Extremal Length, Extremal Regions and Quadratic Differentials.

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Extremal Length, Extremal Regions and Quadratic Differentials

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0. Introduction

This paper is devoted to the proof of two theorems; Theorem 1 is given in Section 9 and Theorem 2 in Section 12. The proofs of these theorems require a number of auxiliary concepts and constructions (given in Sections 1–8 and 10–11), but the theorems themselves can be stated without such preparation.

Let σ be a 1-chain on an open Riemann surface X. We associate to σ (actually, to the $\mathbf{Z}_2$-homology class of σ) a quadratic differential Φ, a family $\mathscr{F}$ of crosscuts on X, and a family $\mathscr{F}^*$ of curves on X. $\mathscr{F}$ consists of all crosscuts which cross σ an odd number of times; $\mathscr{F}^*$ consists of boundaries $\partial\Omega$ where Ω is a Jordan open set containing an element in the $\mathbf{Z}_2$-homology class of σ (see Section 5). Φ is characterized in Proposition 4 (Section 8); if X is the interior of a compact bordered Riemann surface, Φ can also be described as follows: among all curves which are $\mathbf{Z}_2$-homologous to σ, there is a unique one σ_0 which has the greatest extremal distance from the boundary of X (see Proposition 2, Section 8). Let ω be the harmonic measure of σ_0 on $X-\sigma_0$. Then $\Phi=\frac{1}{4}(d\omega+i*d\omega)^2$.

Theorem 1 relates the extremal lengths of $\mathscr{F}$ and $\mathscr{F}^*$ to the integral norm of Φ:

$$\iint_X |\Phi| = \frac{1}{\lambda(\mathscr{F})} = \frac{1}{4}\lambda(\mathscr{F}^*).$$

Theorem 2 is an analogous result for closed surfaces X and for reduced extremal lengths.

In the simplest special case (X is simply connected and σ is an arc), Theorem 1 reduces to an extremal length theorem of J. Hersch [1]. Some applications of Theorems 1 and 2 are discussed briefly in Section 13.

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1. Intersection Numbers for 1-chains

Let γ and δ be singular 1-chains on a Riemann surface X. We wish to define the intersection number $\gamma \times \delta$. Informally, $\gamma \times \delta$ is the algebraic sum of the number of times δ crosses γ from left to right; right to left crossings are counted with negative values. A necessary condition for this intersection number to be defined is $\gamma \subset X - \partial\delta$ and $\delta \subset X - \partial\gamma$. It is bilinear and skew-symmetric and depends only on the homology class of γ on $X - \partial\delta$ and the homology class of δ on $X - \partial\gamma$.

The formal definition of $\gamma \times \delta$ can be given in terms of simplicial approximation. Because γ and δ need not be cycles, care must be taken to prevent the endpoints of one chain from shifting across the other chain. Take a triangulation of X so that the points in $\partial\gamma$ are vertices in the triangulation and the points in $\partial\delta$ are barycenters. This triangulation should be so fine that if a star in the barycentric subdivision contains a point of either $\partial\gamma$ or $\partial\delta$, then that star is disjoint from the other chain. For such triangulations we can replace γ and δ by their simplicial approximations and then define the intersection number in the same manner as for cycles (Ahlfors-Sario [1, pp. 67–72]).

An intersection number can still be defined when one of γ and δ is a relative 1-chain. γ is a relative 1-chain if $\gamma = \sum n_i \gamma_i$ is a countable formal sum where n_i is an integer and for each compact set $K \subset X$ the set of indices i for which $n_i \neq 0$ and $\gamma_i \cap K \neq \emptyset$ is finite. Let δ be a finite 1-chain. Since $\gamma_i \times \delta \neq 0$ for at most finitely many indices i, $\gamma \times \delta = \sum n_i (\gamma_i \times \delta)$ is defined. In this situation $\gamma \times \delta$ depends only on the weak homology class of γ on $X - \partial\delta$.

2. The Covering Surface $\tilde{X}(\sigma)$

Let σ be an integral 1-chain on a Riemann surface X and set $X' = X - \partial\sigma$. We shall construct a two-sheeted (possibly branched) covering surface of X. It will be denoted by $\tilde{X}$ or $\tilde{X}(\sigma)$. In case σ is a simple slit, this construction generalizes the classical technique of cross-identifying the edges of the slits between two copies of $X - \sigma$.

Fix a base point Q on X'. For $\gamma \in \pi_1(X', Q)$ define $h(\gamma)$ to be the mod 2 residue class of $\gamma \times \sigma$. The group homomorphism

$$h : \pi_1(X', Q) \to \mathbf{Z}_2 \tag{1}$$

is trivial or surjective according to whether or not there is a $\gamma \in \pi_1(X', Q)$ with $\gamma \times \sigma \equiv 1 \pmod 2$. We shall express this condition as a homology property of σ. To accomplish this we consider homology on X modulo the ideal boundary β of X with coefficients in $\mathbf{Z}_2$. If δ and τ are 1-chains, then the notation $[\delta]_2 = [\tau]_2 \pmod{\beta}$ will mean that $\delta - \tau$ is a cycle which is $\mathbf{Z}_2$-homologous to a dividing cycle. The following lemma is an analog of the fact that a cycle α on a Riemann surface is dividing if and only if $\alpha\times\gamma=0$ for all cycles γ.

LEMMA 1. *Let σ be a* 1*-chain on X. $[\sigma]_2=0 \pmod{\beta}$ if and only if $\gamma\times\sigma\equiv 0 \pmod 2$ for all closed curves γ on X'.*

Henceforth, we shall assume that the given 1*-chain σ is* **nontrivial** *in the sense that*

$$[\sigma]_2\neq 0 \pmod{\beta}. \tag{2}$$

This means that either σ is not a $\mathbf{Z}_2$-cycle or else σ is a $\mathbf{Z}_2$-cycle which is not $\mathbf{Z}_2$-homologous to a dividing cycle. The homomorphism h in (1) is then surjective and its kernel

$$G(\sigma)=\{\gamma\in\pi_1(X', Q):\gamma\times\sigma\equiv 0 \pmod 2\} \tag{3}$$

is a normal subgroup of $\pi_1(X', Q)$ of index two. There is a normal, smooth, two-sheeted covering surface $\tilde{X}'\xrightarrow{f}X'$ and the fundamental group of $\tilde{X}'$ is isomorphic to $G(\sigma)$. Any $\gamma\in\pi_1(X', Q)$ lifts to a closed curve $\tilde{\gamma}$ on $\tilde{X}'$ if and only if $\gamma\in G(\sigma)$. There is exactly one nontrivial cover transformation $T:\tilde{X}'\to\tilde{X}'$ and T is an involution.

It follows from Lemma 1 that for a planar surface X all finite 1-chains with the same boundary give rise to the same two-sheeted covering surface $\tilde{X}'$.

Next, we extend $\tilde{X}'$ to a two-sheeted ramified covering of X. Let $\partial\sigma=\sum n_i P_i$ where the P_i are distinct points of X and the n_i are nonzero integers. If n_i is even, then a sufficiently small circle centered at P_i lifts to two disjoint circles on $\tilde{X}'$. In such a case we add two points to $\tilde{X}'$ corresponding to the centers of these circles. The projection map f and the cover transformation T both extend continuously to these added points; f maps them to P_i and T interchanges them. If n_i is odd, then a small circle τ centered at P_i does not lift to a closed curve on $\tilde{X}'$, but 2τ does lift to a closed curve. In this case we add a first-order branch point $\tilde{P}_i$ to $\tilde{X}'$, extend the projection map f to $\tilde{P}_i$ by $f(\tilde{P}_i)=P_i$ and extend the cover transformation T so that $\tilde{P}_i$ becomes a fixed point of T.

Let $\tilde{X}(\sigma)$ denote the covering surface $\tilde{X}'$ after all such points have been added. Then $\tilde{X}(\sigma)\xrightarrow{f}X$ is a two-sheeted ramified covering with branch points over each point in the $\mathbf{Z}_2$-boundary of σ. The cover transformation T is involutory and its only fixed points are the branch points. We shall refer to $\tilde{X}(\sigma)$ as *the covering surface of X determined by σ*; if no confusion results we shall write $\tilde{X}$ in place of $\tilde{X}(\sigma)$. Clearly, $\tilde{X}$ is independent of the particular base point Q selected on X'. Also, if $[\sigma]_2=[\sigma_0]_2 \pmod{\beta}$, then $\tilde{X}(\sigma)=\tilde{X}(\sigma_0)$.

3. The Sheet Structure of $\tilde{X}(\sigma)$

The space $\tilde{X}-f^{-1}(\sigma)$ can be partitioned into two sheets both of which are open sets; they need not be connected. Fix a base point $Q\in X'$ and let $\tilde{Q}$ be a point on $\tilde{X}$ lying over Q. Consider all arcs $\delta:[0,1]\to X'$ with $\delta(0)=Q$ and $\delta(1)\notin\sigma$, and let $\tilde{\delta}$ be the unique lift of δ to $\tilde{X}$ with $\tilde{\delta}(0)=\tilde{Q}$. The *lower sheet* of $\tilde{X}(\sigma)$ (with respect to $\tilde{Q}$) consists of all points $\tilde{\delta}(1)$ where δ crosses σ an even number of times; the *upper sheet* is the set of all $\tilde{\delta}(1)$ for δ's which cross σ an odd number of times. These sheets are well defined: if $\tilde{\delta}(1)=\tilde{\delta}_0(1)$, then $\delta_0\delta^{-1}$ is in the group $G(\sigma)$, and so it crosses σ an even number of times. Hence, $\delta\times\sigma$ and $\delta_0\times\sigma$ have the same parity. The distinction between 'upper' and 'lower' depends on the choice of the point $\tilde{Q}$ over Q. If $\tilde{\tilde{Q}}$, the other point on $\tilde{X}$ over Q, were used in place of $\tilde{Q}$, then the two sheets would be interchanged. In any case, the cover transformation T permutes the two sheets and f maps each sheet homeomorphically onto $X-\sigma$. Moreover, the two sheets, except possibly for the labels 'upper' and 'lower', do not depend upon the selection of base point.

The sheets do depend upon the 1-chain σ, and it is necessary to determine the dependence explicitly. Let σ and σ_0 satisfy $[\sigma]_2=[\sigma_0]_2 \pmod\beta$. Then $\tilde{X}(\sigma)=\tilde{X}(\sigma_0)$ as surfaces, yet they may have quite different sheet decompositions. Let $U(\sigma)$ and $L(\sigma)$ denote the upper sheet and lower sheet of $\tilde{X}(\sigma)$. We say that $\tilde{X}(\sigma)$ and $\tilde{X}(\sigma_0)$ have *the same sheet structure* if, for any regular subregion $\Omega\subset X$ such that $\Omega\supset\sigma\cup\sigma_0$, the set $\{U(\sigma)\cap f^{-1}(X-\Omega),\ L(\sigma)\cap f^{-1}(X-\Omega)\}$ is the same as the set $\{U(\sigma_0)\cap f^{-1}(X-\Omega),\ L(\sigma_0)\cap f^{-1}(X-\Omega)\}$.

LEMMA 2. *Let σ and σ_0 be 1-chains on X which satisfy $[\sigma]_2=[\sigma_0]_2 \pmod\beta$. Then $\tilde{X}(\sigma)$ and $\tilde{X}(\sigma_0)$ have the same sheet structure if and only if $[\sigma]_2=[\sigma_0]_2$.*

(Let us now determine the number of covering surfaces $\tilde{X}(\sigma)$ and the number of sheet structures for a given surface X. Fix a 1-chain σ on X. Let $H_1(\beta(X),\mathbf{Z}_2)$ denote the subgroup of $H_1(X,\mathbf{Z}_2)$ generated by the dividing cycles. The homology group modulo dividing cycles, or the relative homology group with respect to the ideal boundary, is $H_1(X,\mathbf{Z}_2)/H_1(\beta(X),\mathbf{Z}_2)$. We have seen that there is a one-to-one correspondence between covering surfaces $\tilde{X}(\sigma_0)$ derived from 1-chains σ_0 with $\partial_2\sigma_0=\partial_2\sigma$ and elements of $H_1(X,\mathbf{Z}_2)/H_1(\beta(X),\mathbf{Z}_2)$. Each of these covering surfaces can generally be partitioned into sheets in many non-equivalent ways. Lemma 2 shows that for a fixed covering surface $\tilde{X}$ there is a one-to-one correspondence between $H_1(\beta(X),\mathbf{Z}_2)$ and partitions of $\tilde{X}$ into two sheets. In case X is compact, there is just one partition of each covering surface $\tilde{X}$ into two sheets. If X is planar, then all 1-chains σ_0 with $\partial_2\sigma_0=\partial_2\sigma$ give rise to the same covering surface; however, there are 2^{n-1} ways to partition it into two sheets when X has connectivity n. Not every two-sheeted covering surface can be obtained in this fashion because all of these have an even number of branch points and there exist two-sheeted coverings with an odd number of branch points.)

If X is an open surface, let β denote the one-point ideal boundary. This one-point compactification of X naturally induces a two-point compactification of $\tilde{X}$; the ideal points are denoted by $\tilde{\beta}_0$ and $\tilde{\beta}_1$. The compactification is defined by stipulating that a sequence $\{\tilde{P}_n\}$ on $\tilde{X}$ converges to $\tilde{\beta}_0$ (respectively, $\tilde{\beta}_1$) if $f(\tilde{P}_n) \to \beta$ and if $\{\tilde{P}_n\}$ is eventually on the lower (respectively, upper) sheet of $\tilde{X}$. The compactification is dependent on the $\mathbf{Z}_2$-homology class of σ. The function f extends to a continuous function from the two-point compactification of $\tilde{X}$ to the one-point compactification of X and the cover transformation T interchanges $\tilde{\beta}_0$ and $\tilde{\beta}_1$.

4. The Function ũ

In this section we shall assume that $X \notin O_G$. Let $\{\Omega\}$ be an exhaustion of X such that $\sigma \subset \Omega$ for each Ω. Let $\beta_\Omega = \partial\Omega$ and let $\tilde{\beta}_{\Omega 0}$ and $\tilde{\beta}_{\Omega 1}$ be the lifts of β_Ω to the lower sheet and upper sheet, respectively. Suppose $\tilde{u}_\Omega$ is the harmonic function on $f^{-1}(\Omega)$ determined by the boundary values

$$\tilde{u}_\Omega(\tilde{P}) = j \quad \text{if} \quad \tilde{P} \in \tilde{\beta}_{\Omega_j} \quad (j = 0, 1). \tag{4}$$

By the maximum principle $\tilde{u}_\Omega$ satisfies

$$\tilde{u}_\Omega(\tilde{P}) + \tilde{u}_\Omega(T(\tilde{P})) \equiv 1 \tag{5}$$

for all $\tilde{P} \in f^{-1}(\Omega)$. As $\Omega \to X$, $\tilde{u}_\Omega$ converges, uniformly on compacta, to a nonconstant harmonic limit function $\tilde{u}$; $\tilde{u} = \lim_{\tilde{\Omega} \to \tilde{X}} \tilde{u}_{\tilde{\Omega}}$ and $D_{\tilde{X}}(\tilde{u}) = \lim_{\tilde{\Omega} \to \tilde{X}} D_{\tilde{\Omega}}(\tilde{u}_{\tilde{\Omega}})$, where $\tilde{\Omega} = f^{-1}(\Omega)$, $D_{\tilde{X}}(\tilde{u})$ is the Dirichlet integral of $\tilde{u}$ over $\tilde{X}$ and $\tilde{u}_{\tilde{\Omega}} = \tilde{u}_\Omega$ is given by (4). The symmetry property (5) continues to hold for the limit function:

$$\tilde{u}(\tilde{P}) + \tilde{u}(T(\tilde{P})) \equiv 1 \tag{6}$$

for all $\tilde{P} \in \tilde{X}$.

$\tilde{u}$ depends upon the $\mathbf{Z}_2$-homology class of σ and the choice of base point. If a different base point were selected, then either the same function $\tilde{u}$ would be obtained or else $1 - \tilde{u} = \tilde{u} \circ T$ would be produced. Both these functions have the same Dirichlet integral.

5. The Curve Families $\mathscr{F}$, $\mathscr{F}^*$

A (general) crosscut on a Riemann surface X is an open arc $\delta: (0, 1) \to X$ such that $\delta(t) \to \beta$ as $t \to 0$ and as $t \to 1$. If X is the interior of a compact bordered surface, there is a classical notion of crosscut which is slightly different. The classical definition requires points Q_0 and Q_1 on the border of X such that $\lim_{t\to j}\delta(t)=Q_j$ for $j=0, 1$. The classical definition is not conformally invariant with respect to conformal mappings of the interior of X. Given a family of general crosscuts on a compact bordered surface, the subfamily of those which are not classical has infinite extremal length. Therefore, the nonclassical crosscuts can be discarded at any stage without changing extremal lengths.

An open subset Ω of X is called a Jordan open set if Ω is relatively compact and its border consists of a finite number of Jordan curves. We shall be interested in the extremal lengths of the following curve families on $X'=X-\partial\sigma$:

$\mathscr{F}$: all crosscuts δ on X such that $\delta\times\sigma\equiv 1 \pmod 2$.
$\mathscr{F}^*$: all $\partial\Omega$, where Ω is a Jordan open subset of X and Ω contains a 1-chain σ_Ω satisfying $[\sigma_\Omega]_2=[\sigma]_2$.

In order to investigate $\mathscr{F}$ and $\mathscr{F}^*$ we introduce two related curve families on $\tilde{X}$:

$\tilde{\mathscr{F}}$: all open arcs on $\tilde{X}$ which tend to $\tilde{\beta}_0$ in one direction and tend to $\tilde{\beta}_1$ in the other direction.
$\tilde{\mathscr{F}}^*$: all curves on $\tilde{X}$ which separate $\tilde{\beta}_0$ and $\tilde{\beta}_1$.

We may apply a well known extremal length theorem (cf. Ahlfors-Sario [1], Marden-Rodin [1], Strebel [1]) to these curve families on $\tilde{X}$ to obtain the following result.

LEMMA 3. *$\lambda(\tilde{\mathscr{F}}^*)=\lambda^{-1}(\tilde{\mathscr{F}})=D_{\tilde{X}}(\tilde{u})$. If $X\in O_G$, then $D_{\tilde{X}}(\tilde{u})=0$; otherwise $X\notin O_G$ and $0<D_{\tilde{X}}(\tilde{u})<\infty$.*

Our goal is to employ Lemma 3 in deriving a corresponding result on X for the curve families $\mathscr{F}$ and $\mathscr{F}^*$. Two of the simplest tools for relating extremal length on $\tilde{X}$ to extremal length on X are given in Lemmas 4 and 5.

LEMMA 4. *Let $\mathscr{G}$ and $\tilde{\mathscr{G}}$ be curve families on X and $\tilde{X}$ respectively. If $f(\tilde{\mathscr{G}})\subset\mathscr{G}$, then $\lambda(\mathscr{G})\leqslant 2\lambda(\tilde{\mathscr{G}})$.*

LEMMA 5. *Let $\mathscr{G}$ and $\tilde{\mathscr{G}}$ be curve families on X and $\tilde{X}$, respectively, such that the extremal metric for $\tilde{\mathscr{G}}$ is T-invariant. If every $\gamma\in\mathscr{G}$ can be lifted to a $\tilde{\gamma}\in\tilde{\mathscr{G}}$, then $\lambda(\mathscr{G})\geqslant 2\lambda(\tilde{\mathscr{G}})$.*

It is easy to see that the families $\mathscr{F}$ and $\tilde{\mathscr{F}}$ satisfy the hypotheses of both Lemmas 4 and 5. We first note that any $\tilde{\delta}\in\tilde{\mathscr{F}}$ goes from one sheet of $\tilde{X}$ to the other; therefore, $f(\tilde{\delta})$ crosses σ an odd number of times. Hence $f(\tilde{\mathscr{F}})\subset\mathscr{F}$. Conversely, if $\delta\in\mathscr{F}$, then both lifts of δ belong to $\tilde{\mathscr{F}}$. Furthermore, the extremal metric for $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{F}}^*$ is

$\widetilde{ds}_0 = |d\tilde{u} + i * d\tilde{u}|$ which is T-invariant. These considerations immediately give us

$$\lambda(\mathscr{F}) = 2\lambda(\tilde{\mathscr{F}}) = 2D_{\tilde{X}}^{-1}(\tilde{u}). \tag{7}$$

Next, we wish to establish a corresponding result for $\mathscr{F}^*$ and $\tilde{\mathscr{F}}^*$.

The families $\mathscr{F}^*$ and $\tilde{\mathscr{F}}^*$ satisfy the lifting property required in Lemma 5. To see this, let $\gamma^* = \partial\Omega \in \mathscr{F}^*$, where Ω is a Jordan open subset of X and contains an element σ_Ω of $[\sigma]_2$. Take Ω_0 to be the union of all noncompact components of $X - Cl\,\Omega$; then Ω_0 is a neighborhood of the ideal boundary β of X. Consider the upper and lower sheets of $\tilde{X}$ defined relative to σ_Ω; these sheets are equivalent to those defined by σ. Ω_0 can be lifted homeomorphically to the lower sheet of $\tilde{X}$; denote the lifted set by $\tilde{\Omega}_0$. Then $\tilde{\Omega}_0$ is a neighborhood of $\tilde{\beta}_0$ and $\tilde{\gamma}_0^* = \partial\tilde{\Omega}_0$ is an element of $\tilde{\mathscr{F}}^*$. Since $\tilde{\gamma}_0^*$ is a lift of some of the contours of γ^* (the lifts of the remaining contours are not relevant), we conclude that γ^* can be lifted to an element in $\tilde{\mathscr{F}}^*$. Lemma 5 now implies that

$$\lambda(\mathscr{F}^*) \geqslant 2\lambda(\tilde{\mathscr{F}}^*) = 2D_{\tilde{X}}(\tilde{u}). \tag{8}$$

We cannot use Lemma 4 to prove the opposite inequality because $\mathscr{F}^*$ and $\tilde{\mathscr{F}}^*$ do not satisfy the projection hypothesis of that lemma. If $\mathscr{F}^*$ could be modified so that the new family satisfied the hypotheses of both Lemmas 4 and 5, then the remaining proof would be easier. We have been unable to find a reasonable modification of this sort. (Of course, one could take $\mathscr{F}^*$ to be the family $f(\tilde{\mathscr{F}}^*)$, but this device would not be useful for applications unless we also possessed an intrinsic description of $f(\tilde{\mathscr{F}}^*)$ directly in terms of X.) Instead, we shall prove the desired inequality for $\lambda(\mathscr{F}^*)$ by investigating the level lines of $\tilde{u}$.

6. Level Lines of ũ

In this section we assume that X is the interior of a compact bordered surface. Then $\tilde{X}$ is also the interior of a compact bordered surface and each level line $\tilde{\tau}_k = \{\tilde{P} \in \tilde{X} : \tilde{u}(\tilde{P}) = k\}$, $0 < k < 1$, is a finite cycle on $\tilde{X}$. Orient $\tilde{\tau}_k$ so that $\tilde{u}$ is less than k to the left of $\tilde{\tau}_k$.

The cover transformation T sends $\tilde{\tau}_k$ to $-\tilde{\tau}_{1-k}$. For $\frac{1}{2} < k < 1$ define τ_k to be $f(\tilde{\tau}_k)$; τ_k is a cycle on $X' = X - \partial\sigma$. The case $k = \frac{1}{2}$ is exceptional; $\tilde{\tau}_{1/2}$ projects to a 1-chain traced once in each direction. We shall define the projection $\tau_{1/2}$ as follows. Observe that all of the branch points of $\tilde{X}$ belong to $\tilde{\tau}_{1/2}$. First, express $\tilde{\tau}_{1/2}$ as a sum $\tilde{\alpha}_0 + \tilde{\alpha}_1$, where $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ are 1-chains which satisfy $T\tilde{\alpha}_0 = -\tilde{\alpha}_1$ ($\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ are not uniquely determined by these conditions). Set $\tau_{1/2} = f(\tilde{\alpha}_0)$; note that the orientation of $\tau_{1/2}$ is ambiguous. This will not matter, however, since we shall be using $\mathbf{Z}_2$-homology.

LEMMA 6. *If X is the interior of a compact bordered Riemann surface, then* $\tau_{1/2} \in [\sigma]_2$.

Proof. We first note that σ and $\tau_{1/2}$ have the same $\mathbf{Z}_2$-boundary. The only possible boundary points of $\tau_{1/2}$ are at the projections P_i of the branch points $\tilde{P}_i$ of $\tilde{X}$. Suppose $\tilde{P}_i$ is a critical point of $\tilde{u}$ of multiplicity n ($n=1$ means that $\tilde{u}$ is regular at $\tilde{P}_i$). One can prove that n must be odd. Hence, P_i is in the $\mathbf{Z}_2$-boundary of $\tau_{1/2}$.

We now demonstrate that $[\sigma]_2 = [\tau_{1/2}]_2$ by showing that $\delta \times \sigma \equiv \delta \times \tau_{1/2} \pmod 2$ for all crosscuts δ on X. Let δ be a crosscut of X parametrized on the interval (0, 1). By deforming δ if necessary we may assume that it intersects $\tau_{1/2}$ in a finite number of transversal crossings. Let $\tilde{\delta}$ be a fixed lift of δ; we can assume that $\lim_{t\downarrow 0}\tilde{\delta}(t) = \tilde{\beta}_0$. Now, $\delta \times \sigma$ is even or odd according to whether $\lim_{t\uparrow 1}\tilde{\delta}(t) = \tilde{\beta}_0$ or $\lim_{t\uparrow 1}\tilde{\delta}(t) = \tilde{\beta}_1$. If $\lim_{t\uparrow 1}\tilde{\delta}(t) = \tilde{\beta}_0$, then $\tilde{u}$ has the value 0 at both ends of $\tilde{\delta}$ so that $\tilde{u}(\tilde{\delta}(t))$ takes the value 1/2 an even number of times for $0<t<1$. This means that $\tilde{\delta} \times \tilde{\tau}_{1/2}$ is even. On the other hand, if $\lim_{t\uparrow 1}\tilde{\delta}(t) = \tilde{\beta}_1$, then $\tilde{u}$ has different values at the endpoints of $\tilde{\delta}$, and $\tilde{u}(\tilde{\delta}(t))$ assumes the value 1/2 an odd number of times. In this case $\tilde{\delta} \times \tilde{\tau}_{1/2}$ is odd. Thus $\delta \times \sigma$ and $\tilde{\delta} \times \tilde{\tau}_{1/2}$ have the same parity. The number of crossings $\tilde{\delta} \times \tilde{\tau}_{1/2}$ is the same as the number of crossings $\delta \times \tau_{1/2}$ (distinct crossing points of $\tilde{\delta}$ and $\tilde{\tau}_{1/2}$ may correspond to multiple crossing points of δ and $\tau_{1/2}$). Therefore, $\delta \times \sigma$ and $\delta \times \tau_{1/2}$ have the same parity.

LEMMA 7. *Let X be the interior of a compact bordered Riemann surface. For* $1/2<k<1$ *define*

$$\tilde{\Omega}_k = \{\tilde{P} \in \tilde{X} : 1-k < \tilde{u}(\tilde{P}) < k\}, \qquad \Omega_k = f(\tilde{\Omega}_k).$$

Then Ω_k is a relatively compact open subset of X, τ_k is its border, and $\tau_{1/2} \subset \Omega_k$.

From Lemmas 6 and 7 we obtain the following result.

LEMMA 8. *If X is the interior of a compact bordered Riemann surface, then* $\tau_k \in \mathscr{F}^*$ *for each* k, $1/2<k<1$.

7. Extremal Length Relations on Open Surfaces

Because of the symmetry (6), $\tilde{u}$ corresponds to a 2-valued harmonic function on X. We can choose a single-valued branch on $X - \tau_{1/2}$. Specifically, for $P \in X - \tau_{1/2}$, let $u(P)$ be the larger of the two values $\tilde{u}(f^{-1}(P))$. u is a harmonic function on $X - \tau_{1/2}$. If X is the interior of a compact bordered Riemann surface, then u is determined by its boundary values $u(P) = 1/2$ if $P \in \tau_{1/2}$, and $u(P) = 1$ if $P \in \beta$. Its Dirichlet integral and flux are related by

$$\tfrac{1}{2}D_{\tilde{X}}(\tilde{u}) = D_X(u) = \tfrac{1}{2}\int_{\beta} * du = \tfrac{1}{2}\int_{\tau_k} * du \qquad (\tfrac{1}{2}<k<1).$$

There is a standard method to obtain an upper bound for extremal length by using Schwarz' inequality. We omit the finitely many τ_k's which pass through a critical point of u and use u to form the real part of a local parameter on the complement of the set of critical points. If $\varrho(z)\,|dz|$ is any linear density on X, then, because $\tau_k \in \mathscr{F}^*$ (Lemma 8), we obtain $L^2(\mathscr{F}^*, \varrho)/A(X, \varrho) \leqslant 4D_X(u) = 2D_{\tilde{X}}(\tilde{u})$, and hence,

$$\lambda(\mathscr{F}^*) \leqslant 2D_{\tilde{X}}(\tilde{u}). \tag{9}$$

We collect the results (7), (8), (9) and obtain:

LEMMA 9. *On any open Riemann surface X, $\lambda(\mathscr{F}^*) \geqslant 2D_{\tilde{X}}(\tilde{u}) = 4\lambda^{-1}(\mathscr{F})$. If X is the interior of a compact bordered Riemann surface, then $\lambda(\mathscr{F}^*) = 2D_{\tilde{X}}(\tilde{u})$. $D_{\tilde{X}}(\tilde{u})$ vanishes if and only if $X \in O_G$.*

We can now state a preliminary form of the extremal length theorem.

PROPOSITION 1. *If X is an open Riemann surface and σ a nontrivial 1-chain on X, then $\lambda(\mathscr{F}^*) = 4\lambda^{-1}(\mathscr{F}) = 2D_{\tilde{X}}(\tilde{u})$.*

Proof. In case X is the interior of a compact bordered Riemann surface, then this is an immediate consequence of Lemma 9. If X is not the interior of a compact bordered Riemann surface, let $\{\Omega_n\}$ be a regular exhaustion of X such that $\sigma \subset \Omega_n$ for all n, each Ω_n is the interior of a compact bordered surface and the sequence $\{\Omega_n\}$ is increasing. Suppose $\tilde{\Omega}_n$ is the covering surface of Ω_n determined by σ. Let $\mathscr{F}_n$ and $\mathscr{F}_n^*$ be the analogous curve families defined for Ω_n; then $\lambda(\mathscr{F}_n^*) = 4\lambda^{-1}(\mathscr{F}_n) = 2D_{\tilde{\Omega}_n}(\tilde{u}_n)$.

Since $\{\Omega_n\}$ is an increasing sequence, the curve families $\mathscr{F}_n^*$ also increase with n. Moreover, $\mathscr{F}^* = \bigcup \mathscr{F}_n^*$ so $\lim \lambda(\mathscr{F}_n^*) = \lambda(\mathscr{F}^*)$. In our construction of $\tilde{u}$ we noted that $\lim D_{\tilde{\Omega}_n}(\tilde{u}_n) = D_{\tilde{X}}(\tilde{u})$. Therefore, $\lambda(\mathscr{F}^*) = 2D_{\tilde{X}}(\tilde{u})$.

The proof that $2\lambda^{-1}(\mathscr{F}) = D_X(\tilde{u})$ is more involved. Every crosscut δ in $\mathscr{F}_{n+1}$ contains a subarc in $\mathscr{F}_n$, so $\lambda(\mathscr{F}_n) \leqslant \lambda(\mathscr{F}_{n+1})$ for all n. Also, each crosscut in $\mathscr{F}$ contains a subarc in $\mathscr{F}_n$ so $\lambda(\mathscr{F}_n) \leqslant \lambda(\mathscr{F})$. Thus, $\lim \lambda(\mathscr{F}_n) \leqslant \lambda(\mathscr{F})$. The opposite inequality may be proved by making use of an extremal length technique attributed to Beurling and first developed by Wolontis [1]. The same method has been used and extended by others (Marden-Rodin [1], Minda [1, 2], Strebel [1], Suita [1]). The technique is very topological; crosscuts from the families $\mathscr{F}_n$ must be pieced together to form a crosscut of X belonging to $\mathscr{F}$. This process will show that $\lim \lambda(\mathscr{F}_n) \geqslant \lambda(\mathscr{F})$. Because the proof is very long and is so similar to the proofs given in the preceding references, especially Strebel [1], it is omitted here. The proof is then completed by observing $\lambda(\mathscr{F}) = \lim \lambda(\mathscr{F}_n) = \lim 2D_{\tilde{\Omega}}^{-1}(\tilde{u}_n) = 2D_{\tilde{X}}^{-1}(\tilde{u})$.

8. Extremal Regions and Quadratic Differentials on Open Surfaces

In this section we discuss extremal properties of the level curve $\tau_{1/2}$ and of a quadratic differential Φ_σ on X which is derived from the function $\tilde{u}$. Proposition 2 is an extension of the Grötzsch Extremal Region Theorem. Proposition 4 characterizes the quadratic differential Φ_σ determined by a 1-chain σ. This result provides a general setting in which to extend the results of Hersch [1] concerning certain elliptic functions and their relations to extremal length problems in a disk.

The extremal problems and uniqueness properties that we obtain refer only to σ and X, not to $\tilde{X}$. As a consequence, we shall be able to reformulate Proposition 1 in terms that are intrinsic to X (see Section 9, Theorem 1).

By the harmonic measure of a closed subset $\alpha \subset X$ is meant the harmonic function ω_α on $X-\alpha$ with boundary values 1 on α and 0 on β. The extremal distance from α to β is denoted by $\lambda(\alpha, \beta)$. It is well-known that $\lambda(\alpha, \beta)=D_X^{-1}(\omega_\alpha)$.

PROPOSITION 2. *Let X be the interior of a compact bordered Riemann surface and let σ be a nontrivial 1-chain on X. Among all 1-chains which are $\mathbf{Z}_2$-homologous to σ there is a unique one σ_0 which has the greatest extremal distance from β. Specifically, σ_0 is the level curve $\tau_{1/2}$ of u; the harmonic measure of σ_0 is $\omega_{\sigma_0}=2(1-u)$. We have*

$$\lambda(\tau_{1/2}, \beta)=\max\{\lambda(\alpha, \beta): \alpha \in [\sigma]_2\}, \tag{10}$$

$$D_X(\omega_{\tau_{1/2}})=\min\{D_X(\omega_\alpha): \alpha \in [\sigma]_2\}, \tag{11}$$

$$\lambda(\tau_{1/2}, \beta)=D_X^{-1}(\omega_{\tau_{1/2}})=\tfrac{1}{4}D_X^{-1}(u)=\tfrac{1}{2}D_{\tilde{X}}^{-1}(\tilde{u}). \tag{12}$$

Proof. Given $\alpha \in [\sigma]_2$ form $\tilde{X}=\tilde{X}(\sigma)=\tilde{X}(\alpha)$ and $\tilde{u}$. Lift the harmonic measure ω_α to $\tilde{X}$ by defining $\tilde{\omega}_\alpha(\tilde{P})=\omega_\alpha(P)$ if $\tilde{P}$ is on the lower sheet of $\tilde{X}(\alpha)$ and $\tilde{P}$ lies over P, and define $\tilde{\omega}_\alpha(T(\tilde{P}))=2-\tilde{\omega}_\alpha(\tilde{P})$. Set $\tilde{\omega}_\alpha(\tilde{P})=1$ if $\tilde{P}$ lies over α. The harmonic function $\tilde{u}$ on $\tilde{X}$ has the same boundary values as the piecewise harmonic function $\frac{1}{2}\tilde{\omega}_\alpha$. From Dirichlet's principle we obtain $D_{\tilde{X}}(\tilde{u}) \leqslant D_{\tilde{X}}(\frac{1}{2}\tilde{\omega}_\alpha)$ and equality holds if and only if $\tilde{u}=\frac{1}{2}\tilde{\omega}_\alpha$. Since $\tau_{1/2} \in [\sigma]_2$ (Lemma 6) and $\tilde{u}=\frac{1}{2}\tilde{\omega}_{\tau_{1/2}}$, it follows that $D_X(\omega_{\tau_{1/2}}) \leqslant D_X(\omega_\alpha)$ and equality holds if and only if $\omega_\alpha=\omega_{\tau_{1/2}}$. Necessary and sufficient for $\omega_\alpha=\omega_{\tau_{1/2}}$ is that α and $\tau_{1/2}$ coincide as $\mathbf{Z}_2$-chains. This establishes (10), (11) and (12).

We now consider an arbitrary open Riemann surface X, not necessarily the interior of a compact bordered surface. We shall need an extremal length characterization of harmonic measures in this general situation. Let β_j be a closed subset of the ideal boundary β of X. Here β is regarded as the Kerékjártó-Stoilow ideal boundary of X. We say that β_j has *positive capacity* if the extremal distance from β_j to a fixed continuum in X is finite.

Suppose β is partitioned into two disjoint nonempty closed subsets β_0 and β_1. The harmonic measure of β_1 is the harmonic function on X which has boundary values 0 on β_0 and 1 on β_1. (The precise definition refers to a limit of the corresponding harmonic measures on an exhaustion of X.) The following characterization of the harmonic measure will be used in the proof of Proposition 4.

PROPOSITION 3. *Let β_0, β_1 be a partition of the ideal boundary of X into disjoint nonempty closed sets of positive capacity. The harmonic measure of β_1 is uniquely determined up to an additive constant as the harmonic function ω on X which satisfies the conditions:*

(i) $D_X(\omega)<\infty$,
(ii) $\int_\delta d\omega=1$ *for almost all crosscuts δ from β_0 to β_1.*

The constructions that we have developed provide a method of associating a quadratic differential $\Phi=\phi(z)\,dz^2$ to a 1-chain σ. Given σ, form $\tilde{X}$ and $\tilde{u}$. Equation (6) shows that $(d\tilde{u}+i*d\tilde{u})^2$ is a T-invariant holomorphic quadratic differential on $\tilde{X}$. The invariance means that it can be transferred to X. In this way we obtain a (meromorphic) quadratic differential $\Phi=\phi(z)\,dz^2$ on X; $\Phi=(du+i*du)^2$ on $X-\sigma$. To indicate the dependence on σ we shall sometimes write $\Phi=\Phi_\sigma=\phi_\sigma(z)\,dz^2$.

On X', ϕ is locally the square of an analytic function. Hence Φ has no poles on X', and all of its zeros there are of even order. At a branch point projection $P\in\partial\sigma$, Φ will have a simple pole if $\tilde{P}=f^{-1}(P)$ is not a critical point of $\tilde{u}$; otherwise, Φ will have a zero of odd order at P. The norm of $\Phi=\phi(z)\,dz^2$ is defined as $\iint_X|\phi(z)|\,dx\,dy=\|\Phi\|$ and we see that $\|\Phi\|=D_X(u)=\frac{1}{2}D_{\tilde{X}}(\tilde{u})$. Furthermore, $\operatorname{Re}\int\sqrt{\phi(z)}\,dz=\int\pm du$.

PROPOSITION 4. *Let σ be a nontrivial 1-chain on a Riemann surface $X\notin O_G$. There is a unique quadratic differential $\Phi=\phi(z)\,dz^2$ on X which satisfies the following conditions:*

(i) $\|\Phi\|<\infty$,
(ii) *There is a germ $\sqrt{\phi(z)}\,dz$ on X' which can be continued along all paths on X' and the continuation satisfies*
  (a) $\operatorname{Re}\int_\gamma\sqrt{\phi(z)}\,dz=0$ *if $\gamma\in\pi_1(X')$ and $\gamma\times\sigma\equiv0\ (\mathrm{mod}\,2)$,*
  (b) $\operatorname{Re}\int_\delta\sqrt{\phi(z)}\,dz=1$ *for almost all crosscuts δ such that $\delta\times\sigma$ is an odd positive integer.*

Proof. The quadratic differential Φ_σ satisfies properties (i), (ii)-a and (ii)-b. To verify the uniqueness, consider another such quadratic differential Φ. Let $w(P)=\operatorname{Re}\int_Q^P\sqrt{\phi(z)}\,dz$. Property (ii)-a shows that w lifts to a single-valued harmonic function $\tilde{w}$ on $\tilde{X}'$. Property (i) shows that $D_{\tilde{X}'}(\tilde{w})<\infty$. Therefore, $\tilde{w}$ has removable singularities at the branch points; we can consider $\tilde{w}$ as a Dirichlet-finite harmonic function on $\tilde{X}$. Property (ii)-b implies that $\int_{\tilde{\delta}}d\tilde{w}=1$ for almost all crosscuts $\tilde{\delta}$ on $\tilde{X}$ which join $\hat{\beta}_0$ to $\hat{\beta}_1$. By Proposition 3, $d\tilde{w}=d\tilde{u}$. Consequently $\Phi=\Phi_\sigma$.

The orthogonal trajectories of a quadratic differential $\phi(z)\,dz^2$ are the solutions of $\phi(z)\,dz^2<0$. The orthogonal trajectories of Φ_σ are therefore the level curves of u. If X is the interior of a compact bordered Riemann surface, then these trajectories are $\tau_k \in \mathscr{F}^*$ $(0<k<\frac{1}{2})$, as well as the special trajectory $\tau_{1/2} \in [\sigma]_2$. We therefore have the following result.

LEMMA 10. *The properties* (i) *and* (ii) *of Proposition* 4 *uniquely determine* Φ_σ. *If X is the interior of a compact bordered Riemann surface, then there is a unique set σ_0 of orthogonal trajectories of Φ_σ such that $\sigma_0 \in [\sigma]_2$; $X-\sigma_0$ is the extremal region of Proposition* 2.

9. The Main Theorem for Open Surfaces

The two-sheeted covering surface $\tilde{X}(\sigma)$ and the associated function $\tilde{u}$ have played an auxiliary role in our results. They were needed as a tool in the proofs; the final result can now be stated without reference to them.

THEOREM 1. *Let σ be a nontrivial* 1*-chain on the hyperbolic Riemann surface X. There is a quadratic differential Φ_σ on X which is uniquely determined by properties* (i) *and* (ii) *of Proposition* 4. *Let $\mathscr{F}$ and $\mathscr{F}^*$ be the curve families defined in Section* 5. *Then*

$$\lambda(\mathscr{F}^*)=4\lambda^{-1}(\mathscr{F})=4\,\|\Phi_\sigma\|\,.$$

If X is the interior of a compact bordered surface, then there is a collection σ_0 of orthogonal trajectories of Φ_σ such that σ_0 is $\mathbf{Z}_2$-homologous to σ. σ_0 has the greatest extremal distance from β among all l-chains which are $\mathbf{Z}_2$-homologous to σ. On $X-\sigma_0$ we have $2\,\mathrm{Re}\,\sqrt{\Phi_\sigma}=d\omega$, where ω is the harmonic measure of σ_0 and where the square root with positive real part is selected.

10. Reduced Extremal Length Relations on Closed Surfaces

Throughout this section we assume that X is a closed Riemann surface. Suppose σ is a 1-chain on X and $\tilde{X}$ is the two-sheeted covering surface of X determined by σ. $\tilde{X}$ is also a closed Riemann surface. Fix a point $P \in X-\sigma$. In this situation we are interested in the following curve families:

$\mathscr{F}_P$: all crosscuts δ on $X-P$ such that $\delta\times\sigma\equiv 1 \pmod 2$.
$\mathscr{F}_P^*$: all $\partial\Omega$, where Ω is a Jordan open subset of $X-P$ and Ω contains a 1-chain σ_Ω satisfying $[\sigma_\Omega]_2=[\sigma]_2$.

We are concerned with the reduced extremal length of the family $\mathscr{F}_P$ and the reduced modulus of the family $\mathscr{F}_P^*$. To aid in our study of these families we define two related curve families on $\tilde{X}$. Let $\tilde{P}_0$ and $\tilde{P}_1$ be the two distinct points on $\tilde{X}$ lying over P.

$\mathscr{F}_P$: all arcs on $\tilde{X}$ which join $\tilde{P}_0$ to $\tilde{P}_1$.
$\mathscr{F}_P^*$: all closed curves on $\tilde{X}$ which separate $\tilde{P}_0$ and $\tilde{P}_1$.

Let us recall the definitions of reduced extremal length and reduced modulus (cf. Ahlfors-Beurling [1, 2], Minda [1, 2]). Suppose that $\tilde{z}_j$ is a local coordinate defined in a neighborhood of $\tilde{P}_j$ such that $\tilde{z}_j(\tilde{P}_j)=0$ $(j=0, 1)$. For $r>0$ and sufficiently small, let $\tilde{B}_j(r)=\{\tilde{Q}\in\tilde{X}:|\tilde{z}_j(\tilde{Q})|\leq r\}$ be the closed disk of radius r centered at $\tilde{P}_j$ and let $\tilde{\alpha}_j(r)=\{\tilde{Q}\in\tilde{X}:|\tilde{z}_j(\tilde{Q})|=r\}$ be its boundary. If $r_0, r_1>0$ are sufficiently small, set $\tilde{X}(r_0, r_1)=\tilde{X}-(B_0(r_0)\cup B_1(r_1))$; $\tilde{X}(r_0, r_1)$ is the interior of a compact bordered Riemann surface. Define

$\mathscr{F}_P(r_0, r_1)$: all arcs on $\tilde{X}(r_0, r_1)$ which connect $\tilde{\alpha}_0(r_0)$ to $\tilde{\alpha}_1(r_1)$,
$\mathscr{F}_P^*(r_0, r_1)$: all closed curves on $\tilde{X}(r_0, r_1)$ which separate $\tilde{\alpha}_0(r_0)$ from $\tilde{\alpha}_1(r_1)$.

The quantity $\lambda(\mathscr{F}_P(r_0, r_1))+(1/2\pi)\log(r_0r_1)$ increases if either r_0 or r_1 decreases;

$$\tilde{\lambda}(\mathscr{F}_P)=\lim_{(r_0, r_1)\to(0,0)}\left[\lambda(\mathscr{F}_P(r_0, r_1))+\frac{1}{2\pi}\log(r_0r_1)\right]$$

is called the reduced extremal length of the family $\mathscr{F}_P$. The number $\tilde{\lambda}(\mathscr{F}_P)$ depends upon the choice of local coordinates at $\tilde{P}_0$ and $\tilde{P}_1$ in such a way that $\exp[-2\pi\tilde{\lambda}(\mathscr{F}_P)]$ $|d\tilde{z}_0|\,|d\tilde{z}_1|$ is an invariant form. Let $M(\mathscr{G})$ denote the modulus of a curve family $\mathscr{G}$, i.e., the reciprocal of the extremal length. The expression $M(\mathscr{F}_P^*(r_0, r_1))+1/(2\pi)$ $\log(r_0r_1)$ increases whenever either r_0 or r_1 decreases;

$$\tilde{M}(\mathscr{F}_P^*)=\lim_{(r_0, r_1)\to(0,0)}\left[M(\mathscr{F}_P^*(r_0, r_1))+\frac{1}{2\pi}\log(r_0r_1)\right]$$

is called the reduced modulus of the family $\mathscr{F}_P^*$. As before, $\exp[-2\pi\tilde{M}(\mathscr{F}_P^*)]$ $|d\tilde{z}_0|$ $|d\tilde{z}_1|$ is an invariant form.

For fixed values of r_0 and r_1, Lemma 3 implies that $\lambda(\mathscr{F}_P(r_0, r_1))=M(\mathscr{F}_P^*(r_0, r_1))$. Therefore, $\tilde{\lambda}(\mathscr{F}_P)=\tilde{M}(\mathscr{F}_P^*)$ whenever both quantities are computed with respect to the same pair of local coordinates. These reduced quantities can be expressed in terms of a canonical harmonic function. Let $\tilde{v}$ be a harmonic function on $\tilde{X}$ with a positive logarithmic singularity at $\tilde{P}_1$ and a negative logarithmic singularity at $\tilde{P}_0$; that is, $\tilde{v}(\tilde{z}_j)+(-1)^{j+1}\log|\tilde{z}_j|$ has a removable singularity at $\tilde{z}_j=0$. The function $\tilde{v}$ is only determined up to an additive constant; however, $\tilde{v}$ is uniquely specified if we require that $\tilde{v}\circ T=-\tilde{v}$. We shall assume that this is true. Then $\lambda(\mathscr{F}_P(r, r))=-(1/\pi)\log r$ $=(1/2\pi)^2 D_{\tilde{X}(r,r)}(\tilde{v})$, which gives $\tilde{\lambda}(\mathscr{F}_P)=(1/2\pi)^2\tilde{D}_{\tilde{X}}(\tilde{v})$, where $\tilde{D}_{\tilde{X}}(\tilde{v})$ is the reduced Dirichlet integral of $\tilde{v}$ defined by

$$\tilde{D}_{\tilde{X}}(\tilde{v})=\lim_{(r_0, r_1)\to(0,0)}[D_{\tilde{X}(r_0, r_1)}(\tilde{v})+2\pi\log(r_0r_1)].$$

That $\tilde{\lambda}(\mathscr{F}_P)=(1/2\pi)^2 D_{\tilde{X}}(\tilde{v})$ is a special case of a known result (Minda [2, Theorem 3]). Clearly, $\exp[-(1/2\pi)\tilde{D}_{\tilde{X}}(\tilde{v})]\,|d\tilde{z}_0|\,|d\tilde{z}_1|$ is an invariant form.

The reduced extremal length $\tilde{\lambda}(\mathscr{F}_P)$ and the reduced modulus $\tilde{M}(\mathscr{F}_P^*)$ are defined as follows. Let z be a local coordinate defined in a neighborhood of P, $B(r)$ the closed disk of radius r centered at P and $\alpha(r)$ the boundary of this disk. Set $X(r)=X-B(r)$ and

$\mathscr{F}_P(r)$: all crosscuts δ on $X(r)$ such that $\delta\times\sigma\equiv 1 \pmod 2$,
$\mathscr{F}_P^*(r)$: all $\partial\Omega$, where Ω is a Jordan open subset of $X(r)$ and Ω contains a 1-chain σ_Ω satisfying $[\sigma_\Omega]_2=[\sigma]_2$.

The quantity $\lambda(\mathscr{F}_P(r))+(4/2\pi)\log r$ (note the presence of the factor 4 in place of 1) increases if r decreases and

$$\tilde{\lambda}(\mathscr{F}_P)=\lim_{r\to 0}\left[\lambda(\mathscr{F}_P(r))+\frac{4}{2\pi}\log r\right]$$

is called the reduced extremal length of the family $\mathscr{F}_P$. $\exp[-(\pi/2)\tilde{\lambda}(\mathscr{F}_P)]\,|dz|$ is an invariant form. The reduced modulus $\tilde{M}(\mathscr{F}_P^*)$ is defined by

$$\tilde{M}(\mathscr{F}_P^*)=\lim_{r\to 0}\left[M(\mathscr{F}_P^*(r))+\frac{1}{2\pi}\log r\right]$$

and $\exp[-2\pi\tilde{M}(\mathscr{F}_P^*)]\,|dz|$ is invariant.

It is easy to relate these various reduced quantities. Fix a local parameter z at P with $z(P)=0$. Let $\tilde{z}_j$ be the lift of this local parameter to a neighborhood of $\tilde{P}_j$ ($j=0, 1$), then $\tilde{z}_0\circ T=\tilde{z}_1$. Relative to these local coordinates it is clear that $\tilde{X}(r, r)$ is the two-sheeted covering surface of $X(r)$ defined by σ. From Equation (7), $\lambda(\mathscr{F}_P(r))=2\lambda(\tilde{\mathscr{F}}_P(r, r))$ so that

$$\mathscr{F}_P(r)+\frac{4}{2\pi}\log r=2\left[\lambda(\tilde{\mathscr{F}}_P(r, r))+\frac{1}{2\pi}\log r^2\right].$$

Consequently, $\tilde{\lambda}(\mathscr{F}_P)=2\tilde{\lambda}(\tilde{\mathscr{F}}_P)$. Similarly, Equation (8) and Lemma 9 together give $M(\mathscr{F}_P^*(r))=\frac{1}{2}M(\tilde{\mathscr{F}}_P^*(r, r))$ which implies that $\tilde{M}(\mathscr{F}_P^*)=\frac{1}{2}\tilde{M}(\tilde{\mathscr{F}}_P^*)$.

If we gather together the results of this section, we obtain the following result which is the analog of Proposition 1 for closed surfaces.

PROPOSITION 5. *If X is a closed Riemann surface, σ is a nontrivial 1-chain on X and $P\in X-\sigma$, then*

$$\tilde{\lambda}(\mathscr{F}_P)=4\tilde{M}(\mathscr{F}_P^*)=\frac{1}{2\pi^2}\tilde{D}_{\tilde{X}}(\tilde{v}).$$

Recall that $\tilde{v}$ satisfies the symmetry condition $\tilde{v}=-\tilde{v}\circ T$. For $-\infty<k<\infty$ define $\tilde{\sigma}_k=\{\tilde{P}\in\tilde{X}:\tilde{v}(\tilde{P})=k\}$. Each $\tilde{\sigma}_k$ is a finite cycle on $\tilde{X}$; orient $\tilde{\sigma}_k$ so that $\tilde{v}$ is less than k to the left of $\tilde{\sigma}_k$. The cover transformation T maps $\tilde{\sigma}_k$ onto $-\tilde{\sigma}_{-k}$ and $\tilde{\sigma}_k\in\mathscr{F}_P^*$ for all k. The projection of $\tilde{\sigma}_0$ onto X is a 1-chain traced once in each direction. As in Section 6 it is possible to define a projection σ_0 of $\tilde{\sigma}_0$ onto X such that $[\sigma_0]_2=[\sigma]_2$. $\tilde{v}$ projects to a 2-valued harmonic function on X; a single-valued branch can be defined on $X-\sigma_0$ by letting $v(Q)$ be the larger of the two values $\tilde{v}(f^{-1}(Q))$. Then v is the Green's function of the surface $X-\sigma_0$ with pole at P. The reduced Dirichlet integral of v is $\tilde{D}_X(v)=\frac{1}{2}\tilde{D}_{\tilde{X}}(\tilde{v})$.

11. Extremal Regions and Quadratic Differentials on Closed Surfaces

In this section we discuss extremal properties of the level curve σ_0 and a quadratic differential Ψ_σ on X which is derived from the function $\tilde{v}$. Proposition 6 is an analog of Proposition 2, while Proposition 7 characterizes the quadratic differential Ψ_σ determined by a 1-chain σ.

Given a nontrivial 1-chain $\alpha\subset X$ and a point $P\in X-\alpha$, let g_α be the Green's function of $X-\alpha$ with logarithmic singularity at P. Thus, g_α has boundary values 0 on α, and if z is a local coordinate at P with $z(P)=0$ then in a neighborhood of $z=0$, $g_\alpha(z)=$ $=w_\alpha(z)-\log|z|$, where w_α is harmonic at $z=0$. Define

$$g_\alpha\langle P\rangle=\lim_{z\to 0}(g_\alpha(z)+\log|z|); \tag{13}$$

the form $\exp(-g_\alpha\langle P\rangle)\,|dz|$ is invariant. $g_\alpha\langle P\rangle$ is related to the classical Robin's constant. The reduced extremal distance from α to P is denoted by $\tilde{\lambda}(\alpha, P)$. It is known that $\tilde{\lambda}(\alpha, P)=(1/2\pi)\,g_\alpha\langle P\rangle$ (Ohtsuka [1]). One can also prove

LEMMA 11. $g_\alpha\langle P\rangle=(1/2\pi)\,\tilde{D}_X(g_\alpha)$.

PROPOSITION 6. *Let X be a closed Riemann surface and σ a nontrivial 1-chain on X. Fix a point $P\in X-\sigma$. Among all 1-chains which are $\mathbf{Z}_2$-homologous to σ on $X-P$, there is a unique one σ_0 which has the greatest reduced extremal distance from P. Specifically, σ_0 is the level curve σ_0 of v and $g_{\sigma_0}=v$ is the Green's function for $X-\sigma_0$ with logarithmic singularity at P. We have*

$$\tilde{\lambda}(\sigma_0, P)=\max\{\tilde{\lambda}(\alpha, P)\colon [\alpha-\sigma]_2=0 \text{ on } X-P\}, \tag{14}$$

$$\tilde{D}_X(g_{\sigma_0})=\max\{\tilde{D}_X(g_\alpha)\colon [\alpha-\sigma]_2=0 \text{ on } X-P\}, \tag{15}$$

$$\tilde{\lambda}(\sigma_0, P)=\left(\frac{1}{2\pi}\right)^2\tilde{D}_X(v)=\left(\frac{1}{2\pi}\right)^2\tilde{D}_X(g_{\sigma_0}). \tag{16}$$

Proof. Given a 1-chain α on $X-P$ which is $\mathbf{Z}_2$-homologous to σ, construct $\tilde{X}=\tilde{X}(\sigma)=\tilde{X}(\alpha)$ and the function $\tilde{v}$. Lift the Green's function g_α to $\tilde{X}$ by defining $\tilde{g}_\alpha(\tilde{Q})=g_\alpha(Q)$ if $\tilde{Q}$ is on the upper sheet of $\tilde{X}(\alpha)$ and $\tilde{Q}$ lies over Q, and $\tilde{g}_\alpha(T(\tilde{Q}),$ $=-\tilde{g}_\alpha(\tilde{Q})$. Set $\tilde{g}_\alpha(\tilde{Q})=0$ if $\tilde{Q}$ lies over α. Then $\tilde{g}_\alpha$ is a piecewise harmonic function on $\tilde{X}$ having a positive logarithmic singularity at $\tilde{P}_1$ and a negative logarithmic singularity at $\tilde{P}_0$, where $\tilde{P}_1$ is the point on the upper sheet lying over P and $\tilde{P}_0$ is the point on the lower sheet over P. Set $\tilde{w}=\tilde{g}_\alpha-\tilde{v}$, then $\tilde{w}$ is piecewise harmonic on all of $\tilde{X}$. Fix a local coordinate z at P and let $\tilde{z}_j$ be the induced coordinate at $\tilde{P}_j$ with $\tilde{z}_0\circ T=\tilde{z}_1$.

Now, we derive an analog of the Dirichlet principle in this setting. From $\tilde{g}_\alpha=\tilde{v}+\tilde{w}$, we get

$$D_{\tilde{X}(r,r)}(\tilde{g}_\alpha)=D_{\tilde{X}(r,r)}(\tilde{v})+D_{\tilde{X}(r,r)}(\tilde{w})+2D_{\tilde{X}(r,r)}(\tilde{v},\tilde{w}).$$

If we apply Stokes' Theorem to the mixed Dirichlet integral and let $r\to 0$, then we obtain

$$\tilde{D}_{\tilde{X}}(\tilde{g}_\alpha)=D_{\tilde{X}}(\tilde{w})+\tilde{D}_{\tilde{X}}(\tilde{v})+4\pi\,[\tilde{w}(P_1)-\tilde{w}(P_0)].$$

The equalities $\tilde{w}(\tilde{P}_j)=\tilde{g}_\alpha\langle\tilde{P}_j\rangle-\tilde{v}\langle\tilde{P}_j\rangle$, $\tilde{g}_\alpha\langle\tilde{P}_j\rangle=(-1)^{j+1}\,g_\alpha\langle P\rangle$, $\tilde{v}\langle\tilde{P}_j\rangle=(-1)^{j+1}$ $v\langle P\rangle$, together with the fact that the reduced Dirichlet integrals on $\tilde{X}$ are twice the corresponding quantity on X, give

$$\tilde{D}_X(g_\alpha)-4\pi g_\alpha\langle P\rangle=D_X(g_\alpha-v)+\tilde{D}_X(v)-4\pi v\langle P\rangle.$$

Making use of Lemma 11, we have

$$\tilde{D}_X(v)=D_X(g_\alpha-v)+\tilde{D}_X(g_\alpha).$$

Consequently, $\tilde{D}_X(v)\geqslant\tilde{D}_X(g_\alpha)$ and equality holds if and only if $v=g_\alpha$. This is sufficient to establish the proposition.

A quadratic differential $\Psi=\psi(z)\,dz^2$ can be associated with a 1-chain σ on a closed Riemann surface X. Given σ, form $\tilde{X}$ and $\tilde{v}$. The fact that $\tilde{v}\circ T=-\tilde{v}$ shows that $(d\tilde{v}+i*d\tilde{v})^2$ is a T-invariant meromorphic quadratic differential on $\tilde{X}$. This invariance condition implies that it can be projected to X. In this manner we obtain a meromorphic quadratic differential $\Psi=\psi(z)\,dz^2$ on X and $\Psi=(dv+i*dv)^2$ on $X-\sigma$. We shall sometimes write $\Psi=\Psi_\sigma=\psi_\sigma(z)\,dz^2$ to indicate the dependence on σ.

Because ψ is locally the square of an analytic function on $X'-P$, Ψ has no poles on $X'-P$ and all of its zeros are of even order. Let $\tilde{Q}$ be a branch point of $\tilde{X}$. Ψ will have a simple pole at $Q=f(\tilde{Q})$ if $\tilde{Q}$ is not a critical point of $\tilde{v}$, and otherwise will have a zero of odd order at Q. The reduced norm of $\Psi=\psi(z)\,dz^2$ is

$$\langle\!\langle \Psi \rangle\!\rangle=\lim_{r\to 0}\left[\iint_{X(r)} |\phi(z)|\,dx\,dy+2\pi\log r\right].$$

Ψ has a pole of order 2 at P and if z is a local parameter at P with $z(P)=0$, then $\psi(z)=(1/z^2)+(a_{-1}/z)+a_0+\cdots$. Clearly, $\langle\!\langle \Psi \rangle\!\rangle=\tilde{D}_X(v)$ and $\operatorname{Re}\int\sqrt{\psi(z)}\,dz=\int\pm dv$.

PROPOSITION 7. *Let σ be a nontrivial* 1*-chain on a closed Riemann surface X. Fix a point $P\in X-\sigma$. There is a unique meromorphic quadratic differential $\Psi=\psi(z)\,dz^2$ on X which satisfies the following conditions:*

(i) *Ψ has a pole of order* 2 *at P and the initial coefficient of the Laurent expansion of $\psi(z)$ is* 1, *Ψ has no poles on $X'-P$ and poles of at most the first order at the points of $\partial\sigma$.*

(ii) *Any germ $\sqrt{\psi(z)}\,dz$ on $X'-P$ can be continued along all paths on $X'-P$ and the continuation satisfies* $\operatorname{Re}\int_\gamma\sqrt{\psi(z)}\,dz=0$ *if $\gamma\in\pi_1(X'-P)$ and $\gamma\times\sigma\equiv 0$* (mod 2).

Proof. The quadratic differential Ψ_σ has properties (i) and (ii). To prove that it is unique, consider another such quadratic differential Ψ. Let $w(Q)=\operatorname{Re}\int_{P_0}^{Q}\sqrt{\psi(z)}\,dz$. Property (ii) shows that w lifts to a single-valued harmonic function $\tilde{w}$ on $\tilde{X}'-\{\tilde{P}_0,\tilde{P}_1\}$, where $\tilde{P}_0$, $\tilde{P}_1$ are the two points lying over P. Property (i) shows that $\tilde{w}$ has a positive logarithmic pole at one of the points $\tilde{P}_0,\tilde{P}_1$ and a negative logarithmic pole at the other. Also, $\tilde{w}$ has removable singularities at the branch points because Ψ has at most simple poles at the projection of the branch points. Thus, $\tilde{w}$ is harmonic on $\tilde{X}$ except for two logarithmic singularities, so it follows that $\tilde{w}=\pm\tilde{v}+$constant. From this we see that $\Psi=\Psi_\sigma$.

The orthogonal trajectories of Ψ_σ are the level curves of v. These are just σ_k $(0<k<\infty)$ together with the special trajectory $\sigma_0\in[\sigma]_2$. The following result has been established.

LEMMA 12. *The properties* (i) *and* (ii) *of Proposition* 7 *uniquely determine Ψ_σ on a closed Riemann surface X. There is a unique set σ_0 of orthogonal trajectories of Ψ_σ such that $\sigma_0\in[\sigma]_2$; $X-\sigma_0$ is the extremal region of Proposition* 6.

12. The Main Theorem for Closed Surfaces

We now gather together our results for closed surfaces.

THEOREM 2. *Let σ be a nontrivial* 1*-chain on the closed Riemann surface X. Fix $P\in X-\sigma$. There is a quadratic differential Ψ_σ on X which is uniquely determined by* *properties* (i) *and* (ii) *of Proposition* 7. *Let* $\mathscr{F}_P$ *and* $\mathscr{F}_P^*$ *be the curve families defined in Section* 10. *Then*

$$\tilde{\lambda}(\mathscr{F}_P)=4\tilde{M}(\mathscr{F}_P^*)=\frac{1}{\pi^2}\langle\!\langle\Psi_\sigma\rangle\!\rangle .$$

There is a collection σ_0 *of orthogonal trajectories of* Ψ_σ *such that* σ_0 *is* $\mathbf{Z}_2$*-homologous to* σ. σ_0 *has the greatest reduced extremal distance from P among all l-chains on* $X-P$ *which are* $\mathbf{Z}_2$*-homologous to* σ. *On* $X-\sigma_0$ *we have Re* $\sqrt{\Psi_\sigma}=dg$, *where g is the Green's function for* $X-\sigma_0$ *with logarithmic singularity at P and where the square root with positive real part is selected.*

13. Examples and Applications

In this section we briefly illustrate the variety of applications and problems suggested by Theorems 1 and 2.

(a) *Estimates of* $\|\Phi_\sigma\|$ *and* $\langle\!\langle\Phi_\sigma\rangle\!\rangle$. In the special case that σ is an arc and X is a simply connected hyperbolic plane region, Theorem 1 reduces to a theorem of J. Hersch [1]. Let h be the hyperbolic distance between the endpoints of σ. If we use Hersch's computations, we find that

$$\|\Phi_\sigma\|=2\nu(e^{-2h}), \tag{17}$$

where $\nu(r)$ is the modulus of the doubly connected domain obtained by removing the interval $[0, r]$ $(0<r<1)$ from the unit disk. Explicitly,

$$\nu(r)=\frac{K(\sqrt{1-r^2})}{4K(r)} \quad \text{where} \quad K(r)=\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}} .$$

The following useful estimates hold (Hersch [1]):

$$\begin{gathered}\frac{4}{\pi}\log\frac{(\sqrt{1+r}+\sqrt{2r})^2}{1-r}\leqslant\frac{1}{\nu(r)}\leqslant\frac{4}{\pi}\log\frac{4(1+r)}{1-r},\\ \nu(r)=\frac{1}{2\pi}\log\frac{4}{r}+0(r^2) \quad \text{as} \quad r\to 0,\\ \nu(r)=\frac{\pi}{4\log(8/1-r)}(1+0(1-r)) \quad \text{as} \quad r\to 1.\end{gathered} \tag{18}$$

We can also express Theorem 2 explicitly in the simply connected case. Let X be the Riemann sphere, let σ be an arc from z_0 to ∞ ($z_0 \neq 0$), and let the origin be selected as base point. One can then show that

$$\langle\!\langle \Psi_\sigma \rangle\!\rangle = 2\pi \log 4|z_0|. \tag{19}$$

We know of no estimates comparable to those above in case σ is a union of arcs or in case X is not simply connected. That such estimates might be useful can be seen from the applications below, and from the applications given in Hersch [1].

(b) *Wolff's lemma.* Let σ be an arc in X, and let f be a K-quasiconformal homeomorphism of X onto Y. Let $\mathscr{F}_X$ (respectively, $\mathscr{F}_Y$) be the family of crosscuts on X (respectively, Y) which cross σ (respectively, $f(\sigma)$) an odd number of times. Then $K^{-1}\lambda(\mathscr{F}_Y) \leqslant \lambda(\mathscr{F}_X) = \|\Phi_\sigma\|^{-1} \leqslant K\lambda(\mathscr{F}_Y)$. Given a conformal metric on Y, let m be the infimum of the lengths of the crosscuts in $\mathscr{F}_Y$ and let A be the area of Y. Then $K^{-1}(m^2/A) \leqslant K^{-1}\lambda(\mathscr{F}_Y) \leqslant \|\Phi_\sigma\|^{-1}$, so that

$$m \leqslant \sqrt{\frac{KA}{\|\Phi_\sigma\|}}. \tag{20}$$

We shall show that (20) is a sharper, as well as more general, form of Wolff's lemma (J. Wolff [1, pp. 217–218]). To reduce (20) to Wolff's lemma, choose X to be the unit disk, choose σ to be an arc from the origin to δ $(0 < \delta < 1)$, choose f to be conformal ($K = 1$), and choose Y to be a bounded plane region with the euclidean metric. Thus m is the infimum of the lengths of crosscuts in Y which separate $f(0)$ from $f(\delta)$. From (20) and (17) we obtain

$$m^2 \leqslant \frac{A}{2v(e^{-2h})} = \frac{A}{v((1-\delta)/(1+\delta))} = 4Av(\delta), \tag{21}$$

where we have used the identity $v(r)\,v((1-r)/(1+r)) = \frac{1}{8}$ (Hersch [1, p. 317]). We now have $m \leqslant 2\sqrt{Av(\delta)}$; if we estimate $v(\delta)$ from (18),

$$v(\delta) \leqslant \frac{\pi}{4} \frac{1}{\log[(\sqrt{1+\delta} + \sqrt{2\delta})^2/1-\delta]} < \frac{\pi}{4} \frac{1}{\log(1/1-\delta)},$$

we obtain, essentially, the original form of Wolff's lemma:

$$m < \sqrt{\frac{\pi A}{\log(1/1-\delta)}}.$$

(c) *Extremal regions.* Theorems 1 and 2 are relevant to the subject of extremal regions and distortion theorems. In the simplest special case (X is simply connected and σ is an arc) Theorem 1 yields Grötzsch's extremal region theorem: Among all arcs in $\{|z|>1\}$ which join $z=R$ to $z=\infty$ $(1<R<\infty)$ the segment $\sigma=[R, \infty]$ has the greatest extremal distance from the circle $\{|z|=1\}$, and this extremal distance is $\frac{1}{4}\|\Phi\|^{-1}=v(R^{-1})$.

In the simplest special case Theorem 2 yields the related result: Among all arcs which join $z=R$ to $z=\infty$ $(0<R<\infty)$, the segment $\sigma=[R, \infty]$ has the greatest reduced extremal distance from $z=0$; the value with respect to the identity local coordinate is $\tilde{\lambda}(0, \sigma)=(1/2\pi)\log 4R$. (The Koebe one-quarter theorem is an easy consequence.)

Theorems 1 and 2 might be applied to more general extremal regions and distortion theorems by letting X be nonsimply connected or letting σ be a union of arcs.

(d) *Conformal metrics.* Let P and Q be points on an open Riemann surface X. We use Theorem 1 to define $d(P, Q)=\inf\lambda(\mathscr{F}^*(\sigma))=4\inf\lambda^{-1}(\mathscr{F}(\sigma))=4\inf\|\Phi_\sigma\|$, where the infimum is taken over all arcs σ on X with $\partial\sigma=P-Q$. It can be shown that $d(P, Q)$ is a metric on X. The relationship between this metric and the metric of S. Gál [1, 2] and of M. Lavrentieff [1] are striking but are not yet completely understood. Properties and uses of this metric will be treated in a future paper.

(e) *Geometric inequalities.* In general, extremal length and conjugate extremal length statements lead to length-area inequalities of differential geometry (Rodin [1]). The inequalities that arise in this way from Theorem 1 are especially interesting because the extremal metric $|\sqrt{\Phi}|$ cannot be realized as the euclidean metric on a plane region. A number of problems arise concerning the sharpness of the inequalities and how they can be improved with regard to the euclidean metric.

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