest of this section, we finish the proof of the Theorem 2, assuming the existence of  $P_{i,A}$  satisfying (1)–(6). Consider  $f \in \text{Diff}(\mathbf{R}^n, r, \alpha)$  with support in  $\text{int } D_n$ . We wish to show  $f$  is in the commutator subgroup if it is sufficiently close to  $\text{id}$ .

For any  $u \in \text{Diff}(\mathbf{R}^n, r, \alpha)$  with support in  $\text{int } D_{0,A}$  we try to define

$$u_0 = A^{-1} f u A, \quad u_1 = P_{n,A}(u_0), \quad u_2 = P_{n-1,A}(u_1), \dots, u_n = P_{1,A}(u_{n-1}).$$

If  $u$  and  $f$  are sufficiently close to the identity in the  $C^1$  topology, these will actually be defined, by properties (1)–(3) of  $P_{i,A}$ .

It is easily seen that  $u_0$  is conjugate to  $fu$  in  $\text{Diff}(\mathbf{R}^n, r, \alpha)$ . Thus,  $[u_0] = [fu]$  in the commutator quotient group of  $\text{Diff}(\mathbf{R}^n, r, \alpha)$ . Then

$$[fu] = [u_n], \quad (*)$$

by (5) and the definition of the  $u_i$ .

**LEMMA.** *Suppose the hypotheses of Theorem 2 are satisfied. There exists  $A_0$  such that if  $A \geq A_0$ , then for  $\varepsilon > 0$  sufficiently small,  $\mu_{r,\alpha}(u) \leq \varepsilon$  and  $\mu_{r,\alpha}(f) \leq \varepsilon$  imply  $\mu_{r,\alpha}(u_n) \leq \varepsilon$ .*

Assuming this lemma, one can prove Theorem 2 in exactly the same way as Theorem 2 in [2] was proved there. Since there is no change in this proof, we will say nothing further about it.

*Proof of the Lemma.* Exactly as in the proof of the lemma in §3 of [2], we have that if  $\varepsilon > 0$  is sufficiently small, and  $\mu_{r,\alpha}(f) < \varepsilon$ ,  $\mu_{r,\alpha}(u) < \varepsilon$ , then  $\mu_{r,\alpha}(fu) < 3\varepsilon$ .

From the definition of  $u_0$ , we have  $\mu_{r,\alpha}(u_0) \leq A^r \mu_{r,\alpha}(fu)$  and  $\mu_{r,\alpha}(u_0) \leq A^{r-1+\beta} \mu_{r,\alpha}(fu)$ , if  $\alpha$  is defined on all of  $[0, \infty)$  and  $\alpha(tx) \leq t^\beta \alpha(x)$ , for  $t \geq 1$ .

From condition (6) on the mappings  $P_{i,A}$ , and the definition of  $u_1, \dots, u_n$ , it follows that if  $\varepsilon > 0$  is sufficiently small, then  $\mu_{r,\alpha}(u_n) \leq 3C^n A^{r-n} \varepsilon$  and  $\mu_{r,\alpha}(u_n) \leq 3C^n A^{r-n-1+\beta} \varepsilon$ , if  $\alpha$  is defined on all of  $[0, \infty)$  and  $\alpha(tx) \leq t^\beta \alpha(x)$ , for  $t \geq 1$ .

Under the hypotheses of the lemma, the exponent of  $A$  is negative, so by taking  $A$  sufficiently large we may arrange that  $3C^n A^{r-n} < 1$  or  $3C^n A^{r-n-1+\beta} < 1$ , according to the case. In either case, we have  $\mu_{r,\alpha}(u_n) \leq \varepsilon$ . Q.E.D.

### §3. Construction of the Mappings $P_{i,A}$

We consider the problem: given  $u$  with support in  $\text{int } D_{i,A}$ , find  $v$  with support in  $\text{int } D_{i-1,A}$  such that  $\tau_i u$  and  $\tau_i v$  are conjugate. We also wish  $v$  to satisfy an estimate of the form

$$\mu_{r,\alpha}(v) \leq C A^{-1} \mu_{r,\alpha}(u) \quad (1)$$

where  $C$  is a constant (independent of  $A$ ). This estimate should be satisfied for  $u$  in a neighborhood of the identity. (This neighborhood may depend on  $A$ .)

Our method is similar to the method of [2, §5], where we solved the “inverse” problem. In particular, since  $\text{supp } u \subset \text{int } D_{i,A} \subset \text{int } D_{i-1,A}$ , we may and do construct  $h$  in the same way as there, provided  $u$  is sufficiently close to the identity. Let  $\tilde{h}$  denote the unique diffeomorphism of  $\mathbf{R}^n$  such that  $\pi \tilde{h} = h \pi$  and  $\tilde{h}$  is the identity on  $\{x_i = 0\}$ , where  $\pi$  denotes the projection of  $\mathbf{R}^n$  on  $C_i$ .



We let  $B$  be a positive integer, which will be specified below. If  $h$  is sufficiently  $C^1$  close to the identity,  $\tilde{h}$  will be close enough to the identity for us to define diffeomorphisms  $\tilde{h}_1, \dots, \tilde{h}_B$  of  $\mathbf{R}^n$  by the formulae

$$\tilde{h}_1(x) - x = \frac{1}{B} (\tilde{h}(x) - x)$$

$$\tilde{h}_2(x) - x = \frac{1}{B-1} (\tilde{h} \circ \tilde{h}_1^{-1}(x) - x)$$

$$\tilde{h}_3(x) - x = \frac{1}{B-2} (\tilde{h} \circ \tilde{h}_1^{-1} \circ \tilde{h}_2^{-1}(x) - x)$$

...

$$\tilde{h}_{B-1}(x) - x = \frac{1}{2} (\tilde{h} \circ \tilde{h}_1^{-1} \circ \dots \circ \tilde{h}_{B-2}^{-1}(x) - x)$$

$$\tilde{h}_B(x) - x = \tilde{h} \circ \tilde{h}_1^{-1} \circ \dots \circ \tilde{h}_{B-1}^{-1}(x) - x.$$

Then  $\tilde{h} = \tilde{h}_B \circ \tilde{h}_{B-1} \circ \dots \circ \tilde{h}_1$ .

Let  $\zeta$  be a real  $C^\infty$  function on  $\mathbf{R}$ , periodic of period 1, equal to 1 in a neighborhood of the integers, equal to zero in a neighborhood of the half-integers, and satisfying  $0 \leq \zeta \leq 1$  everywhere.

For  $j=1, \dots, B$ , and  $k=0, 1$ , we define  $\tilde{h}_{jk}$  (for  $k=0, 1$ ) by the formulae  $\tilde{h}_{j0}(x) - x = \zeta(x_i)(\tilde{h}_j(x) - x)$  where  $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ , and  $\tilde{h}_{j1} = \tilde{h}_j \tilde{h}_{j0}^{-1}$ . These will be diffeomorphisms if  $h$  is  $C^1$  close to id. We have  $\tilde{h}_j = \tilde{h}_{j1} \tilde{h}_{j0}$  and  $\tilde{h} = \tilde{h}_{B1} \tilde{h}_{B0} \tilde{h}_{B-1,1} \tilde{h}_{B-1,0} \dots \tilde{h}_{1,1} \tilde{h}_{1,0}$ .

This is the decomposition of  $\tilde{h}$  we need in order to construct  $v$ .

**CONSTRUCTION of  $v$ .** We first construct two sequences  $E_-^1, \dots, E_-^B, E_+^1, \dots, E_+^B$  of strips in  $\mathbf{R}^n$ . These are defined as follows. Let  $a$  be the least half integer  $\geq -2A$ . Let

$$E_-^j = \{x \in \mathbf{R}^n : a + 3j - 3 \leq x_i \leq a + 3j - 2\}$$

$$E_+^j = \{x \in \mathbf{R}^n : a + 3j - 3/2 \leq x_i \leq a + 3j - 1/2\}.$$

We let  $B$  be the greatest integer such that  $a + 3B - 1/2 \leq 2A$ .

In terms of increasing values of  $x_i$ , the strips occur in the following order:  $E_-^1, E_+^1, E_-^2, E_+^2, \dots, E_-^B, E_+^B$ . Moreover they are disjoint, they all lie in the set  $\{-2A \leq x_i \leq 2A\}$ , and their sides are defined either by  $x_i = \text{half-integer}$  (the  $E_-^j$ ) or  $x_i = \text{integer}$  (the  $E_+^j$ ). They are squeezed as closely together as possible, compatibly with these properties.

We let  $v|_{E_-^j}$  be the unique diffeomorphism of  $E_-^j$  onto itself such that  $v|_{E_-^j} = \tilde{h}_{j0}|_{E_-^j}$ . We let  $v|_{E_+^j}$  be the unique diffeomorphism of  $E_+^j$  onto itself such that  $v|_{E_+^j} = \tilde{h}_{j1}|_{E_+^j}$ . We let

$$v|_{(\mathbf{R}^n - \bigcup_j (E_-^j \cap E_+^j))} = \text{id}.$$

If  $u$  is sufficiently close to the identity this is a well defined diffeomorphism of  $\mathbf{R}^n$ .

Moreover, it is easily seen that  $\Gamma_v^i = h$ . Hence  $\Gamma_v^i (\Gamma_u^i)^{-1} \in \mathcal{G}$ , and it follows from the lemma in [2, §4] that  $\tau_i u$  and  $\tau_i v$  are conjugate in  $\text{Diff}(\mathbf{R}^n, r, \alpha)$ , if  $u$  is in  $\text{Diff}(\mathbf{R}^n, r, \alpha)$  and sufficiently  $C^1$  close to the identity.

This completes the construction of  $v$ . Our estimate (1) will be proved in the next section.

We set  $P_{i,A}(u) = v$ . Of the properties (1)–(6) of  $P_{i,A}$  listed in §2, properties (1)–(3) are obvious, we may arrange for (4) to hold by replacing the domain of  $P_{i,A}$  by a smaller neighborhood of  $\text{id}$ , (5) is a consequence of the fact that  $\tau_i u$  and  $\tau_i v$  are conjugate, and (6) is equivalent to the inequality (1) of this section, which will be proved in the next section.

#### §4. Estimate for $v$

In this section, we will complete the proof of Theorem 2 by proving the estimate (1) in §3. The proof is based on the following five estimates.

(1) If  $u$  is  $C^{r,\alpha}$  and sufficiently near the identity, then

$$\mu_{r,\alpha}(\Gamma_u) \leq 8\mu_{r,\alpha}(u).$$

(2) If  $u$  is  $C^{r,\alpha}$  and sufficiently near the identity, then

$$\mu_{r,\alpha}(h) \leq 3\mu_{r,\alpha}(\Gamma_u).$$

(3) There exists a constant  $C_1 > 0$ , independent of  $A$ , such that if  $u$  is  $C^{r,\alpha}$  and sufficiently near the identity, then

$$\mu_{r,\alpha}(\tilde{h}_i) \leq C_1 A^{-1} \mu_{r,\alpha}(h), \quad i = 1, \dots, B.$$

(4) There exists a constant  $C_2$ , independent of  $A$ , such that if  $u$  is  $C^{r,\alpha}$  and sufficiently close to the identity, then

$$\mu_{r,\alpha}(\tilde{h}_{i,j}) \leq C_2 \mu_{r,\alpha}(\tilde{h}_i), \quad j = 0, 1$$

(5) We have

$$\mu_{r,\alpha}(v) = \max \{ \mu_{r,\alpha}(\tilde{h}_{ij}) \}.$$

All but one of these estimates is obvious or is in [2, §6] in slightly different guise. Thus, estimate (1) is essentially a special case of (1) in [2, §6]. Here,  $\text{supp } u \subset \text{int } D_{i-1,A}$ , whereas there, we had only the weaker condition  $\text{supp } u \subset \text{int } D_{i,A}$ . This explains why we may omit  $A$  from the right side of the inequality here: the width of  $D_{i-1,A}$  in the  $i$ th coordinate is 4, while the width of  $D_{i,A}$  is  $4A$ . The proof of (1) here is exactly the same as the proof of (1) in [2, §6].

Estimate (2) is exactly the same as (2) in [2, §6].

Estimate (3) is the new result. It will be proved below.

Estimate (4) is proved by the same argument that was used in Step 3 in [2, §6]. The equation (5) is obvious from the definitions.

From the estimates (1)–(5) of this section, we get that (1) of the previous section holds, with  $C = 24C_1C_2$ .

The estimate (3) is a consequence of the following lemma. Let  $\mathcal{A}$  denote the group of  $C^1$  diffeomorphisms  $U$  of  $\mathbf{R}^n$  such that  $U(x) = x$  if  $|x_j| \geq 2A$  for some  $j \neq i$  or  $x_i$  is an integer. For example,  $\tilde{h}$  and the  $\tilde{h}_i$  are in  $\mathcal{A}$ .

LEMMA. *Let  $0 < \lambda < 1$ . For any  $U \in \mathcal{A}$ , define  $V$  by*

$$V(x) - x = \lambda(U(x) - x)$$

*and let  $W = UV^{-1}$  (provided  $V^{-1}$  exists, which is the case when  $U$  is sufficiently close to the identity). Then there exists  $\delta > 0$  (small) and  $C > 0$  (large) such that for any  $U \in \mathcal{A}$  satisfying  $\mu_{r,\alpha}(U) < \delta$ , we have*

$$\mu_{r,\alpha}(W) \leq (1 - \lambda) \mu_{r,\alpha}(U) + C \mu_{r,\alpha}(U)^2. \quad (*)$$

We will prove this lemma below. First, however, we prove (3), assuming the lemma. Clearly

$$\mu_{r,\alpha}(\tilde{h}_1) = \frac{1}{B} \mu_{r,\alpha}(\tilde{h}) = \frac{1}{B} \mu_{r,\alpha}(h).$$

Applying the lemma with  $U = \tilde{h}$  and  $\lambda = 1/B$ , we get

$$\mu_{r,\alpha}(\tilde{h} \circ \tilde{h}_1^{-1}) \leq \frac{B-1}{B} \mu_{r,\alpha}(h) + O(\mu_{r,\alpha}(h)^2).$$

Then, it is clear that

$$\mu_{r,\alpha}(\tilde{h}_2) \leq \frac{1}{B} \mu_{r,\alpha}(h) + O(\mu_{r,\alpha}(h)^2).$$

Applying the lemma a second time, with  $U = \tilde{h} \circ \tilde{h}_1^{-1}$  and  $\lambda = (B-1)^{-1}$ , we get

$$\begin{aligned} \mu_{r,\alpha}(\tilde{h} \circ \tilde{h}_1^{-1} \circ \tilde{h}_2^{-1}) &\leq \frac{B-2}{B-1} \mu_{r,\alpha}(\tilde{h} \circ \tilde{h}_1^{-1}) + O(\mu_{r,\alpha}(\tilde{h} \circ \tilde{h}_1^{-1})^2) \\ &\leq \frac{B-2}{B} \mu_{r,\alpha}(h) + O(\mu_{r,\alpha}(h)^2). \end{aligned}$$

Then, it is clear that  $\mu_{r,\alpha}(\tilde{h}_3) \leq (1/B) \mu_{r,\alpha}(h) + O(\mu_{r,\alpha}(h)^2)$ .

Continuing in this way, we see that we have

$$\mu_{r,\alpha}(\tilde{h}_i) \leq \frac{1}{B} \mu_{r,\alpha}(h) + O(\mu_{r,\alpha}(h)^2)$$

for  $i=1, 2, \dots, B$ . However, it is clear from the definition of  $B$  that  $B \geq \frac{1}{2}A$ , so we get the estimate (3).

*Proof of the Lemma.* We give different proofs depending on whether  $r=1$  or  $r>1$ . If  $f$  is a mapping of  $\mathbf{R}^n$  into itself, we define

$$v_r(f) = \sup_{x \in \mathbf{R}^n} \|D^r f(x)\|.$$

*Case  $r>1$ .* We write  $R_{f,g}$  for the sum of the “other terms” in formula (2) in §2 of [2]. Thus,

$$D^r(f \circ g) = (D^r f \circ g) \cdot (Dg)^r + (Df \circ g) \cdot D^r g + R_{f,g}. \quad (6)$$

If  $f$  and  $g$  are  $C^r$ , then  $R_{f,g}$  is  $C^1$  and there exists  $\delta>0$  and  $C>0$  such that if  $f, g \in \mathcal{A}$  and  $v_r(f), v_r(g) < \delta$ , then

$$v_1(R_{f,g}) \leq C v_r(f) v_r(g).$$

Applying (6) to  $g=f^{-1}$ , we get

$$D^r(f^{-1}) = -((Df)^{-1} \circ f^{-1}) \cdot (D^r f \circ f^{-1}) \cdot ((Df)^{-1} \circ f^{-1})^r + R_f, \quad (7)$$

where there exists  $\delta>0$  and  $C>0$  such that if  $f \in \mathcal{A}$  and  $v_r(f) < \delta$ , then

$$v_1(R_f) \leq C v_r(f)^2.$$

Then

$$\begin{aligned} D^r W &= (D^r U \circ V^{-1}) \cdot (DV^{-1})^r + (DU \circ V^{-1}) \cdot D^r V^{-1} + R_{U,1} \\ &= (D^r U \circ V^{-1}) - \lambda(D^r U \circ V^{-1}) + S_U + R_{U,2} \end{aligned}$$

where

$$\begin{aligned} S_U &= (D^r U \circ V^{-1}) \cdot (DV^{-1})^r - D^r U \circ V^{-1} \\ &\quad + (D^r V \circ V^{-1} - ((DU \circ V^{-1}) \cdot ((DV)^{-1} \circ V^{-1}) \cdot (D^r V \circ V^{-1}) \cdot ((DV)^{-1} \circ V^{-1})^r)) \end{aligned}$$

and  $R_{U,i}$  ( $i=1, 2$ ) has the property that there exists  $\delta>0$  and  $C>0$  such that if  $U \in \mathcal{A}$  and  $v_r(U) < \delta$ , then  $v_1(R_{U,i}) < C v_r(U)^2$ .

Then it is easily seen that there exist  $\delta>0$  and  $C>0$ , such that if  $U \in \mathcal{A}$  and  $\mu_{r,\alpha}(U) < \delta$ , we have

$$\mu_\alpha(S_U + R_{U,2}) \leq C \mu_{r,\alpha}(U)^2.$$

The lemma (in the case  $r > 1$ ) follows immediately.

Case  $r = 1$ . We have

$$DW = (DU \circ V^{-1}) \cdot D(V^{-1}) = (DU \circ V^{-1}) \cdot ((DV)^{-1} \circ V^{-1}) = (DU \cdot (DV)^{-1}) \circ V^{-1}. \quad (8)$$

Then

$$\begin{aligned} (DV)^{-1} &= (I + (DV - I))^{-1} = I - (DV - I) + (DV - I)^2 - \dots \\ &= I - \lambda(DU - I) + \lambda^2(DU - I)^2 - \dots. \end{aligned}$$

Therefore

$$DU \cdot (DV)^{-1} = I + (1 - \lambda)(DU - I) + a_2(DU - I)^2 + a_3(DU - I)^3 + \dots \quad (9)$$

where  $a_2 z^2 + a_3 z^3 + \dots$  is a convergent power series (for  $|z| < \lambda^{-1}$ ). The lemma, in the case  $r = 1$ , follows immediately from formulas (8) and (9).

## §5. Application to Haefliger's Classifying Space

This is just like §7 of [2]. We get that  $FG_n^r$  is  $(n+1)$ -connected if  $n \geq r \geq 1$ . Likewise  $FG_n^{r, \alpha}$  is  $(n+1)$ -connected under the hypotheses of Theorem 2. These assertions follow in exactly the same way from the main results of this paper as the assertions in §7 of [2] followed from the main results of [2], so we will not repeat the proofs here.

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*Dept. of Mathematics*  
*Harvard University*  
*1 Oxford Street*  
*Cambridge, Mass. 02138*  
*U.S.A.*

Received June 18, 1974