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Semisimple Algebras and a Cancellation Law

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Introduction

Let R be a commutative ring and Λ be an R-algebra which is finitely generated as R-module.[1]) For any ideal I of R the image of the canonical mapping $I \otimes \Lambda \to \Lambda$ is an ideal of Λ, denoted by $I\Lambda$ and called a scalar ideal. The algebra Λ is called an ideal algebra if $I \to I\Lambda$ defines a bijection from the ideals of R to the ideals of Λ [8]. Λ is called a semisimple R-algebra if any finitely generated Λ-module (left as well as right) is (Λ, R) projective in the sense of Hochschild's relative homology. A theorem of Endo-Watanabe [2] states that for central semisimple algebras the maximal ideals are of the form $m\Lambda$, where m is a maximal ideal of R. Since Azumaya algebras are semisimple and also ideal algebras, it is natural to ask whethei or not central, projective, semisimple algebras are ideal algebras. In [8], it was shown that this is indeed the case, if R is a Dedekind domain.

In this note, we obtain a class of ideals which must be scalar ideals. We show that any ideal of a central, projective, semisimple algebra, lying over a radical ideal, is a scalar ideal. If in addition R is noetherian, we prove that any ideal lying over an irreducible ideal, must necessarily be a scalar ideal.

Finally, we prove two types of Skolem-Noether theorems over semilocal rings, and deduce a "Cancellation Law" for a class of semisimple algebras. In particular, we show that if Λ_1 and Λ_2 are projective, central, semisimple algebras, then $M_n(\Lambda_1) \simeq M_n(\Lambda_2)$ implies $\Lambda_1 \simeq \Lambda_2$. Such "Cancellation Laws" for Azumaya algebras were studied by Knus [5], Ojanguren-Sridharan [6] and Roy-Sridharan [9].

For standard concepts and results regarding semisimple algebras and Azumaya algebras, we refer to Hattori [4], Endo-Watanabe [2, 3] and Bass [1].

1. Ideals of a Semisimple Algebra

We begin with a slight generalization of a result of Endo-Watanabe [2].

LEMMA 1.1. *Let (R, m) be a local ring such that every finitely generated ideal*

[1]) Throughout this paper we assume that all algebras are finitely generated as modules. We call an R-algebra projective if it is projective as R-modele. $\otimes$ will stand for $\otimes_R$. By an ideal, we always mean a two sided ideal.

has a non trivial annihilator. Let Λ be a central algebra over R with a unique maximal ideal $m\Lambda$. Then Λ is Azumaya if it is a generator as R-module.

Proof. Since $\Lambda/m\Lambda$ is simple, it suffices to show that R/m is the center of $\Lambda/m\Lambda$. Let $\bar{\lambda}$ be in the center of $\Lambda/m\Lambda$ and λ be a representative of $\bar{\lambda}$ in Λ. Then we have $\lambda\alpha-\alpha\lambda\in m\Lambda$ for all $\alpha\in\Lambda$. As Λ is finitely generated as R-module, it follows that $\lambda\alpha-\alpha\lambda\in I\Lambda$ for all $\alpha\in\Lambda$, where I is a finitely generated ideal of R. Let $0\neq s\in R$ such that $sI=0$. Then we have $s(\lambda\alpha-\alpha\lambda)\in sI\Lambda=0$ for all α and therefore $s\lambda\in R$.

As Λ is a generator as R-module, $\Lambda=R\oplus M$ for some R-module M. If $\lambda=r+v$, it is clear that $sv=0$. Therefore v generates a proper ideal in Λ and must necessarily be contained in $m\Lambda$. So we have $\bar{\lambda}=\bar{r}\in R/m$.

PROPOSITION 1.2. *Let R be either a ring of Krull dimension zero or a total quotient ring of a noetherian ring. If Λ is a projective, central, semisimple R-algebra, then it is Azumaya and in particular an ideal algebra.*

Proof. By [2], the maximal ideals of Λ are of the form $m\Lambda$, where m is a maximal ideal of R. Also every finitely generated ideal of R_m has a non trivial annihilator. By Lemma 1.1., Λ is locally Azumaya and therefore is an Azumaya algebra.

The following is easy to verify

LEMMA 1.3. *Let R_S be the total quotient ring of R. If Λ is a projective R-algebra such that any non zero ideal of Λ_S contains a non zero element of R_S, then every non zero ideal of Λ also contains a non zero element of R.*

PROPOSITION 1.4. *Let Λ be a projective, central, semisimple R-algebra. Any ideal of Λ, lying over a radical ideal of R, is a scalar ideal.*

Proof. First suppose that $\mathfrak{A}$ is any prime ideal of Λ. Let $\mathfrak{A}\cap R=P$. Since Λ_p is again an R_p-central semisimple algebra [4]. $\Lambda_p/P\Lambda_p$ is a simple ring. As $\Lambda/P\Lambda$ is R/P-projective and R_p/PR_p is the quotient field of R/P, by Lemma 1.3. every non zero ideal of $\Lambda/P\Lambda$ contains a non zero element of R/P. Thus the image of $\mathfrak{A}$ in $\Lambda/P\Lambda$ is zero, i.e. $\mathfrak{A}=P\Lambda$.

Now, let $\mathfrak{A}$ be an ideal of Λ such that $\mathfrak{A}\cap R=I$ is a radical ideal ($\sqrt{I}=I$). Let P be any prime ideal of R containing I. $\mathfrak{A}_p\cap R_p=I_p$ and hence, $\mathfrak{A}_p$ is a proper ideal of Λ_p. $P\Lambda_p$, being the unique maximal ideal of Λ_p, must contain $\mathfrak{A}_p$. Contracting to Λ, we have $\mathfrak{A}\subset P\Lambda$. Thus $\mathfrak{A}\subset\bigcap P\Lambda$ where the intersection is taken over all prime ideals containing I. Because Λ is R-projective, $\mathfrak{A}\subset\bigcap P\Lambda=(\bigcap P)\,\Lambda=I\Lambda$, the desired conclusion follows.

LEMMA 1.5. *Let (R, m) be a commutative quasi-Frobenius local ring. Λ be a projective algebra over R with a unique maximal ideal $m\Lambda$. Then Λ is also quasi-Frobenius. Furthermore every non zero ideal of Λ contains a non zero element of R.*

Proof. We have $\mathrm{Ext}_R^1(R/m, R)=0$. Λ, being finitely generated as R-module, is artinian. Also Λ is R-free. Therefore $\Lambda\otimes\mathrm{Ext}_R^1(R/m, R)\simeq\mathrm{Ext}_\Lambda^1(\Lambda/m\Lambda, \Lambda)=0$. $\Lambda/m\Lambda$ is an artinian simple ring and hence any two simple left Λ-modules are isomorphic. Moreover, $\Lambda/m\Lambda\simeq S\oplus\cdots\oplus S$ where S is a simple left Λ-module. Thus $\mathrm{Ext}_\Lambda^1(S, \Lambda)=0$. Therefore Λ is left Λ-injective [7].

To prove the second assertion, consider the annihilator of $m\Lambda$ in Λ. It is precisely the annihilator of m in Λ. Since Λ is R-projective, we have $(0:_\Lambda m)=(0:_R m)\,\Lambda=I\Lambda$, where I is the annihilator of m in R. As Λ is quasi-Frobenius, it satisfies the "annihilator condition". Therefore $I\Lambda$, the annihilator of the unique maximal ideal, must necessarily be the unique minimal ideal of Λ. Thus every non zero ideal of Λ contains at least I.

Finally, here is the main result of this section:

PROPOSITION 1.6. *Suppose R is a noetherian ring and let Λ be a projective, central, semisimple R-algebra. Then any ideal of Λ, lying over an irreducible ideal of R, is a scalar ideal.*

Proof. Let $\mathfrak{A}$ be an ideal of Λ such that $\mathfrak{A}\cap R=I$ is a P-irreducible ideal. It is enough to show that every non zero ideal of $\Lambda/I\Lambda$ contains a non zero element of R/I.

It is clear that IR_p is a PR_p-irreducible ideal of R_p and hence, R_p/IR_p is an artinian ring with a unique minimal ideal. Therefore it is quasi-Frobenius [Kaplanski's "commutative rings"]. Then the R_p/IR_p-algebra $\Lambda_p/I\Lambda_p$ satisfies all the conditions of Lemma 1.5. Thus every non zero ideal of $\Lambda_p/I\Lambda_p$ contains a non zero element of R_p/IR_p. Since R_p/IR_p is the total quotient ring of R/I, by Lemma 1.3 we have the desired conclusion.

Remark 1. It is known that if Λ is an R-ideal algebra, Λ is necessarily a faithfully flat R-module [8]. Since there exists non projective, central, semisimple algebras over artinian rings [3], not every central semisimple algebra is an ideal algebra.

Remark 2. In Lemma 1.5., we have not assumed that Λ is semisimple. Thus we have actually shown that, if $P\Lambda$ is a prime ideal of a projective R-algebra Λ, then any ideal of Λ, lying over a P-irreducible ideal of R, is necessarily a scalar ideal.

2. $\otimes$ with an Azumaya Algebra

LEMMA 2.1. *Let A be an Azumaya algebra over R. Then there exists an Azumaya R-algebra B such that $A\otimes B\simeq M_n(R)$.*

Proof. Since A is a faithfully projective R-module, there exists a faithfully projective R-module Q, such that $A\otimes Q\simeq R^n$ for some n [1]. Setting $B=A^0\otimes\mathrm{End}_R(Q, Q)$, we have $A\otimes B\simeq M_n(R)$.

LEMMA 2.2. *Let Λ and A be R-algebras (not necessarily finitely generated) and also assume that A is faithfully flat as R-module. Then $(\mathfrak{A}\otimes A)\cap\Lambda=\mathfrak{A}$ for any ideal $\mathfrak{A}$* of Λ.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathfrak{A} & \to & \Lambda & \to & \Lambda/\mathfrak{A} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathfrak{A}\otimes A & \to & \Lambda\otimes A & \to & \Lambda/\mathfrak{A}\otimes A & \to 0 \end{array}$$

The lower row is exact and the vertical maps $(a \to a\otimes 1)$ are monomorphisms because A is faithfully flat. The assertion follows by diagram chasing.

The following is a natural generalization of a classical result and may be of some independent interest

PROPOSITION 2.3. *Let Λ be an R-algebra (not necessarily finitely generated) and A be an Azumaya R-algebra. Then $\mathfrak{A} \to \mathfrak{A}\otimes A$ defines a bijection from the ideals of Λ to the ideals of $\Lambda\otimes A$.*

Proof. As A is a faithfully projective R-module, the above map is a monomorphism [Lemma 2.2]. Let J be any ideal of $\Lambda\otimes A$. Suppose B is an Azumaya R-algebra such that $A\otimes B \simeq M_n(R)$ [Lemma 2.1]. Then $J\otimes B$ is an ideal of $\Lambda\otimes M_n(R)$ and hence, must be of the form $\mathfrak{A}\otimes M_n(R)$ where $\mathfrak{A}=(J\otimes B)\cap\Lambda$. Thus $J\otimes B=\mathfrak{A}\otimes A\otimes B$. Since B is also faithfully projective, we have $J=\mathfrak{A}\otimes A$.

PROPOSITION 2.4. *Suppose Λ is a faithful, semisimple algebra over R whose maximal ideals are scalar ideals. For any Azumaya R-algebra A, $\Lambda\otimes A$ is also a faithful, semisimple algebra whose maximal ideals are scalar ideals.*

Proof. It is known that $\Lambda\otimes A$ is a semisimple algebra [4]. The rest of the assertion is an immediate consequence of the above proposition.

3. Skolem-Noether Theorems

Throughout this section, R denotes a semilocal ring. Also all semisimple algebras considered in this section are faithful semisimple algebras whose maximal ideals are scalar ideals. Of course any central semisimple algebra is of this type. This class also includes some non central semisimple algebras.

LEMMA 3.1. *Let $m_1, m_2, \ldots, m_n$ be the maximal ideals of R and $J=\bigcap m_i$. If Λ is a semisimple algebra over R, then $\Lambda/J\Lambda \simeq \Lambda/m_1\Lambda \times \cdots \times \Lambda/m_n\Lambda$.*

Proof. Since $m_i\Lambda$ are the maximal ideals of Λ, the lemma is a consequence of the Chinese Remainder Theorem, provided that $J\Lambda=\bigcap m_i\Lambda$.

$\Lambda/J\Lambda$ is a semisimple R/J-algebra [4]. Since R/J is a classical semisimple ring, it is easy to verify that $\Lambda/J\Lambda$ must also be a classical semisimple ring. Hence, its Jacobson redical, which is zero, must be the intersection of its maximal ideals. Therefore $JΛ=\bigcap m_iΛ$.

PROPOSITION 3.2. *Suppose Λ is a semisimple algebra over R. If P_1 and P_2 are two finitely generated R-projective left Λ-modules such that* $\operatorname{rank}_R P_1=\operatorname{rank}_R P_2$, *then* $P_1\simeq P_2$.

Proof. Since $Λ$ is semisimple, P_1 and P_2, being R-projective, are $Λ$-projective [4]. So it is sufficient to show that $P_1/JP_1\simeq P_2/JP_2$ where J is the Jacobson radical of R. Thus we can assume that $R=R_1\times\cdots\times R_n$, where each R_i is a field.

Lemma 3.1 implies that $Λ=Λ_1\times\cdots\times Λ_n$ where each $Λ_i$ is a simple R_i-algebra. This induces a decomposition on P_1 and P_2. Let $P_1=P_{11}\oplus\cdots\oplus P_{1n}$ and $P_2=P_{21}\oplus\cdots\oplus P_{2n}$. Then P_{1i} and P_{2i} are $Λ_i$-modules which have the same dimension over R_i (because of rank condition) and hence, must be $Λ_i$-isomorphic. Thus $P_1\cong P_2$.

THEOREM 1. *Let Λ be a semisimple algebra over R and A be an Azumaya R-algebra. Suppose $f, g:Λ\to A$ are R-algebra mappings. Then there exists an inner automorphism θ of A such that $g=\theta\circ f$.*

Proof. It is enough to show the existence of a unit u of A such that $uf(\lambda)=g(\lambda)u$ for all $\lambda\in Λ$. Let ${}_fA$ and ${}_gA$ denote the left $Λ$-module structures on A through f and g respectively. Consider the left $Λ\otimes A^0$-modules ${}_fA_A$ and ${}_gA_A$. Since $Λ\otimes A^0$ is again semisimple, whose maximal ideals are scalar ideals [Prop. 2.4], and A is R-projective, by Prop. 3.2, there exists $\eta:{}_fA_A\to{}_gA_A$ which is a $Λ\otimes A^0$-isomorphism. η, being a right A-linear isomorphism, must be of the form $\eta(\bar{\lambda})=u\bar{\lambda}$ where u is a unit. The left $Λ$-linearity implies $uf(\lambda)=g(\lambda)u$ for all $\lambda\in Λ$.

THEOREM 2. *Let A be a projective semisimple algebra over R and Λ be an Azumaya R-algebra. Let $f, g:A\to Λ$ be R-monomorphisms. Then there exists an inner automorphism θ of Λ such that $g=\theta\circ f$.*

Proof. Since it is assumed that A is R-projective one has only to consider the left $A\otimes Λ^0$-modules ${}_fΛ_Λ$ and ${}_gΛ_Λ$ and imitate the proof of Theorem 1.

CANCELLATION LAW: *Let $Λ_1$ and $Λ_2$ be projective semisimple algebras over R. Suppose A is an Azumaya R-algebra such that $A\otimes Λ_1\simeq A\otimes Λ_2$. Then $Λ_1\simeq Λ_2$.*

Proof. Let $h:A\otimes Λ_1\to A\otimes Λ_2$ be an isomorphism. We have two monomorphisms of A into $A\otimes Λ_2$ namely, $f:A\to A\otimes 1\subset A\otimes Λ_2$ and $g:A\to A\otimes 1\subset A\otimes Λ_1\to A\otimes Λ_2$. By the above theorem, there exists an automorphism θ of $A\otimes Λ_2$ such that $g=\theta\circ f$. Hence the commutants of $f(A)$ and $g(A)$ in $A\otimes Λ_2$ are isomorphic. But these commutants are isomorphic to $Λ_2$ and $Λ_1$ respectively, which proves the proposition.

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