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# The Hurwitz Matrix Equations and Multifour Groups 

Samuel S. H. Young ${ }^{1}$ )

## §0. Introduction

Let $\Omega_{s}$ be the set of $4^{s} s$-tuples $\left(\lambda_{1}, \ldots, \lambda_{s}\right)$ where each $\lambda_{i}$ stands for $e, \alpha, \beta$, or $\gamma$, the elements of Klein's four group $V_{4}$, which satisfy the following relations:

$$
\begin{array}{ll}
\alpha \circ \alpha=\beta \circ \beta=\gamma \circ \gamma=e, & \alpha \circ \beta=\beta \circ \alpha=\gamma, \\
\beta \circ \gamma=\gamma \circ \beta=\alpha, & \gamma \circ \alpha=\alpha \circ \gamma=\beta .
\end{array}
$$

If we define multiplication in $\Omega_{s}$ componentwise, i.e.,

$$
\left(\lambda_{1}, \ldots, \lambda_{s}\right) \circ\left(\mu_{1}, \ldots, \mu_{s}\right)=\left(v_{1}, \ldots, v_{s}\right)
$$

where $v_{i}$ is the group product $\lambda_{i} \circ \mu_{i}$ in $V_{4}$, then $\Omega_{s}$ becomes a commutative group which we denote by $G\left(\Omega_{s}\right)$. While $G\left(\Omega_{s}\right)$ is simply the direct product of $s$ copies of $V_{4}$, the properties of certain of its substructures turn out to be useful in the explicit construction of solutions of systems of matrix equations of the following form:

$$
\begin{equation*}
B_{h}^{2}= \pm I, \quad B_{h} B_{k} \pm B_{k} B_{h}=0, \quad B_{h} \pm B_{h}^{\prime}=0, \quad(h, k=1,2, \ldots ; h \neq k) \tag{*}
\end{equation*}
$$

where the unknown is a set of unspecified number of $n \times n$ matrices $\left(B_{1}, \ldots, B_{q}\right)$ with entries in a given field $F, B_{h}^{\prime}$ is the transpose of $B_{h}$, and each of the ambiguity signs can be + or - . When the signs of the last two equations are positive and that of the first equation negative, (*) reduces to the well known system of Hurwitz matrix equations, first proposed and solved in the complex field by A. Hurwitz in connection with his problem on the composition of quadratic forms [2]. Further investigations were made by Radon [5], Eckmann [1], Lee [3], Wong [7] and others and several far reaching results were obtained. The topic is still of current interest as can be seen in the recent works by Porteous [4] and Semple and Tyrell [6].

Geometrically, the system of Hurwitz matrix equations plays an important role in the study of isoclinic $n$-planes in Euclidean $2 n$-space and the Clifford-parallel $(n-1)$-planes in elliptic $(2 n-1)$-space, a work which was initiated by Y. C. Wong [7]. As to other systems of matrix equations of the form (*) aside from the Hurwitz

[^0]equations, it should be interesting to investigate their corresponding geometric meanings.

In this paper, we shall be concerned mainly with the properties of certain substructures of $G\left(\Omega_{s}\right)$. We shall indicate briefly how the results can be utilized to construct explicitly by means of matrix representations the maximal real solutions of the Hurwitz matrix equations thus verifying the original theorem due to Wong [7]. Our treatment is more elementary compared with the method given by Eckmann [1] and the employment of tools from the representation theory of finite groups is not needed. Using similar technique, the construction of solutions of (*) with signs arbitrarily chosen can also be achieved in many instances. We shall not go into details here.

## §1. Definitions and Basic Lemmas

DEFINITION 1.1. The direct product of $s$ copies of Klein's four group $V_{4}$, denoted by $G\left(\Omega_{s}\right)$, is called the multifour group of order $4^{s}$.

We shall use Roman capital letters to denote the elements in $\Omega_{s}$, thus we may write $L$ for $\left(\lambda_{1}, \ldots, \lambda_{s}\right)$ and $M$ for $\left(\mu_{1}, \ldots, \mu_{s}\right)$. In particular, we shall denote $(e, \ldots, e)$ in $\Omega_{s}$ by $I_{s}$. If $L=\left(\lambda_{1}, \ldots, \lambda_{h}\right)$ and $M=\left(\mu_{1}, \ldots, \mu_{k}\right)$ are elements in $\Omega_{h}$ and $\Omega_{k}$ respectively, then $N=\left(\lambda_{1}, \ldots, \lambda_{h}, \mu_{1}, \ldots, \mu_{k}\right) \in \Omega_{h+k}$ will be denoted by $(L, M)$. If $h=k$, then $\left(\lambda_{1} \circ \mu_{1}, \ldots, \lambda_{h} \circ \mu_{k}\right) \in \Omega_{h}$ will be denoted by $L \circ M$. Thus, $L^{2}=L \circ L=I_{s}$ for every $L \in \Omega_{s}$.

DEFINITION 1.2. Let $L=\left(\lambda_{1}, \ldots, \lambda_{s}\right)$ and $M=\left(\mu_{1}, \ldots, \mu_{s}\right)$ be elements in $\Omega_{s}$. The unordered pair $\left[\lambda_{i}, \mu_{i}\right]$ is called the $i$ th component pair of $L, M$.

DEFINITION 1.3. Two elements $L$ and $M$ in $\Omega_{s}$ are said to be odd-related (resp. even-related) if they have an odd (resp. even) number of component pairs which are of the forms $[\alpha, \beta],[\alpha, \gamma]$ and $[\beta, \gamma]$. We write $L \tau M$ or $L \pi M$ according as they are odd-related or even-related. More generally, two subsets $\Sigma_{1}, \Sigma_{2}$ of $\Omega_{s}$, one of which may be singleton, are said to be odd-related (resp. even-related) if each element of $\Sigma_{1}$ is odd-related (resp. even-related) to each element of $\Sigma_{2}$. Or in symbol, $\Sigma_{1} \tau \Sigma_{2}$ (resp. $\Sigma_{1} \pi \Sigma_{2}$ ).

LEMMA 1.4. For any $L, M, N \in \Omega_{s}$, we have
(i) $L \tau M \Leftrightarrow L \tau L \circ M, \quad L \pi M \Leftrightarrow L \pi L \circ M$;
(ii) $L \tau M, L \tau N \Rightarrow L \pi M \circ N$, $L \pi M, L \pi N \Rightarrow L \pi M \circ N$, $L \tau M, L \pi N \Rightarrow L \tau M \circ N$.
The proof of the above lemma is straight forward.
LEMMA 1.5. Let $L \neq I_{s}$ be an element in $\Omega_{s}$. Then $L$ is odd-related to $4^{s} / 2$ elements and even-related to the remaining $4^{s} / 2$ elements in $\Omega_{s}$.

Proof. Our lemma can be easily verified for $s=1$ and 2 .
Let us assume that our lemma holds for $\Omega_{s-1}, s>1$, and write $L=\left(H, \lambda_{s}\right)$ where $H \in \Omega_{s-1}$. Then by induction assumption, there exist $4^{s-1} / 2$ elements $H_{i} \in \Omega_{s-1}$ such that $H_{i} \tau H$ and $4^{s-1} / 2$ elements $K_{i} \in \Omega_{s-1}$ such that $K_{i} \pi H$, where $i=1, \ldots, 4^{s-1} / 2$.

Case 1. If $\lambda_{s}=e$, then for each $i,\left(H_{i}, \mu\right) \tau\left(H, \lambda_{s}\right)$ for any $\mu$ and $\left(K_{i}, v\right) \pi\left(H, \lambda_{s}\right)$ for any $v$. Hence, there are altogether $4 \cdot 4^{s-1} / 2=4^{s} / 2$ elements which are odd-related to $L$.

Case 2. If $\lambda_{s} \neq e$, we may take $\lambda_{s}=\alpha$ without loss of generality. Then $\left(H_{i}, \mu\right) \tau$ $\left(H, \lambda_{s}\right)$ if and only if $\mu=e$ or $\alpha$ and $\left(K_{i}, v\right) \tau\left(H, \lambda_{s}\right)$ if and only if $v=\beta$ or $\gamma$. Again, there are $4^{s-1}+4^{s-1}=4^{s} / 2$ elements in $\Omega_{s}$ which are odd-related to $L$.

In either case, the remaining $4^{s} / 2$ elements in $\Omega_{s}$ must be even-related to $L$.

## §2. The Group $\widetilde{G}(\Sigma)$

Let $\Sigma$ be a subset of elements in $\Omega_{s}$. We shall denote by $\langle\Sigma\rangle$ the subgroup generated by $\Sigma$ in $G\left(\Omega_{s}\right)$, and by $|\Sigma|$ the number of elements in $\Sigma$.

DEFINITION 2.1. A subset $\Sigma$ of $\Omega_{s}$ is called an independent set if for each $N \in \Sigma, N \notin\langle\Sigma \backslash N\rangle$. A singleton distinct from $I_{s}$ is considered as independent.

Let $\Sigma$ be an independent set in $\Omega$ with $|\Sigma|=t$. Then $|\langle\Sigma\rangle|=1+\sum_{r=1}^{t}\binom{t}{r}=2^{t}$, and it follows that $t \leqq 2 s$ since $\Omega_{s}$ has only $2^{2 s}$ elements.

Let us denote by $(\Sigma)_{e}$ the set of all elements in $\Omega_{s}$ which are even-related to each element of an independent set $\Sigma$. By Lemma 1.4 (ii), it is clear that $(\Sigma)_{e}$ is a subgroup of $G\left(\Omega_{s}\right)$. Since $G\left(\Omega_{s}\right)$ is a commutative group, $(\Sigma)_{e}$ is normal in $G\left(\Omega_{s}\right)$, and so we may form the quotient group $G\left(\Omega_{s}\right) /(\Sigma)_{e}$ which we shall denote by $\widetilde{G}(\Sigma)$.

LEMMA 2.2. Let $\Sigma=\{L, M\}$ be an independent set in $\Omega_{s}$. Then the group $\widetilde{G}(\Sigma)$ is isomorphic to $V_{4}$.

Proof. The subgroup $(\Sigma)_{e}$ of $G\left(\Omega_{s}\right)$ is the set $\{N: N \pi L, N \pi M\}$. Let us denote by $P \circ(\Sigma)_{e}$ the coset of $(\Sigma)_{e}$ in $G\left(\Omega_{s}\right)$ consisting of all elements $P \circ N$ with $N \in(\Sigma)_{e}$. We have two distinct cases:

Case 1. $L \tau M$. For any $S \in \Omega_{s}, S \in L \circ(\Sigma)_{e}\left(\right.$ resp. $\left.S \in M \circ(\Sigma)_{e}\right)$ if and only if $S \pi L$ and $S \tau M$ (resp. $S \pi M, S \tau L$ ) by Lemma 1.4. Furthermore, for any $T \in \Omega_{s}, T \in L \circ M \circ(\Sigma)_{e}$ if and only if $T \tau L$ and $T \tau M$. Since no element in $\Omega_{s}$ can belong to a coset distinct from $(\Sigma)_{e}, L \circ(\Sigma)_{e}, M \circ(\Sigma)_{e}$ and $L \circ M \circ(\Sigma)_{e}$, these are just the four cosets of $(\Sigma)_{e}$ in $G\left(\Omega_{s}\right)$. The representatives $\left\{I_{s}, L, M, L \circ M\right\}$ can be identified with $V_{4}$ and it follows that $\widetilde{G}(\Sigma)$ is isomorphic to $V_{4}$ in this case.

Case 2. $L \pi M$. The proof is similar.

LEMMA 2.3. Let $\Sigma$ be an independent set in $\Omega_{s}$ with $|\Sigma|=t \leqq 2 s$. For any partition of $\Sigma$ as the union of $\Sigma_{1}$ and $\Sigma_{2}$, one of which may be empty, there exists some $Q \in \Omega_{s}$ such that $Q \pi \Sigma_{1}$ and $Q \tau \Sigma_{2}$.

Proof. We prove by induction up to $t=2 s$.
For $t=1$ and 2, the lemma follows from Lemma 1.5 and the proof of Lemma 2.2. Let us assume that our lemma holds for all $\Sigma^{\prime}$ with $2 \leqq|\Sigma|<t \leqq 2 s$.

For any partition of $\Sigma$ in $\Omega_{s}$ as the disjoint union of $\Sigma_{1}$ and $\Sigma_{2}$, where we may assume that $\Sigma_{1}$ is non-empty, the partition gives rise to a partition of $\Sigma \backslash S$, where $S \in \Sigma_{1}$, as the disjoint union of $\Sigma_{1} \backslash S$ and $\Sigma_{2}$. By induction assumption, there exists $T \in \Omega_{s}$ such that $T \pi \Sigma_{1} \backslash S, T \tau \Sigma_{2}$. By Lemma 2.2, there exists $P \in \Omega_{s}$ such that $P \pi S$ and $P \pi T$. Then $Q=P \circ T \in \Omega_{s}$ is the element satisfying our lemma.

PROPOSITION 2.4. Let $\Sigma$ be an independent set in $\Omega_{s}$ with $|\Sigma|=t \leqq 2 s$. For any partition of $\Sigma$ as the union of $\Sigma_{1}$ and $\Sigma_{2}$, one of which may be empty, there exist $2^{2 s-t}$ elements in $\Omega_{s}$ each of which is even-related to $\Sigma_{1}$ and odd-related to $\Sigma_{2}$.

Proof. By Lemma 2.3, there exists $Q \in \Omega_{s}$ such that $Q \pi \Sigma_{1}$ and $Q \tau \Sigma_{2}$. Then $Q \circ(\Sigma)_{e}$ is the coset of $(\Sigma)_{e}$ in $G\left(\Omega_{s}\right)$ each element of which is related to $\Sigma_{1}$ and $\Sigma_{2}$ in the same manner as $Q$. Since $\Sigma$ can be partitioned in $2^{t}$ ways, we obtain $2^{t}$ cosets of $(\Sigma)_{e}$ in $G\left(\Omega_{s}\right)$ which are elements of $\widetilde{G}(\Sigma)$. Clearly, each element in $\Omega_{s}$ must belong to one of these cosets and it follows that each coset consists of $2^{2 s-t}$ elements.

COROLLARY 2.5. Let $\Sigma$ be an independent set in $\Omega_{s}$ with $|\Sigma|=2 s$. Then there exists exactly one element $P \in \Omega_{s}$ such that $P \tau \Sigma$ and there is no element in $\Omega_{s}$ distinct from $I_{s}$ which is even-related to $\Sigma$.

PROPOSITION 2.6. Let $\Sigma$ be an independent set in $\Omega_{s}$ with $|\Sigma|=t \leqq 2 s$. Then according as $t=2 k$ or $2 k+1$, the group $\widetilde{G}(\Sigma)$ is isomorphic to $G\left(\Omega_{k}\right)$ or $C_{2} \times G\left(\Omega_{k}\right)$, where $C_{2}$ denotes the cyclic group $\langle\delta\rangle$ of order 2.

Proof. Case 1. $t=2 k$. Let $\Sigma=\left\{S_{1}, T_{1}, \ldots, S_{k}, T_{k}\right\}$. A correspondence between the cosets of $(\Sigma)_{e}$ in $G\left(\Omega_{s}\right)$ (i.e., the elements of $\tilde{G}(\Sigma)$ ), and the elements of $G\left(\Omega_{k}\right)$ can be set up in the following manner: For any representative $Q$ of a given coset of $(\Sigma)_{e}$ in $G\left(\Omega_{s}\right)$, we let $Q \circ(\Sigma)_{e}$ correspond to the element $\left(v_{1}, \ldots, v_{k}\right)$ in $G\left(\Omega_{k}\right)$ where

$$
\begin{array}{lll}
v_{i}=e & \text { if } Q \pi S_{i} \text { and } Q \pi T_{i} \\
v_{i}=\alpha & \text { if } Q \pi S_{i} \text { and } Q \tau T_{i} \\
v_{i}=\beta & \text { if } Q \tau S_{i} \text { and } Q \pi T_{i}
\end{array}
$$

and

$$
v_{i}=\gamma \text { if } Q \tau S_{i} \text { and } Q \tau T_{i}
$$

That the above correspondence is in fact a group isomorphism is easily verified.

Case 2. $t=2 k+1$. Let $\Sigma=\left\{R, S_{1}, T_{1}, \ldots, S_{k}, T_{k}\right\}$ and let $P$ be a representative of any given coset of $(\Sigma)_{e}$ in $G\left(\Omega_{s}\right)$. If we let $P \circ(\Sigma)_{e}$ correspond to the element $\left(\varrho, v_{1}, \ldots, v_{k}\right)$ in $C_{2} \times G\left(\Omega_{k}\right)$ by setting $\varrho=e$ or $\delta$ according as $P \pi R$ or $P \tau R$, and $\nu=e, \alpha, \beta$, or $\gamma$ in the same way as in Case 1, then the correspondence is a group isomorphism between $\widetilde{G}(\Sigma)$ and $C_{2} \times G\left(\Omega_{k}\right)$.

## 83. 0-sets and $E$-sets in $\Omega_{s}$

DEFINITION 3.1. Let $\Sigma$ be an independent set in $\Omega_{s}$ and $P$ the product (meaning group product) of all the elements in $\Sigma$. Then $P \notin \Sigma$ and we call the set $\bar{\Sigma}=\Sigma \cup\{P\}$ (or simply $\Sigma \cup P$ ) a complete set in $\Omega_{s}$. (In the sequel, the symbol " $\cup$ " will denote disjoint union.)

It follows immediately from definition that if $\bar{\Sigma}$ is a complete set in $\Omega_{s}$, then the product of all the elements in $\bar{\Sigma}$ is equal to $I_{s}$ and every element in this set is the product of the remaining elements in the set.

DEFINITION 3.2. An independent or complete set consisting of two or more mutually odd-related (resp. even-related) elements in $\Omega_{s}$ is called an 0 -set (resp. $E$-set) in $\Omega_{s}$. An 0 -set (resp. $E$-set) in $\Omega_{s}$ is said to be maximal if it is not a proper subset of a larger 0 -set (resp. $E$-set) in $\Omega_{s}$.

PROPOSITION 3.3. An 0 -set in $\Omega_{s}$ is maximal if and only if it is complete.
Proof. Let $\bar{\Phi}=\Phi \cup P$ be a complete 0 -set in $\Omega_{s}$. If $|\Phi|=2 s$, then by Corollary 2.5, $P$ is the only element such that $P \tau \Phi$ and so $\Phi$ is maximal. If $|\Phi|<2 s$, then by Proposition 2.4, there exists some $Q \in \Omega_{s}$ distinct from $P$ such that $Q \tau \Phi$. Since $P \tau \Phi,|\Phi|$ must be even, and this implies that $Q \pi P$ by Lemma 1.4. Therefore, for any such $Q$, the enlarged set $\Phi \cup P \cup Q$ is not an 0 -set showing that $\bar{\Phi}$ is also maximal in this case.

On the other hand, let $\Phi$ be any 0 -set in $\Omega_{s}$ which is not complete. Since $\Phi$ is then an independent set, we have $|\Phi| \leqq 2 s$. By Lemma 2.3, there exists some $Q \in \Omega_{s}$ such that $\Phi \cup Q$ is an 0 -set. Hence, $\Phi$ is not maximal.

PROPOSITION 3.4. Let $\bar{\Phi}=\Phi \cup S \cup T$ be an 0 -set in $\Omega_{s}$ and $|\Phi \cup S \cup T|=2 k+1$ where $k \leqq s$. Then $\bar{\Phi}$ is a complete 0 -set if and only if $\widetilde{\mathcal{G}}(\Phi \cup S)$ and $\widetilde{\mathcal{G}}(\Phi \cup T)$ are identical.

Proof. If $\bar{\Phi}$ is complete, then $T$ can be expressed as the product of all the elements in $\Phi \cup S$ which are even in number. Hence, for any $Q \in \Omega_{s}, Q \pi \Phi \cup S$ if and only $Q \pi \Phi \cup T$. This means that $(\Phi \cup S)_{e}=(\Phi \cup T)_{e}$ and so the two groups $\tilde{G}(\Phi \cup S)$ and $\tilde{G}(\Phi \cup T)$ are identical.

To prove the converse, we observe that $T \circ(\Phi \cup S)_{e} \in \tilde{G}(\Phi \cup S)$ is odd-related to $\Phi \cup S$. Since the two given groups are identical, $T_{\circ}(\Phi \cup S)_{e}$ appears as $T_{\circ}(\Phi \cup T)_{e}$ in
$\tilde{G}(\Phi \cup T)$, and in $T \circ(\Phi \cup T)_{e}$, every element is even-related to $T$. It follows that $T$ is the only element in $T \circ(\Phi \cup S)$ which is odd-related to $\Phi \cup S$, showing that $\bar{\Phi}$ is maximal and hence complete.

PROPOSITION 3.5. In $\Omega_{s}$, there exist complete 0 -sets with $2 k+1$ elements for $k=1, \ldots, s$.

The proof is straight forward.
PROPOSITION 3.6. Let $\bar{\Phi}$ be a complete 0 -set in $\Omega_{s}$ with $2 s+1$ elements. For any $P \in \bar{\Phi}, \bar{\Phi} \backslash P$ is a set of generators of $G\left(\Omega_{s}\right)$.

Proof. Since $\langle\bar{\Phi} \backslash P\rangle$, the group generated by $\bar{\Phi} \backslash P$, is of order $4^{s}$, it must coincide with $G\left(\Omega_{s}\right)$.

Similar results concerning complete $E$-sets can be easily derived. We state without proof three propositions as follows.

PROPOSITION 3.7. An E-set in $\Omega_{s}$ is maximal if and only if it is complete.
PROPOSITION 3.8. In $\Omega_{s}$, there exist complete $E$-set with $k$ elements for $k=3,4, \ldots, s+1$.

PROPOSITION 3.9. Let $\bar{\Psi}$ be a complete $E$-set in $\Omega_{s}$ with $s+1$ elements. Then any $s$ elements in $\bar{\Psi}$ constitute a set of generators of the identity element $(\Psi)_{e}$ in $\widetilde{G}(\bar{\Psi} \backslash Q)$ where $Q$ is the element deleted from $\bar{\Psi}$.

## §4. Complete 0-sets which are Mutually Even-related and Complete E-sets which are Mutually Odd-related

PROPOSITION 4.1. For any positive integers $j, k$ such that $1 \leqq j, k<s$ and $j+k \leqq s$, there exist in $\Omega_{s}$ complete 0 -sets $\bar{\Phi}_{1}$ and $\bar{\Phi}_{2}$ with $2 j+1$ and $2 k+1$ elements such that $\bar{\Phi}_{1} \pi \bar{\Phi}_{2}$.

Proof. Since $j+k \leqslant s$, there exist 0 -sets $\Phi_{1}, \Phi_{2}$ with $2 j$ and $2 k$ elements such that $\Phi_{1} \cup \Phi_{2}$ is an independent 0 -set. Let $P$ (resp. $Q$ ) be the product of all the elements in $\Phi_{1}\left(\right.$ resp. $\Phi_{2}$ ). Then $\bar{\Phi}_{1}=\Phi_{1} \cup P$ and $\bar{\Phi}_{2}=P \circ \Phi_{2} \cup Q$ are two complete 0 -sets which are even-related.

PROPOSITION 4.2. For any positive integer $t$ such that $1<t \leqq s$, there exist in $\Omega_{s}$ 0 -sets $\bar{\Phi}_{1}, \ldots, \bar{\Phi}_{t}$ which are complete, disjoint, mutually even-related, and such that $\sum_{i=1}^{t}\left|\bar{\Phi}_{i}\right| \leqq 2 s+t$.

Proof. Let $\bar{\Phi}_{1}=\Phi_{1} \cup P_{1}$ where $P_{1}$ is any element in $\bar{\Phi}_{1}$, and let $\left|\Phi_{1}\right|=2 j$. By Proposition 4.1, there exists an independent 0 -set $\Phi$ with $2(s-j)$ elements such that $\Phi \pi \Phi_{1}$.

Since $\langle\Phi\rangle \subset\left(\Phi_{1}\right)_{e}$, the identity element of $\widetilde{G}\left(\Phi_{1}\right)$, we have $\langle\Phi\rangle=\left(\Phi_{1}\right)_{e}$ since they have the same number of elements.

The group $G\left(\Omega_{s-j}\right)$ is generated by an independent 0 -set $\Sigma_{1}$ with $2(s-j)$ elements by Proposition 3.6. A correspondence between $G\left(\Omega_{s-j}\right)$ and $\left(\Phi_{1}\right)_{e}$ can be set up by associating the elements in $\Sigma_{1}$, the generators of $G\left(\Omega_{s-j}\right)$, with those in $\Phi$, the generators of $\left(\Phi_{1}\right)_{e}$, in one-to-one manner. Clearly, this correspondence is a group isomorphism which preserves the odd and even relations.

Now let $\Sigma_{2}$ be an independent set with $2 k$ elements in $\Omega_{s-j}$ where $1 \leqq k<s-j$. Then there exists an independent 0 -set $\Sigma_{3}$ in $\Omega_{s-j}$ with $\left|\Sigma_{3}\right| \leqq 2(s-j-k)$ such that $\Sigma_{2} \pi \Sigma_{3}$. By the isomorphism given in the above paragraph, there exist 0 -sets $\Phi_{2}$ and $\Phi_{3}$ in $\left(\Phi_{1}\right)_{e}$ corresponding to $\Sigma_{2}$ and $\Sigma_{3}$ respectively such that $\Phi_{2} \pi \Phi_{3}$. Since $\Phi_{2}$ and $\Phi_{3}$ are in $\left(\Phi_{1}\right)_{e}$, it follows that $\Phi_{1}, \Phi_{2}$ and $\Phi_{3}$ are mutually even-related. Then $\bar{\Phi}_{1}, \bar{\Phi}_{2}$ and $\bar{\Phi}_{3}$ are complete 0 -sets which are mutually even-related and $\Sigma\left|\Phi_{i}\right| \leqq 2 s+3$.

Our proposition then follows by repeating the above arguments.
We state without proof the following propositions concerning complete $E$-sets in $\Omega_{s}$.

PROPOSITION 4.3. Let $\bar{\Psi}_{1}=\left\{L_{1}, L_{2}, P\right\}$, and $\bar{\Psi}_{2}=\Psi \cup P$ be two complete $E$-sets in $\Omega_{s}$. Then $\left\{L_{1}, L_{2}\right\} \tau \Psi$.

PROPOSITION 4.4. Let $\Psi_{1}$ be an independent $E$-set in $\Omega_{s}$ and $\left|\Psi_{1}\right|<s-1$. There exist E-sets $\Psi_{2}$ and $\Psi_{3}$ such that $\Psi_{1} \cup \Psi_{2} \cup \Psi_{3}$ is an independent set and the three sets are mutually odd-related.

## §5. Construction of Complete 0-Sets and Complete $E$-Sets in $\Omega_{s}$ from those in $\Omega_{s-1}$

Let $\Sigma=\left\{L_{1}, \ldots, L_{t}\right\}$ be an arbitrary set of elements in $\Omega_{s-1}$, where $s \geqslant 2$. For simplicity, we shall use the notation $\{\Sigma, \lambda\}$ to denote the set of elements $\left\{\left(L_{1}, \lambda\right), \ldots,\left(L_{i}, \lambda\right)\right\}$ in $\Omega_{s}$.

By virtue of the propositions given in §3 and §4, the following constructions can be achieved. The proofs are omitted.

PROPOSITION 5.1. The following are complete 0 -sets in $\Omega$ :
(i) $\left\{\left(I_{s-1}, \alpha\right),\left(I_{s-1}, \beta\right),\left(I_{s-1}, \gamma\right)\right\}$,
(ii) $\left\{\left(I_{s-1}, \alpha\right),(L, \beta),(L, \gamma)\right\} \quad$ for any $L \in \Omega_{s-1}$.

PROPOSITION 5.2. Let $\bar{\Phi}$ be a complete 0 -set in $\Omega_{s-1}$. Then the following are complete 0 -sets in $\Omega_{s}$ :
(i) $\{\bar{\Phi}, e\}$,
(ii) $\left\{\left(I_{s-1}, \alpha\right),\left(I_{s-1}, \beta\right),\{\bar{\Phi}, \gamma\}\right\}$.

PROPOSITION 5.3. Let $\bar{\Phi}_{1}, \bar{\Phi}_{2}, \bar{\Phi}_{3}$ be complete 0 -sets in $\Omega_{s-1}$ which are mutually even-related. Then the following are complete 0 -sets in $\Omega_{s}$ :
(i) $\left\{\left(I_{s-1}, \alpha\right),\left\{\bar{\Phi}_{1}, \beta\right\},\left\{\bar{\Phi}_{2}, \gamma\right\}\right\}$,
(ii) $\left\{\left\{\bar{\Phi}_{1}, \alpha\right\},\left\{\bar{\Phi}_{2}, \beta\right\},\left\{\bar{\Phi}_{3}, \gamma\right\}\right\}$.

PROPOSITION 5.4. Let $\Phi \cup \Phi^{\prime}$ be a complete 0 -set in $\Omega_{s-1}$ such that $|\Phi|$ is even. Then the following are complete 0 -sets in $\Omega_{s}$ :
(i) $\left\{\{\Phi, \alpha\},\left\{\Phi^{\prime}, e\right\}\right\}$
(ii) $\left\{\{\Phi, e\},(P, \alpha),(P, \beta),\left\{\Phi^{\prime}, \gamma\right\}\right\}$
(iii) $\{\{\Phi, e\},(P, \alpha),(P, \beta),(P, \gamma)\}$,
where $P$ is the product of all the elements in $\Phi^{\prime}$.
PROPOSITION 5.5. Let $\bar{\Psi}$ be a complete E-set in $\Omega_{s-1}$. For any partition of $\bar{\Psi}$ as the union of $\Psi_{1}$ and $\Psi_{2}$, where $\left|\Psi_{1}\right|$ is even and $\Psi_{2}$ may be singelton or empty, the following is a complete $E$-set in $\Omega_{s}$ :
$\left\{\left\{\Psi_{1}, \alpha\right\},\left\{\Psi_{2}, e\right\}\right\}$.
PROPOSITION 5.6. Let $\Psi_{1} \cup P$ and $\Psi_{2} \cup P$ be complete $E$-sets in $\Omega_{s-1}$ such that $\left|\Psi_{1}\right|$ and $\left|\Psi_{2}\right|$ are both even and $\Psi_{1} \tau \Psi_{2}$. Then the following is a complete $E$-set in $\Omega_{s}$ :
$\left\{\left\{\Psi_{1}, \alpha\right\},\left\{\Psi_{2}, \beta\right\}\right\}$.
In particular, if $\Psi_{1}=\{L, M\}$, the following $E$-set in $\Omega_{s}$ is complete:

$$
\left\{(L, \alpha),(M, \alpha),\left\{\Psi_{2}, \beta\right\}\right\}
$$

PROPOSITION 5.7. Let $\Psi_{1}, \Psi_{2}, \Psi_{3}$ be E-sets in $\Omega_{s-1}$ which are mutually oddrelated and such that their union is an independent set. Let $P$ be the product of all the elements in the three sets. If the number of elements in $\Psi_{i}$ are all odd or all even, then the following is a complete $E$-set in $\Omega_{s}$ :

$$
\left\{(P, e),\left\{\Psi_{1}, \alpha\right\},\left\{\Psi_{2}, \beta\right\},\left\{\Psi_{3}, \gamma\right\}\right\}
$$

Remark. In the propositions of this section, the last components $\alpha, \beta$ and $\gamma$ may be interchanged. Thus, by Proposition 5.3 (i), $\left\{\left(I_{s-1}, \beta\right),\left\{\bar{\Phi}_{1}, \alpha\right\},\left\{\Phi_{2}, \gamma\right\}\right\}$ and $\left\{\left(I_{s-1}, \gamma\right)\right.$, $\left.\left\{\bar{\Phi}_{1}, \alpha\right\},\left\{\bar{\Phi}_{2}, \beta\right\}\right\}$ are also complete 0 -sets in $\Omega_{s}$.

There are other possible constructions but they are not useful in connection with the solution of matrix equations.

## §6. Maximal 0-sets in $\Omega_{s}^{*}(\beta)$

Let $\Omega_{s}^{*}(\beta)$ be the subset of $\Omega_{s}$ defined by the following condition: $L=\left(\lambda_{1}, \ldots, \lambda_{s}\right)$
$\in \Omega_{s}^{*}(\beta)$ if and only if among the components $\lambda_{i}$ of $L$, the element $\beta$ in $V_{4}$ appears an odd number of times. Then $\Omega_{s} \backslash \Omega_{s}^{*}(\beta)$ is the subset of $\Omega_{s}$ consisting of those elements, among the components of each of which, $\beta$ appears an even number of times or does not appear at all. The determination of maximal 0 -sets and maximal $E$-sets with elements lying entirely in $\Omega_{s}^{*}(\beta)$ or entirely in $\Omega_{s} \backslash \Omega_{s}^{*}(\beta)$ are especially useful in applications. We shall confine our discussion to maximal 0 -sets in $\Omega_{s}^{*}(\beta)$.

LEMMA 6.1. If $L, M$ are both in $\Omega_{s}^{*}(\beta)$ or both in $\Omega_{s} \backslash \Omega_{s}^{*}(\beta)$ and $L \tau M$, then $L \circ M \in \Omega_{s}^{*}(\beta)$.

Proof. Since $L, M$ are both in $\Omega_{s}^{*}(\beta)$ or both in $\Omega_{s} \backslash \Omega_{s}^{*}(\beta)$, the total number of their component pairs of the forms $[e, \beta],[\alpha, \beta]$ and $[\gamma, \beta]$ is even. On the other hand, $L \tau M$ implies that the total number of their component pairs of the forms $[\alpha, \beta],[\gamma, \beta]$ and $[\alpha, \gamma]$ is odd. It follows that the total number of component pairs of the forms $[e, \beta]$ and $[\alpha, \gamma]$ is odd and consequently $L \circ M$ has an odd number of components equal to $\beta$. Our lemma is proved.

LEMMA 6.2. If $\Phi_{1}$ is an independent 0 -set in $\Omega_{s}^{*}(\beta)$ with $2 k$ elements, where $k<s$, then an element $N \in \Omega_{s}^{*}(\beta)$ can be chosen so that $N \pi \Phi_{1}$.

Proof. It suffices to consider the case when $k=s-1$. The subgroup $\left(\Phi_{1}\right)_{e}$ in $G\left(\Omega_{s}\right)$ is isomorphic to $V_{4}$ according to the proof given in Proposition 4.2. Let $\left(\Phi_{1}\right)_{e}$ $=\left\{I_{s}, L, M, N\right\}$. Then if $L, M \notin \Omega_{s}^{*}(\beta), N$ must be in $\Omega_{s}^{*}(\beta)$ by Lemma 6.1.

LEMMA 6.3. If there exists a maximal 0 -set $\Phi$ in $\Omega_{s}^{*}(\beta)$ with $2 k+1$ elements, where $k<s$, then there exists a maximal 0 -set in $\Omega_{s+2}^{*}(\beta)$ with $2 k+5$ elements.

Proof. If $k<s$, then there exists $N \in \Omega_{s}^{*}(\beta)$ such that $N \pi \Phi$, by Lemmas 6.1 and 6.2. Then $\left\{\left(I_{s}, \beta\right),(N, \alpha),(N, \gamma)\right\}$ is a maximal 0 -set in $\Omega_{s+1}^{*}(\beta)$ which is even-related to $\{\Phi, e\}$ in $\Omega_{s+1}^{*}(\beta)$. By Proposition 5.3 (i), the following is a maximal 0 -set in $\Omega_{s+2}^{*}(\beta)$ with $2 k+5$ elements:

$$
\left\{\left(I_{s+1}, \beta\right),\left(\left(I_{s}, \beta\right), \alpha\right),((N, \alpha), \alpha),((N, \gamma), \alpha),\{\{\Phi, e\}, \gamma\}\right\}
$$

PROPOSITION 6.4. In $\Omega_{s}^{*}(\beta), s \geqslant 2$, there exist maximal 0 -sets with $k$ elements, where $k=3,7, \ldots, 4[s / 2]-1$, and $[s / 2]$ denotes the greatest integer not exceeding $s / 2$.

Proof. There exist maximal 0 -sets in $\Omega_{2}^{*}(\beta)$ and $\Omega_{3}^{*}(\beta)$ with 3 elements, for instance, $\{(e, \beta),(\beta, \alpha),(\beta, \gamma)\}$ and $\{(e, e, \beta),(a, \beta, \alpha),(a, \beta, \gamma)\}$. The proposition then follows from Proposition 5.2 (i) and Lemma 6.3.

PROPOSITION 6.5. In $\Omega_{s}^{*}(\beta)$, where $s=4 k+3$, there exists a maximal 0 -set with $2 s+1$ elements.

Proof. In $\Omega_{2}^{*}(\beta), \Phi_{1}=\{(e, \beta),(\beta, \alpha),(\beta, \gamma)\}$ and $\Phi_{2}=\{(\beta, e),(\alpha, \beta),(\gamma, \beta)\}$ are the
only two maximal 0 -sets which are even-related. Hence, $\Phi_{3}=\left\{\left(I_{2}, \beta\right),\left\{\Phi_{1}, \alpha\right\},\left\{\Phi_{2}, \gamma\right\}\right\}$ is a maximal 0 -set in $\Omega_{3}^{*}(\beta)$ with 7 elements. Now $\left\{I_{3}, \Phi_{3}\right\} \pi\left\{\Phi_{3}, I_{3}\right\}$ in $\Omega_{6}^{*}(\beta)$, it follows that the following is a maximal 0 -set in $\Omega_{7}^{*}(\beta)$ with 15 elements:

$$
\Phi_{4}=\left\{\left(I_{6}, \beta\right),\left\{\left\{I_{3}, \Phi_{3}\right\}, \alpha\right\},\left\{\left\{\Phi_{3}, I_{3}\right\}, \gamma\right\}\right\}
$$

Our proposition is thus true for $k=0,1$.
Assume that our proposition holds for all $\Omega_{t}^{*}(\beta)$ with $t=4 h+3,0 \leqslant h<k$. We write $k=h_{1}+h_{2}+1, h_{1}, h_{2} \geqslant 0$. By induction assumption, there exist maximal 0 -sets $\Phi_{1}^{\prime} \in \Omega_{t_{1}}^{*}(\beta), \Phi_{2}^{\prime} \in \Omega_{t_{2}}^{*}(\beta)$, where $t_{i}=4 h_{i}+3$, such that $\left|\Phi_{i}^{\prime}\right|=2 t_{i}+1$. Then $\left\{I_{t_{1}}, \Phi_{2}^{\prime}\right\}$ $\pi\left\{\Phi_{1}^{\prime}, I_{t_{2}}\right\}$ in $\Omega_{t_{1}+t_{2}}^{*}(\beta)$, and so we can construct the following maximal 0 -set in $\Omega_{s}^{*}(\beta)$

$$
\left\{\left(I_{s-1}, \beta\right),\left\{\left\{I_{t_{1}}, \Phi_{2}^{\prime}\right\}, \alpha\right\},\left\{\left\{\Phi_{1}^{\prime}, I_{t_{2}}\right\}, \gamma\right\}\right\}
$$

which has $2\left(t_{1}+t_{2}\right)+3=2 s+1$ elements.
PROPOSITION 6.6. In $\Omega_{s}^{*}(\beta)$ where $s=4 k$, there exists a maximal 0 -set with $2 s$ elements which is not complete.

Proof. By Proposition 6.5, there exists a maximal 0 -set $\Phi$ in $\Omega_{s-1}^{*}(\beta)$ with $2 s-1$ elements. By Proposition 5.2 (ii), $\bar{\Phi}=\left\{\left(I_{s-1}, \alpha\right),\left(I_{s-1}, \beta\right),\{\Phi, \gamma\}\right\}$ is a complete 0 -set in $\Omega_{s}$ with $2 s+1$ elements, which after deleting the element $\left(I_{s-1}, \alpha\right) \notin \Omega_{s}^{*}(\beta)$, gives rise to a maximal 0 -set in $\Omega_{s}^{*}(\beta)$ with $2 s$ elements which is not complete.

Remark. If $\Phi \in \Omega_{s-1}^{*}(\beta)$ has less than $2 s-1$ elements, then $\left\{\left(I_{s-1}, \beta\right),\{\Phi, \gamma\}\right\}$ is not maximal in $\Omega_{s}^{*}(\beta)$ because it is contained in the maximal0-set $\left\{\left(I_{s-1}, \beta\right),\{\Phi, \gamma\},\left\{\Phi^{\prime}, \alpha\right\}\right\}$ where $\Phi^{\prime}$ is another maximal 0 -set in $\Omega_{s-1}^{*}(\beta)$ such that $\Phi \pi \Phi^{\prime}$.

PROPOSITION 6.7. In $\Omega_{s}^{*}(\beta)$, where $s=4 k+1$, there exists a maximal 0 -set with $2 s-1$ elements which is not complete.

Proof. By Proposition 6.6, $\left\{\left(I_{s-2}, \beta\right),\{\Phi, \gamma\}\right\}$ is a maximal 0 -set in $\Omega_{s-1}^{*}(\beta)$ with $2 s-2$ elements, where $\Phi$ is a maximal 0 -set in $\Omega_{s-2}^{*}(\beta)$ with $2 s-3$ elements. Then the following 0 -set in $\Omega_{s}^{*}(\beta)$ has $2 s-1$ elements:

$$
\left\{\left(I_{s-1}, \beta\right),\left(\left(I_{s-2}, \beta\right), \alpha\right),\{\{\Phi, \gamma\}, \alpha\}\right\}
$$

Clearly, this 0 -set is maximal but not complete.

## §7. The Associative Algebra $A\left(\Omega_{s}\right)$ and its Matrix Representation

We may consider the $4^{s} s$-tuples $\left(\lambda_{1}, \ldots, \lambda_{s}\right)$ in $\Omega_{s}$ as the base elements of a vector space over a field $F$. If we define the product of two base elements $L$ and $M$, denoted by $L \cdot M$, to be their group product $L \circ M$ in $G\left(\Omega_{s}\right)$ multiplied by a structure
constant $C_{L, M} \in F$, where $C_{L, M}$ need not equal to $C_{M, L}$, this vector space becomes an algebra over $F$. When $F$ is the complex field, it is convenient to choose the structure constants as follows: For each pair of elements $L$ and $M$ not equal to $I_{s}$, we set $C_{L, M}=(\sqrt{-1})^{j+k}$ (or $-(\sqrt{-1})^{j+k}$ ), where $j$ is the number of component pairs of $L, M$ which are of the forms $[e, \beta]$ and $[\alpha, \gamma]$, and $k=0$ if exactly one of $L, M$ lies in $\Omega_{s}^{*}(\beta)$ and $k=1$ if otherwise. Once $C_{L, M}$ is fixed, we set $C_{M, L}=C_{L, M}$ or $-C_{L, M}$ according as $L \pi M$ or $L \tau M$. Also, we set $C_{I_{s}, I_{s}}=C_{I_{s}, L}=C_{L, I_{s}}=1$ and $C_{L, L}=-1$, where $L \neq I_{s}$. With proper choice of $\operatorname{sign}$ for $C_{L, M}$ for each pair $L, M$, we obtain an associative algebra over the complex field. The algebra so defined will be denoted by $A\left(\Omega_{s}\right)$ and it can be represented by matrices with entries in the complex field in the following manner.

For $s \geqslant h \geqslant 1$, we represent the elements $I_{h},\left(I_{h-1}, \alpha\right),\left(I_{h-1}, \beta\right)$ and $\left(I_{h-1}, \gamma\right)$ in $\Omega_{h}$ by

$$
\left(\begin{array}{cc}
J & \\
& J
\end{array}\right), \quad \sqrt{-1}\left(\begin{array}{cc}
J & \\
& -J
\end{array}\right), \quad\left(\begin{array}{cc} 
& J \\
-J &
\end{array}\right), \quad \text { and } \sqrt{ }-1\left(\begin{array}{ll} 
& J \\
J &
\end{array}\right)
$$

respectively, where $J$ stands for the identity matrix of order $2^{h-1} m$ ( $m$ odd). Note that when $h=1,\left(I_{h-1}, \alpha\right)$, etc. mean simply $\alpha$, etc. If $L\left(\neq I_{h}\right) \in \Omega_{h}$ is represented by a matrix $A$ of order $t$, we represent the elements $(L, e),(L, \alpha),(L, \beta)$ and $(L, \gamma)$ in $\Omega_{h+1}$ respectively the following matrices of order $2 t$ :

$$
\left(\begin{array}{cc}
A & \\
& A
\end{array}\right) \cdot \quad\left(\begin{array}{cc}
A & \\
& -A
\end{array}\right), \quad \sqrt{-1}\left(\begin{array}{ll} 
& A \\
-A &
\end{array}\right), \quad \text { and }\left(\begin{array}{ll} 
& A \\
A &
\end{array}\right)
$$

In this way, when any element $\left(\lambda_{1}, \ldots, \lambda_{s}\right) \in \Omega_{s}$ is given, we can start from its first component and construct step by step a matrix of order $2^{s} m$ which is its representation. Since the elements of $\Omega_{s}$ are the base elements of $A\left(\Omega_{s}\right)$, a faithful representation of this algebra is achieved.

DEFINITION 7.1. The $4^{s}$ matrices of order $n=2^{s} m$ ( $m$ odd) representing the elements of $\Omega_{s}$ obtained in the manner described above are called the basic matrices of order $n$.

## §8. Maximal Real Solutions of Hurwitz Matrix Equations

As an illustration of the application of our results, we proceed to show how the maximal real solutions of the following system of Hurwitz matrix equations can be constructed explicitly:

$$
\begin{align*}
& B_{h}^{2}=-I, \quad B_{h} B_{k}+B_{k} B_{h}=0, \quad B_{h}+B_{h}^{\prime}=0, \quad(h, k=1,2, \ldots ; h \neq k)  \tag{**}\\
& \quad \text { order of } \quad B_{h}=2^{s} m \quad(m \text { odd })
\end{align*}
$$

Let $B_{h}, B_{k}$ be basic matrices which are representations of two distinct elements $L_{h}, L_{k}$ in $\Omega_{s}$. It is clear that $L_{h} \tau L_{k}$ if and only if $B_{h} B_{k}=-B_{k} B_{h}$. If $L_{h} \neq I_{s}$, then the relation $B_{h}^{2}=-I$ is always satisfied. Furthermore, $B_{h}$ is real if and only if $L_{h}$ lies in $\Omega_{s}^{*}(\beta)$, in which case, we also have $B_{h}=-B_{h}^{\prime}$. From these observations, we conclude that if $\Sigma=\left\{B_{1}, \ldots, B_{q}\right\}$ is a set of basic matrices representing a maximal 0 -set in $\Omega_{s}^{*}(\beta)$, then $\Sigma$ is a maximal set of real solutions of (**), and conversely. Since the constructions as given in Propositions 6.4, 6.5, 6.6 and 6.7 exhaust all possible types of maximal 0 -sets in $\Omega_{s}^{*}(\beta)$, we are led to the following proposition discovered originally by Wong [7].

PROPOSITION 8.1. There exist $q$-dimensional maximal real solutions of Hurwitz matrix equations of order $2^{s} m$ ( $m$ odd) for the following values of $q$ and $s$ :

$$
\begin{array}{llll}
s \equiv 1(\bmod 4): & q=4 k+3, & k=0,1, \ldots,(s-3) / 2 ; & q=2 s-1 . \\
s \equiv 2(\bmod 4): & q=4 k+3, & k=0,1, \ldots,(s-2) / 2 . & \\
s \equiv 3(\bmod 4): & q=4 k+3, & k=0,1, \ldots,(s-1) / 2 . & \\
s \equiv 0(\bmod 4): & q=4 k+3, & k=0,1, \ldots,(s-2) / 2 ; & q=2 s .
\end{array}
$$

It was proved by Wong [7] that every set of maximal real solutions of (**) is unitary similar to what he called a maximal set of quasisolutions. Basing on this fact, we have the following result.

PROPOSITION 8.2. Every maximal real solution of Hurwitz matrix equations of order $2^{s} m$ ( $m$ odd) is orthogonally similar to a maximal solution consisting of real basic matrices which are representations of a maximal 0 -set in $\Omega_{s}^{*}(\beta)$.

We may consider maximal $E$-sets instead of maximal 0 -sets, or we may restrict such sets to other specified subsets of $\Omega_{s}$ such as $\Omega_{s} \backslash \Omega_{s}^{*}(\beta)$ instead of $\Omega_{s}^{*}(\beta)$, in order to construct real or complex solutions of (*) given in the Introduction with signs differ from those appear in (**). In some cases, the use of other types of matrix representations may be necessary. We leave the details to interested readers for their own investigation.

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