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An Equivariant Pinching Theorem[1])

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1. Introduction

Local qualitative properties of a riemannian manifold imply global topological or differentiable properties. For example, if M is a simply connected, complete riemannian manifold whose curvature tensor is close to the curvature tensor of the sphere S^n, then M is diffeomorphic to S^n, and the action of the isometry group on M is differentiably equivalent to the standard linear action of a subgroup of $O(n+1)$ on S^n, compare [4, 5]. Isometric actions occur frequently. Of special interest is the action of the fundamental group of a manifold on its universal covering space.

In [4, 5] the assumptions on the curvature tensor are rather strong. Here we prove that the assumptions of [5, Theorem 4.0], at least for large dimension, are already sufficient to prove the result on the isometry group. In fact, we give a new proof of the diffeomorphism theorem in which all the constructions are invariant under isometries.

In the formulation of the theorem below we call a riemannian manifold δ-pinched, if the sectional curvature K satisfies $\delta < K \leqslant 1$.

1.1. THEOREM. *There exists a decreasing sequence δ_n with limit $\delta_n \to 0.68$ as n tends to infinity such that the following assertion holds:*

If M is a simply connected, complete, δ_n-pinched riemannian manifold, and $\mu: G \times M \to M$ is an isometric action of the Lie group G on M, then

(i) *There exists a diffeomorphism $F: M \to S^n$.*

(ii) *There exists a homomorphism $\varphi: G \to O(n+1)$ such that*

(iii) $\varphi(g) = F \circ \mu(g, \cdot) \circ F^{-1}$ *for all $g \in G$.*

As in [4, 5] this theorem has the following fairly immediate consequences.

1.2. COROLLARY. *If M is a complete, δ_n-pinched, n-dimensional riemannian manifold (δ_n as above), then*

(i) *M is diffeomorphic to a space of constant positive sectional curvature.*

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(ii) *The isometry group of M is isomorphic to a subgroup of the isometry group of the corresponding constant curvature manifold.*

1.3. COROLLARY. (Compare [8, p. 122]). *There exists a decreasing sequence δ_n with limit $\delta_n \to 0.90$ as n tends to infinity, such that any δ_n-Kähler pinched Kähler manifold M^{2n} is holomorphically equivalent to $\mathbf{C}P^n$.*

The main idea in the proof is to define a procedure which leads from a connection with small curvature on the stabilized tangent bundle to a flat connection on this bundle. The curvature assumptions imply the existence of a connection with small curvature. The resulting flat connection, via a generalized Gauss map, yields the result.

It is a curious fact that our result improves with increasing dimension. This is in contrast to earlier results on this subject [2]. This effect appears in our proof because, for large n, a small neighborhood of any great $n-1$-sphere contains almost all of the volume of the n-sphere.

In the following chapter we give the proof of the main theorem except the construction of a flat connection and the necessary estimates. These topics are discussed in chapters three to five.

We wish to thank Wilfried Köhler for improving inequality 3.6.

2. The Proof

In this chapter we define a G-invariant connection ∇^0 on the stabilized tangent bundle and prove that the existence of a flat G-invariant connection ∇ near ∇^0 implies the assertions of the theorem. We postpone the proof of the existence of ∇ to the following chapter.

Let $E = T(M) \oplus 1(M)$, where $T(M)$ and $1(M)$ are tangent bundle and trivial line bundle $M \times \mathbf{R}$ respectively, denote the stabilized tangent bundle. The bundle E carries a euclidean fiber metric. The isometric action of G on M, together with the trivial action on $\mathbf{R}$, extends to an action on E, i.e., E is a G-vector bundle and $g \in G$ maps the fiber E_p isometrically onto $E_{g(p)}$.

As in [9] we define the connection ∇^0 on E as follows:

$$\begin{aligned} \nabla^0_X Y &= D_X Y - c\langle X, Y\rangle e \\ \nabla^0_X e &= cX \end{aligned} \tag{2.1}$$

where X and Y are vector fields on M, $e: M \to E$ is the section $p \mapsto (0, 1)_p \in T_p(M) \oplus \mathbf{R}$, $c = \sqrt{\frac{1}{2}(1+\delta)}$, $\langle\ ,\ \rangle$ and D respectively are riemannian metric and connection on M. It is easy to check [4, 9] that ∇^0 is a metric connection, is invariant under G-action, and has curvature tensor $R^0 = R - c^2 R_s$, where R is the riemannian curvature tensor on M and R_s is the algebraic expression of the curvature tensor on S^n in terms of the riemannian metric on M. The main point here is that R^0 is small in terms of a suitably chosen norm.

This concludes the discussion of ∇^0. In the second half of this chapter we use the existence of a flat invariant connection ∇ near ∇^0 to prove the theorem. Because M is simply connected, the flat connection ∇ is equivalent to a trivialization $E \cong M \times \mathbf{R}^{n+1}$, i.e., we have a parallel field of frames u, where $u_p = (e_1, \dots e_{n+1})_p$ is an orthonormal frame in the fiber E_p over $p \in M$. With such a frame field u we define the diffeomorphism F and the group homomorphism φ as follows:

$$F: M \to S^n, \qquad p \mapsto \langle e, u \rangle_p, \tag{2.2}$$

where $\langle e, u \rangle_p$ is the component vector of $e = (0, 1) \in T_p(M) \oplus \mathbf{R}$ in terms of the orthonormal basis u_p. To prove that F is a diffeomorphism we need an estimate on dF. For this estimate we introduce the following norm on the difference $\nabla - \nabla^0$. We recall that $\nabla - \nabla^0$, with proper identifications, is a linear map of the tangent space of M into the Lie algebra $\mathfrak{o}(n+1)$ of the structure group $O(n+1)$ of E. We define

$$\|\nabla - \nabla^0\| = \operatorname{Max} |(\nabla_X - \nabla_X^0)\, U|, \tag{2.3}$$

where $|\ |$ is the euclidean norm in E and the maximum is taken over unit vectors X and U in $T(M)$ and E respectively.

The definition 2.2, together with the fact that u is parallel under ∇, implies the following expression for the differential dF:

$$dFX = \langle \nabla_X e, u \rangle = \langle \nabla_X^0 e, u \rangle + \langle (\nabla_X - \nabla_X^0)\, e, u \rangle.$$

This expression for dF, together with $\nabla_X^0 e = cX$, (compare 2.1) leads immediately to the inequality

$$|dFX| \geqslant |X| (c - \|\nabla - \nabla^0\|) \tag{2.4}$$

This inequality will give the main numerical restriction for the pinching constant δ. In fact, $\|\nabla - \nabla^0\| < c$ is sufficient to prove that dF is non singular; therefore, since M is simply connected, F is a diffeomorphism.

To define the homomorphism $\varphi: G \to O(n+1)$ we again utilize the parallel frame field u associated to the flat connection ∇ and recall that the frame u_p in the fiber E_p over $p \in M$ is equivalent to an isomorphism $\mathbf{R}^{n+1} \to E_p$. We define

$$\begin{aligned} &\varphi: G \to O(n+1) \\ &\quad g \mapsto u^{-1} \circ g \circ u: \mathbf{R}^{n+1} \to E_p \to E_{g(p)} \to \mathbf{R}^{n+1}, \end{aligned} \tag{2.5}$$

where $\varphi(g)$, as a composition of isometries, lies in $O(n+1)$ and is independent of the point $p \in M$ as we will see shortly. To prove that φ is a group homomorphism it is sufficient to prove that the definition of φ is in fact independent of the point $p\in M$. This is so because E is a G-vector bundle. The main ingredient in the proof of the independence of φ on $p\in M$ is the G-invariance of ∇. This invariance is obvious from the constructions of the next chapter. Let φ' be the map defined in 2.5 where p is replaced by $p'\in M$, let α be a path from p to p', and let τ_α denote parallel translation with respect to ∇ along this path. Then,

$$\varphi'(g)=u_{g(p')}^{-1}\circ g\circ u_{p'}=u_{g(p)}^{-1}\circ\tau_{g(\alpha)}^{-1}\circ g\circ\tau_\alpha\circ u_p=u_{g(p)}^{-1}\circ g\circ u_p=\varphi(g).$$

The definitions of F and φ, compare 2.2 and 2.5 respectively, imply immediately the assertion (iii) of Theorem 1.1. The main point to observe here is that in both definitions the fibers are identified with $\mathbf{R}^{n+1}$ by means of the same frame u.

3. Construction of the Connection ∇

Here we construct the flat G-invariant connection ∇ used in the previous chapter. The first step is the construction of a sequence ∇^i of invariant connections, whose limit is a flat C^0-connection ∇^∞. Then we obtain the C^∞-connection ∇ by smoothing ∇^∞.

The sequence ∇^i starts with the connection ∇^0 of 2.1. We define ∇^{i+1} inductively as a certain average of locally defined flat connections obtained from ∇^i. The main point is that the curvature of an average of flat connections is small in terms of a suitable norm. We observe that ∇^0, the iteration from ∇^i to ∇^{i+1}, as well as the smoothing process is invariant under the isometric action of G.

We compute with connection form ω^i and curvature form Ω^i instead of connection ∇^i and curvature R^i. These forms are naturally defined on the principal bundle. However, since we perform local calculations, we deal with their pull backs by means of a cross section (frame field). Since we are interested in norms of curvature forms only it is not necessary to specify the frames as long as they are orthonormal.

The computations of this chapter are based on Jacobi field estimates on M. The following chapter contains the comparison theorems which justify the use of estimates valid on the standard sphere. For the estimates on curvature forms we must specify a norm in the appropriate space. In principle, any norm will do but the numerical result depends on the choice of the norm.

3.1. *The iteration* $\nabla^i\to\nabla^{i+1}$. For any $q\in M$ we define a connection $\nabla^{i,q}$ with connection form $\omega^{i,q}$ in a neighborhood of q as follows: Let $u^q(q)$ be an orthonormal frame in the fiber E_q and $u^q(p)$ be the frame in E_p obtained by parallel translation of $u^q(q)$ along the unique shortest geodesic from q to p. This way we obtain a field of frames in a neighborhood of q. The corresponding flat connection and connection form are denoted by $\nabla^{i,q}$ and $\omega^{i,q}$ respectively. Next, we define the connection form ω^{i+1}

corresponding to ∇^{i+1} as an average over the connection forms $\omega^{i,q}$. The weight function η in the following definition will be constructed in chapter 5.

$$\omega^{i+1} = \int_M \omega^{i,q} \eta(\cdot, q)\, dq \tag{3.2}$$

The form ω^{i+1} is well defined because $\eta(\cdot, q): M \to \mathbf{R}$ has its support in the domain of definition of $\omega^{i,q}$. Since $\int \eta(\cdot, q)\, dq = 1$, ω^{i+1} is again a connection form.

To obtain an estimate for the curvature form Ω^{i+1} we express Ω^{i+1} in terms of ω^{i+1} by means of the Cartan identity. The following equations hold:

$$\begin{aligned}\Omega^{i+1} &= \Omega^{i+1} - \int \Omega^{i,q} \eta\, dq \\ &= \int (\omega^{i,q} \wedge d_p \eta)\, dq + \left(\int \omega^{i,q} \eta\, dq\right) \wedge \left(\int \omega^{i,q} \eta\, dq\right) - \int (\omega^{i,q} \wedge \omega^{i,q})\, \eta\, dq \\ &= \int (\omega^{i,q} \wedge d_p \eta)\, dq - \int (\omega^{i,q} - \omega^{i+1}) \wedge (\omega^{i,q} - \omega^{i+1})\, \eta\, dq,\end{aligned} \tag{3.3}$$

where $d_p \eta$ is the differential of η in the first variable.

For the following estimates we need norms on the spaces of $\mathfrak{o}(n+1)$-valued differential forms. The choices are dictated by the definition 2.3. First we define a norm on the Lie algebra $\mathfrak{o}(n+1)$ as follows:

$$|A| = \operatorname{Max} |AU|,$$

where $A \in \mathfrak{o}(n+1)$, $U \in \mathbf{R}^{n+1}$ has length one, and $|\ |$ is the standard norm on $\mathbf{R}^{n+1}$. For $\mathfrak{o}(n+1)$-valued 1-forms ω we define

$$\|\omega\| = \operatorname{Max} |\omega(X)|, \tag{3.4}$$

where the maximum is taken over unit vectors in $T(M)$, and $|\ |$ is the above norm on $\mathfrak{o}(n+1)$.

In the same way we define for $\mathfrak{o}(n+1)$-valued 2-forms Ω

$$\|\Omega\| = \operatorname{Max} |\Omega(X, Y)|, \tag{3.5}$$

where X and Y are unit vectors in $T(M)$. With these definitions one has the inequality:

$$\|\omega_1 \wedge \omega_2\| \leqslant 2 \|\omega_1\|\, \|\omega_2\|.$$

Now equation 3.3 yields:

$$\|\Omega^{i+1}\| \leqslant \left\| \int (\omega^{i,q} \wedge d_p \eta)\, dq \right\| + \left\| \int (\omega^{i,q} - \omega^{i+1}) \wedge (\omega^{i,q} - \omega^{i+1})\, \eta\, dq \right\|.$$

We define $\beta^{i,q}=\omega^{i,q}-\omega^{i+1}$ and chose unit vectors X, $Y\in T(M)$, U, $V\in\mathbf{R}^{n+1}$ such that

$$\left\|\int (\beta^{i,q}\wedge\beta^{i,q})\,\eta dq\right\|=\left\langle\left(\int (\beta^{i,q}\wedge\beta^{i,q})\,\eta dq\right)(X, Y)(U), V\right\rangle$$
$$\leqslant\int |\langle\beta^{i,q}(X)\,\beta^{i,q}(Y)(U), V\rangle|\,\eta dq+\int |\langle\beta^{i,q}(Y)\,\beta^{i,q}(X)(U), V\rangle|\,\eta dq.$$

The Cauchy-Schwarz inequality yields:

$$\int |\langle\beta^{i,q}(X)\,\beta^{i,q}(Y)(U), V\rangle|\,\eta dq=\int |\langle\beta^{i,q}(Y)(U), \beta^{i,q}(X)^T(V)\rangle|\,\eta dq$$
$$\leqslant\left(\int |\beta^{i,q}(Y)(U)|^2\,\eta dq\right)^{1/2}\left(\int |\beta^{i,q}(X)^T(V)|^2\,\eta dq\right)^{1/2}.$$

Due to the construction of $\omega^{i,q}$ it is more convenient to estimate $\alpha^{i,q}=\omega^{i,q}-\omega^i$ instead of $\beta^{i,q}$. Since $\beta^{i,q}=\alpha^{i,q}-\int\alpha^{i,q}\eta dq$ we have

$$\int |\beta^{i,q}(Y)(U)|^2\,\eta dq\leqslant\int |\alpha^{i,q}(Y)(U)|^2\,\eta dq.$$

Let the symbol $\|\alpha^i\|$ denote the maximum of $\|\alpha^{i,q}\|=\|\alpha^{i,q}\|_q$ over $q\in M$, where $\|\ \|_q$ indicates that in the definition (3.4) only vectors $X\in T(M)\,|\,\operatorname{supp}\eta(\cdot, q)$ have to be considered. Thus

$$\int |\alpha^{i,q}(Y)(U)|^2\,\eta dq\leqslant\|\alpha^i\|^2.$$

With the observation $\int(\omega^i\wedge d_p\eta)\,dq=0$ we finally obtain

$$\|\Omega^{i+1}\|\leqslant\|\alpha^i\|\int |d_p\eta|\,dq+2\|\alpha^i\|^2. \tag{3.6}$$

3.7. *The computationof* $\|\alpha^i\|$. We recall that $\alpha^{i,q}=\omega^{i,q}-\omega^i=\nabla^{i,q}-\nabla^i$, where $\omega^{i,q}$ is the connection form of the flat connection defined by the frame field obtained by parallel translation (with respect to ∇^i) of a frame in E_q along geodesics starting at $q\in M$.

To estimate the value of $\alpha^{i,q}$ applied to a unit vector $X\in T_pM$, we choose a path $\gamma=\gamma(s)$ in M with initial tangent vector $\dot\gamma(0)=X$, and define a corresponding path $a=a(s)$ on the orthogonal group $O(n+1)$ by setting $a(s)$ equal to the orthogonal transformation defined by parallel translation, with respect to ∇^i, along the triangle $(q, p, \gamma(s))$. The definition of $\nabla^{i,q}$ implies $\nabla^{i,q}_X-\nabla^i_X=\dot a(0)$. To estimate $\dot a(0)$ we introduce the norm $|a|=\max|aU-U|$ on $O(n+1)$, where U is a unit vector in $\mathbf{R}^{n+1}$. This choice is dictated by the previous choice of a norm in the Lie algebra of $O(n+1)$. We observe that, as in [9, p. 131], $|a(s)|$ is majorized by the product of the area of the triangle $(q, p, \gamma(s))$ with a suitable norm of Ω^i. We obtain $|a(s)| \leqslant \text{Area}\,(q, p, \gamma(s)) \times \|\Omega^i\|$. Since $|\dot{a}(0)| = \lim_{s\to 0} |a(s)|/s$ we are interested in an explicit expression for this limit. We obtain it in terms of the Jacobi field Y along the geodesic c $(c(0)=q,$ $c(t)=p)$ with boundary values $Y(0)=0$ and $Y(t)=X$. The result is

$$|\alpha^{i,q}(X)| = |\dot{a}(0)| \leqslant \frac{\int_0^t |Y|}{|Y(t)|} \|\Omega^i\|,$$

and Corollary 4.2 yields

$$\|\alpha^i\| \leqslant \frac{1-\cos r}{\sin r} \|\Omega^i\|, \tag{3.8}$$

where $r \geqslant d(p, q) = t$ is the radius of the ball in which $\nabla^{i,q}$ is defined. We will choose r to minimize the final estimate.

3.9. *The connection* ∇^∞. The inequalities (3.6), (3.8) and the estimate (5.5) for $\int |d_p\eta|\, dq$ yield immediately:

$$\|\Omega^{i+1}\| < \text{const}\, n \sin^{n-1} \sqrt{\delta r}\, \frac{1-\cos r}{\sin r} \|\Omega^i\| + 2\left(\frac{1-\cos r}{\sin r} \|\Omega^i\|\right)^2. \tag{3.10}$$

We utilize the above recursion formula to prove that, for a suitable choice of $\delta = \delta_n$, the connection forms ω^i converge in the C^0-topology to a connection form ω^∞. Actually we need to prove a more precise result. Equation 2.4 shows that we must prove $\|\omega^0 - \omega^\infty\| < c$, where c is the constant of definition 2.1. Since $\|\omega^{i+1} - \omega^i\| < \|\alpha^i\|$, we must prove $\sum_{i=0}^{\infty} \|\alpha^i\| < c$, or, in view of (3.8),

$$\sum_{i=0}^{\infty} \|\Omega^i\| < \frac{c}{d}, \quad \text{where} \quad d = \frac{1-\cos r}{\sin r}. \tag{3.11}$$

For the estimate of the first term in this series we observe that, with proper identifications, the form Ω^0 coincides with the curvature tensor R' of [10, p. 127]. We obtain $\|\Omega^0\| \leqslant \frac{2}{3}(1-\delta)$.

Now, for a rough estimate of (3.11) we observe that (3.10) is of the form

$$\|\Omega^{i+1}\| < a\|\Omega^i\| + b\|\Omega^i\|^2 = (a + b\|\Omega^i\|)\, \|\Omega^i\|.$$

The series (3.11) converges as soon as $a + b\|\Omega^0\| < 1$. We can satisfy this condition for $n \geqslant 3$ because for a suitable choice of r and δ close to 1 the constant a is arbitrarily small, and $b\|\Omega^0\|$ is arbitrarily small as well for δ close enough to 1. Therefore, the series (3.11) is majorized by a geometric series with small first term, and the condition (3.11) is satisfied.

For manifolds of large dimension n it is easy to give a much better estimate as follows:
We observe that the term $n\sin^{n-1}\sqrt{\delta}r$ converges to zero as $n\to\infty$ if $\sqrt{\delta}r\neq\pi/2$. Since the estimate 5.5 holds for $\sqrt{\delta}r>\pi/2$ only, we choose r such that $\sqrt{\delta}r$ is greater than but arbitrarily close to $\pi/2$. Then, we compute the sum of the first few terms in (3.11) and estimate the rest with the rough method above. A computation shows that, for $n=\dim M$ large enough, the condition $\delta>0.68$ implies that the inequality (3.11) holds.

For low dimensions precise estimates are more cumbersome to obtain because the optimal choice of r in 3.10 is not obvious. To compare the present result with that of [5] we determine the dimensions for which the pinching constant $\delta=0.98$ is sufficient to prove the theorem with the present method. A computation shows that $\delta=0.98$ is sufficient for $n>5$.

In any case, (3.11) holds under suitable assumptions on δ_n. Therefore, the sequence ω^i of connection forms converges in the C^0-topology to a connection form ω^∞. Let ∇^∞ denote the corresponding connection. Since ∇^∞ is continuous only, the curvature form Ω^∞ is not defined, but parallel translation with respect to ∇^∞ is independent of the path because ∇^∞ is the limit of connections with curvatures converging to zero. In this sense ∇^∞ is flat.

3.12. *The connection* ∇. Now we smooth ∇^∞ to the C^∞-connection ∇. Since ∇^∞ is equivalent to a C^1-section u^∞ in the principal bundle P of orthonormal frames associated to E, we smooth the section u^∞ to a C^∞-section $u: M\to P$ and define ∇ to be the corresponding connection. To obtain u we average u^∞ by means of the center of mass construction [3, proposition 3.1] as follows. Let P_p denote the fiber of P over $p\in M$ and B_p a ball with center p and sufficiently small radius. Let $v_p: B_p\to P_p$ be the map u^∞ followed by parallel translation, with respect to ∇^0, along the unique shortest geodesic to p. We define $u: M\to P$ by the assignment $p\mapsto$center of v_p with respect to the measure on B_p obtained by multiplying the riemannian measure with the smoothing kernel η of chapter 5.

The section $u: M\to P$ is of class C^∞ and depends on the parameter r in the definition of η. If r converges to zero, i.e., η converges to the Dirac measure, then u converges in the C^1-topology to u^∞, compare [7]. Therefore ∇ can be chosen arbitrarily close to ∇^∞.

4. Inequalities for Jacobi Fields and Volume Functions

In this chapter we prove inequalities for Jacobi fields. For greater convenience we state the results for a riemannian manifold M with curvature restrictions $0<a^2\leqslant K\leqslant b^2$. However, with slight modifications, similar results are true for zero or negative curvature bounds.

4.1. PROPOSITION. *Let c be a geodesic on M with $\|\dot{c}\|=1$ and Y a Jacobi field along c with $Y(0)=0$ and $\langle Y, \dot{c}\rangle=0$. For real munbers s, t with $0\leqslant s\leqslant t<\pi/b$ we have*

$$\frac{\sin as}{\sin at}\leqslant\frac{|Y(s)|}{|Y(t)|}\leqslant\frac{\sin bs}{\sin bt}.$$

4.2. COROLLARY. *Under the same assumptions we have*

$$\frac{1-\cos at}{a\sin at}\leqslant\frac{\int_0^t |Y(s)|\,ds}{|Y(t)|}\leqslant\frac{1-\cos bt}{b\sin bt}.$$

Proof. The proposition is an easy consequence of the following inequality (compare [1, p. 253]).

$$a\frac{\cos at}{\sin at}\geqslant\frac{|Y(t)|'}{|Y(t)|}\geqslant b\,\frac{\cos bt}{\sin bt}. \tag{4.3}$$

To prove (4.3) we assume for simplicity $|Y(t)|=1$. Then we have

$$|Y(t)|'=\langle Y', Y\rangle(t)=I^t(Y)=\int_0^t |Y'|^2-K(Y,\dot{c})\,|Y|^2.$$

Since Jacobi fields minimize the index form it follows that $I^t(Y)\leqslant I^t(fE)$ for any C^∞-function f with $f(0)=0$ and $f(t)=1$, where E is the parallel field along c with $E(t)=Y(t)$. By introducing the lower curvature bound we obtain $I^t(fE)\leqslant \int_0^t(f')^2-a^2f^2$. This integral is known to be minimal for the function $\sin as/\sin at$, the corresponding value of the integral being $a(\cos at/\sin at)$.

For the second part of the inequality we again start with $|Y(t)|'=I^t(Y)$. Introducing the upper curvature bound we get

$$I^t(Y)\geqslant\int_0^t |Y'|^2-b^2|Y|^2\geqslant\int_0^t (|Y|')^2-b^2|Y|^2.$$

The last integral has $b(\cos bt/\sin bt)$ as minimal value, which completes the proof of (4.3).

The proposition then follows by integration and exponentiation. Now we turn to the estimation of volumes of balls. We still assume that the curvature of M satisfies $0<a^2\leqslant K\leqslant b^2$ and consider balls of radius less than π/b. Let $V(p,t)$ denote the volume of the ball $\{q\in M \mid d(p,q)\leqslant t\}$. In order to compute $V(p,t)$ we introduce geodesic polar coordinates around p and get

$$V(p,t)=\int_S\int_0^t J(s,u)\,ds\,du\,,$$

where S is the unit sphere in $T_p(M)$ and $J(s,u)$ is the jacobian of $\exp: T_p(M)\to M$, multiplied by s^{n-1}. For a fixed $u\in S$ the function $J(s)=J(s,u)$ can be expressed as follows. Consider the geodesic c with $c(0)=p$, $\dot c(0)=u$ and choose Jacobi fields $Y_1,\dots,Y_{n-1}$ along c with $Y_i(0)=0$, $Y'(0)\perp u$ and $\{Y'(0)\mid 1\leqslant i\leqslant n-1\}$ linearly independent. Then we have

$$J(s)=\frac{|Y_1(s)\wedge\cdots\wedge Y_{n-1}(s)|}{|Y_1'(0)\wedge\cdots\wedge Y_{n-1}'(0)|}.$$

4.4. PROPOSITION (compare [6]). *Let $u\in S\subset T_pM$ and $J(s)=J(s,u)$ be as above. For s, t with $0\leqslant s\leqslant t<\pi/b$ we have*

$$\left(\frac{\sin as}{\sin at}\right)^{n-1}\leqslant\frac{J(s)}{J(t)}\leqslant\left(\frac{\sin bs}{\sin bt}\right)^{n-1}.$$

Proof. Choose the Jacobi fields $Y_1,\dots,Y_{n-1}$ to be orthonormal at t, then

$$\frac{J(s)}{J(t)}\leqslant\prod_{i=1}^{n-1}\frac{|Y_i(s)|}{|Y_i(t)|}.$$

Now proposition 4.1 yields $J(s)/J(t)\leqslant(\sin bs/\sin bt)^{n-1}$. The proof for the lower bound is similar.

4.5. COROLLARY. *Under the same assumption for s and t we get*

$$\frac{s_n(at)}{a\sin^{n-1}at}\leqslant\frac{\int_S\int_0^t J(s,u)\,ds\,du}{\int_S J(t,u)\,du}\leqslant\frac{s_n(bt)}{b\sin^{n-1}bt},$$

where s_n denotes the function $s_n(t)=\int_0^t\sin^{n-1}s\,ds$.

Proof. The corollary follows easily from the proposition, since only the middle term of the inequality depends on u. Let $V'(p, t)$ denote the derivative of $V(p, t)$ with respect to t. Obviously $V'(p, t)=\int_S J(t, u)\,du$ with $S\subset T_p(M)$, and the corollary may be rewritten in the form

$$\frac{a\sin^{n-1}at}{s_n(at)}\geqslant\frac{V'(p,t)}{V(p,t)}\geqslant\frac{b\sin^{n-1}bt}{s_n(bt)}\,. \tag{4.6}$$

5. The Weight Function

In this chapter we return to our δ-pinched manifold M, i.e., we assume $\delta<K\leqslant 1$, and define the weight function η used in chapter 3. For fixed real numbers r and ϱ with $\pi/2\sqrt{\delta}<r<r+\varrho<\pi$ we choose a monotone C^∞-function h with $h(t)=1$ for $0\leqslant t\leqslant r$, $h(t)=0$ for $t\geqslant r+\varrho$, and $|h'(t)|\leqslant\text{const}/\varrho$. For $p, q\in M$ we set $h(p,q)=h(d(p,q))$.

In order to get a normalized weight function, we form

$$\eta(p,q)=\frac{h(p,q)}{H(p)}\,,$$

where

$$H(p)=\int_M h(p,q)\,dq=\int_S\int_0^{r+\varrho} h(t)\,J(t,u)\,dt\,du\,.$$

We want to estimate the integral $\int_M |d_p\eta(p,q)|\,dq$. Differentiation of η yields

$$d_p\eta=\frac{d_p h}{H}-\frac{dH}{H}\eta\,,$$

therefore we get

$$\int_M |d_p\eta(p,q)|\,dq\leqslant\frac{1}{H(p)}\int_M |d_p h(p,q)|\,dq+\frac{|dH(p)|}{H(p)}\,. \tag{5.1}$$

Obviously

$$\int_M |d_p h(p,q)|\,dq=\int_S\int_r^{r+\varrho} |h'(t)|\,J(t,u)\,dt\,du\,.$$

By proposition 4.4 and our choice of r the function $J(t, u)$ is monotone decreasing between r and $r+\varrho$, hence

$$\int_r^{r+\varrho} |h'(t)|\, J(t, u)\, dt \leqslant \operatorname{const} J(r, u),$$

and we have

$$\frac{1}{H(p)} \int_M |d_p h(p, q)|\, dq \leqslant \frac{\operatorname{const}}{H(p)} \int_S J(r, u)\, du = \operatorname{const} \frac{V'(p, r)}{H(p)} < \operatorname{const} \frac{V'(p, r)}{V(p, r)}. \tag{5.2}$$

From the definition of $H(p)$ we obtain through integration by parts

$$H(p) = \int_S \int_0^{r+\varrho} h(t)\, J(t, u)\, dt\, du = \int_0^{r+\varrho} h(t)\, V'(p, t)\, dt$$
$$= h(t)\, V(p, t)\,|_0^{r+\varrho} - \int_0^{r+\varrho} h'(t)\, V(p, t)\, dt = -\int_r^{r+\varrho} h'(t)\, V(p, t)\, dt.$$

Therefore the differential is

$$dH(p) = -\int_r^{r+\varrho} h'(t)\, d_p V(p, t)\, dt$$

and

$$|dH(p)| \leqslant \int_r^{r+\varrho} |h'(t)|\, |d_p V(p, t)|\, dt.$$

A geometric argument shows $|d_p V(p, t)| \leqslant \frac{1}{2} V'(p, t)$, which can be estimated as before by $\frac{1}{2} V'(p, r)$ for $r \leqslant t \leqslant r+\varrho$. We now get

$$|dH(p)| \leqslant \tfrac{1}{2} V'(p, r) \int_r^{r+\varrho} |h'(t)|\, dt = \tfrac{1}{2} V'(p, r).$$

Again using $H(p) > V(p, r)$ we finally get

$$\frac{|dH(p)|}{H(p)} < \frac{1}{2} \frac{V'(p, r)}{V(p, r)}. \tag{5.3}$$

Combining the above estimates (5.1) to (5.3) and applying (4.6) we find

$$\int_M |d_p\eta(p, q)|\, dq < \text{const}\, \frac{V'(p, r)}{V(p, r)} \leqslant \text{const} \sqrt{\delta}\, \frac{\sin^{n-1}\sqrt{\delta}r}{s_n(\sqrt{\delta}r)}. \tag{5.4}$$

Since $\sqrt{\delta}r > \pi/2$ we have $s_n(\sqrt{\delta}r) > s_n(\pi/2) = \frac{1}{2}s_n(\pi)$, where

$$s_n(\pi) = \int_0^\pi \sin^{n-1} t\, dt = \sqrt{\pi}\Gamma\left(\frac{n}{2}\right) \Big/ \Gamma\left(\frac{n+1}{2}\right).$$

An elementary argument shows $s_n(\pi)/2 > \sqrt{\pi/n}$, hence

$$\int_M |d_p\eta(p, q)|\, dq < \text{const}\, n \sin^{n-1} \sqrt{\delta}r. \tag{5.5}$$

Remark. We need to know the numerical value of the constant in (5.5) only for the explicit estimates in low dimensions. An easy argument shows that the constant in (5.4) can be chosen arbitrarily close to $\frac{3}{2}$. Therefore we get as final constant in (5.5) a number close to $\frac{3}{2}\sqrt{\delta/\pi}$, e.g. 0.85.

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