Homotopy-equivariance.

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Homotopy-equivariance

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1. Introduction

Let X be a G-space, where G is a discrete[1]) group. The classification of real G-vector bundles over X by equivariant isomorphism leads to the Grothendieck ring[2]) $K_G(X)$ of equivariant K-theory, as described in [15]. The notion of *homotopy-equivariant isomorphism*, which we shall define, leads to a quotient ring $K/G(X)$ of $K_G(X)$, and the corresponding notion for fibre homotopy equivalence leads to a factor group $J/G(X)$ of $K/G(X)$. When G is trivial $K/G(X)$ reduces to $K(X)$ and $J/G(X)$ to[3]) $J(X)$, the well-known functor studied by Adams and Atiyah. We discuss, with examples, methods of calculating $K/G(X)$ and $J/G(X)$, and use the results to solve a problem about cross-sections of Stiefel manifolds.

The basic notion of homotopy-equivariance is as follows. For any G-space X we denote by $g_\#: X \to X$ the action of an element $g \in G$. If X and Y are G-spaces we describe a map $f: X \to Y$ as a *homotopy-G map* if $g_\# f g_\#^{-1} \simeq f$, for all $g \in G$. If f is a homotopy G-map and a homotopy equivalence then any homotopy inverse of f is also a homotopy-G map. In that case we describe f as a *homotopy-G equivalence* and say that X and Y have the same *homotopy-G type*. Also we describe a G-space X as *homotopy-G trivial* if $g \simeq 1$ for all $g \in G$. This is the case, for example, if X is contractible with any G-structure, or if X is a sphere with orientation-preserving G-structure.

Now consider the category of G-spaces $E, F, \ldots$ over a given G-space X. In the terminology of [10], the set of overhomotopy classes of overmaps $f: E \to F$ will be denoted by $\pi_X(E, F)$. We describe f as a *homotopy-G overmap* if $g_\# f g_\#^{-1}$ is overhomotopic to f for all $g \in G$. When the overspaces are fibre spaces the term *fibre-preserving homotopy-G map* may be used instead. The other homotopy-equivariant notions are extended to the category of spaces over X in the obvious way.

For any G-module M we denote by $\mathbf{M}$ the G-vector bundle $X \times M$ with the natural projection and product G-structure. Given an equivalence relation $\sim$ between G-vector bundles over X we say that U, V are stably equivalent if $U \oplus \mathbf{M} \sim V \oplus \mathbf{M}$ for some M.

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[1]) This restriction is more a matter of convenience than necessity; the situation when G is topological will be considered in a separate note.

[2]) We write K_G rather than KO_G since we have no occasion to consider complex vector bundles.

[3]) We use J in the unreduced sense, taking dimension into account, and denote the Adams-Atiyah functor by $\tilde{J}$.

We describe the relation as *insensitive* if for every G-module M there exists a G-module N such that $M \oplus N$ is equivalent to a trivial G-module. In that case we can, of course, take M to be trivial in the definition of stable equivalence.

Let U, V be G-vector bundles over X. We say that an isomorphism $f: U \to V$ of vector bundles is a *homotopy-G isomorphism* if f and $g_\# f g_\#^{-1}$ are homotopic through isomorphisms, for all $g \in G$. If such an isomorphism exists we say that U and V are *homotopy-G isomorphic*. This equivalence relation is insensitive since $M \oplus M$ is homotopy-G isomorphic to a trivial G-module for any G-module M. Notice that if U, V, W are G-vector bundles over X with U homotopy-G isomorphic to V then $U \oplus W$ is homotopy-G isomorphic to $V \oplus W$ and $U \otimes W$ is homotopy-G isomorphic to $V \otimes W$.

From now on it is convenient to assume that X is a finite complex. The Grothendieck ring $K_G(X)$ is defined in the usual way. Factor out the ideal consisting of elements of the form $[U]-[V]$, where U and V are stably equivalent G-vector bundles in the sense of homotopy-G isomorphism. The quotient ring thus obtained is denoted by $K/G(X)$. The homomorphism $f^*: K_G(Y) \to K_G(X)$ induced by a G-map $f: X \to Y$ determines a homomorphism $f^*: K/G(Y) \to K/G(X)$. Since f^* depends only on the G-homotopy class of f it follows that $K/G(X)$, like $K_G(X)$, is an invariant of the G-homotopy type. However $K/G(X)$ is not an invariant of the homotopy-G type, as we shall see in a moment.

Note that $K/G(pt) \approx Z \oplus \mathrm{Hom}(G, Z_2)$, as a group. Following the practice in equivariant K-theory we denote by $\tilde{K}/G(X)$ the cokernel of the homomorphism $c^*: K/G \times (pt) \to K/G(X)$ induced by the constant map. If the action of G on X is pointed then $b^* c^* = 1$, where $b: pt \to X$ gives the basepoint, and hence $K/G(X) \approx K/G(pt) \oplus \tilde{K}/G(X)$, as a group.

Without real loss of generality we can assume that every G-vector bundle V over X is equipped with a G-invariant euclidean structure, so that the associated sphere-bundle $S(V)$ is defined as a G-space over X. We now say that two G-vector bundles are equivalent if their associated sphere-bundles have the same fibre homotopy-G type, and we define $J/G(X)$ to be the factor group of $K_G(X)$ (as a group) by the subgroup of elements of the form $[U]-[V]$, where U and V are stably equivalent in this sense. Alternatively we can define $J/G(X)$ as a factor group of $K/G(X)$. The natural projection from $K_G(X)$ to $K/G(X)$ is denoted by K/G, and the natural projection from either $K_G(X)$ or $K/G(X)$ to $J/G(X)$ by J/G. Note that $J/G: K/G(pt) \approx J/G(pt)$. The cokernel of $c^*: J/G(pt) \to J/G(X)$ is denoted by $\tilde{J}/G(X)$.

To illustrate these definitions we shall, in §2, calculate $K/G(X)$ and $J/G(X)$ in case X is a trivial G-space with $G = Z_2$. Similar calculations can be made whenever G acts trivially. Methods which can be used when G acts non-trivially are described in §4, 5, after digressing in §3 to discuss the transfer in relation to our two functors. Finally we show in §6 how Atiyah's theory of Thom spaces [3] can be extended to the homotopy-equivariant case, with J/G playing the role of J. This enables us to reexamine recent work [12] of Sutherland and myself on stunted projective spaces and in particular to solve the following problem which was raised in §4 of [12].

Consider the Stiefel manifold $V_{m,k}$ of orthonormal k-frames in R^m, where $1<k<m$. We fibre $V_{m,k}$ over S^{m-1} in the usual way and describe a cross-section $s: S^{m-1} \to V_{m,k}$ as *simple* if $Ts \simeq s$, where $T: V_{m,k} \to V_{m,k}$ denotes the involution which changes the sign of the last vector in each k-frame. If k is odd then $T \simeq 1$, since m is even, and so every cross-section is simple. It is noted in [12] that a cross-section of $V_{m,k+1}$ projects into a simple cross-section of $V_{m,k}$. Moreover when $k=2$, 4 or 8 it is shown in §4 of [12] that $V_{m,k}$ admits a simple cross-section if and only if m is an even multiple of k. This result is included in

THEOREM (1.1). *Let $k>2$ and $k \equiv 2 \bmod 4$. Then $V_{m,k}$ admits a simple cross-section if (and only if!) $V_{m,k}$ admits a cross-section.*

THEOREM (1.2). *Let $k=2$ or $k \equiv 0 \bmod 4$. Then $V_{m,k}$ admits a simple cross-section if and only if $V_{m,k+1}$ admits a cross-section.*

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2. Homotopy-symmetry

Throughout this section we take X to be a trivial Z_2-space. We denote the non-trivial 1-dimensional representation of Z_2 by L, the trivial by R. Following [11] we say that V is *linearly homotopy-symmetric* if V and $V \otimes L$ are homotopy-Z_2 isomorphic, *homotopy-symmetric* if $S(V)$ and $S(V \otimes L)$ have the same fibre homotopy-Z_2 type. The corresponding stable notions are defined in the obvious way. We shall need

LEMMA (2.1). *Let U, V be J-equivalent vector bundles over X. Then U is stably homotopy-symmetric if V is.*

To prove (2.1) consider the automorphisms v of V and w of $V \oplus V$ which are given by $vx = -x$ and $w(x, y) = (y, x)$, where $x, y \in V$. Consider also the homotopy $h_t: V \oplus V \to V \oplus V$ which is given by

$$h_t(x, y) = \left(x \cos\frac{\pi}{2}t + y \sin\frac{\pi}{2}t,\ x \sin\frac{\pi}{2}t - y \cos\frac{\pi}{2}t\right). \tag{2.2}$$

Write $S(V)=E$ so that $S(V \oplus V) = E * E$, and consider the function

$$E_{\#}: \pi_X(E, E) \to \pi_X(E * E, E * E)$$

given by the fibre join with the identity on E. Clearly V is homotopy-symmetric if and only if $S(v): E \to E$ is fibre homotopic to the identity. Let us say that V is *homotopy-invertible* if $S(w): E * E \to E * E$ is fibre homotopic to the identity. Now $S(1 \oplus v)$ is fibre homotopic to $S(w)$ under $S(h_t): E * E \to E * E$. Hence V is homotopy-invertible if V is homotopy-symmetric, and the converse holds in the stable range since $E_{\#}$ is bijective, by (5.1) of [10]. Stably, therefore, homotopy-symmetry is equivalent to homotopy-invertibility and since the latter condition depends only on the fibre homotopy type of the associated sphere-bundle we obtain (2.1).

The classes of linearly homotopy-symmetric vector bundles form a subring $\Phi(X) \subset K(X)$. We recall from §1 of [11] that $\Phi(X)$ is precisely the image of the Grothendieck group of complex vector bundles, under the realification homomorphism. Using (2.1) we see that the classes of homotopy-symmetric vector bundles form a subgroup $\Psi(X) \subset J(X)$, the determination of which is the subject of the main theorem of [8]. Of course $J\Phi(X) \subset \Psi(X)$ and it turns out that equality holds when X is a sphere or a real, complex or quaternionic projective space.

After these preliminaries we are ready to determine $K/Z_2(X)$ and $J/Z_2(X)$. By (2.2) of [15] an isomorphism

$$\theta: K(X) \oplus K(X) \to K_{Z_2}(X)$$

is given by $\theta([U], [V]) = [U] - [V \otimes L]$, where U, V are vector bundles over X. Hence it follows that the sequences

$$\begin{cases} \Phi(X) \overset{\delta}{\rightarrowtail} K(X) \oplus K(X) \overset{\phi}{\twoheadrightarrow} K/Z_2(X), \\ \Psi(X) \overset{\delta}{\rightarrowtail} J(X) \oplus J(X) \overset{\psi}{\twoheadrightarrow} J/Z_2(X), \end{cases} \tag{2.3}$$

are exact, where δ is given by the diagonal and ϕ, ψ are induced by θ. The same is true when K, J are replaced by $\tilde{K}$, $\tilde{J}$ and Φ, Ψ by their images $\tilde{\Phi}$, $\tilde{\Psi}$ in $\tilde{K}$, $\tilde{J}$, respectively.

For example, take $X = S^n = S((n+1)R)$. Then $\tilde{\Phi}(S^n) = 2\tilde{K}(S^n)$ for $n \equiv 0$ or $1 \bmod 8$ while $\tilde{\Phi}(S^n) = \tilde{K}(S^n)$ for $n \equiv 2$ or $4 \bmod 8$. Using (2.3), therefore, we obtain the following table

$n \pmod 8$	0	1	2	3	4	5	6	7
$\tilde{K}/Z_2(S^n)$	$Z \oplus Z_2$	$Z_2 \oplus Z_2$	Z_2	0	Z	0	0	0

Since $\tilde{\Psi}(S^n) = J\tilde{\Phi}(S^n)$ it follows that the table for $\tilde{J}/Z_2(S^n)$ can be obtained from this by replacing the infinite cyclic summands which occur when $n \equiv 0 \bmod 4$ by the finite cyclic group $\tilde{J}(S^n)$.

In contrast, consider the sphere $S^n = S((n+1)L)$. The natural projection $S^n \to S^n/Z_2 = P^n$ determines an isomorphism $K(P^n) \approx K_{Z_2}(S^n)$, as shown in (2.1) of

[15]. Thus every element of $\tilde{K}_{Z_2}(S^n)$ can be represented by a Z_2-vector bundle of the form r L, for some $r \geqslant 0$. In this case, therefore, $\tilde{K}/Z_2(S^n)=0$, hence $\tilde{J}/Z_2(S^n)=0$. However $S((n+1)L)$ and $S((n+1)R)$ have the same homotopy-Z_2 type when n is even and so, comparing the results of this paragraph with those in the previous one, we see that neither $K/G(X)$ nor $J/G(X)$ is an invariant of the homotopy-G type of X.

3. The Transfer

Suppose that G is finite of order n, say $G=(g^1,\dots,g^n)$. If V is a vector bundle over the G-space X then the transfer V', as defined in §2 of [4], is a G-vector bundle over X, which can be constructed as follows. We are given a vector bundle with projection $p: V \to X$, say. Considei the direct sum $V_1 \oplus \cdots \oplus V_i \oplus \cdots \oplus V_n$, where V_i $(i=1,\dots,n)$ has the same total space as V but projection $g^i_{\#} p: V \to X$. We make G act on this vector bundle by permuting the factors according to the regular representation, and thus obtain a G-vector bundle V'. In this way a homomorphism $\tau: K(X) \to K_G(X)$ is defined such that

$$\varrho\tau=(g^1_{\#})^*+\cdots+(g^n_{\#})^*, \tag{3.1}$$

where $\varrho: K_G(X) \to K(X)$ ignores G-structure. If $S(V)=E$ then $S(V')=E_1 * \cdots * E_i * \cdots * E_n$, where $E_i=S(V_i)$ and G permutes the factors of the multiple join. Thus the fibre homotopy type of $S(V)$ determines the fibre G-homotopy type and *a fortiori* the fibre homotopy-G type of $S(V')$. In this way a homomorphism $\tau: J(X) \to J/G(X)$ is defined such that $\tau J=I/G\tau$, as shown below, where $\tau: K(X) \to K/G(X)$ is obtained by composing $\tau: K(X) \to K_G(X)$ with the natural projection.

$$\begin{array}{ccccc} K(X) & \xrightarrow{\tau} & K/G(X) & \xrightarrow{\varrho} & K(X) \\ \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle J/G} & & \downarrow{\scriptstyle J} \\ J(X) & \xrightarrow[\tau]{} & J/G(X) & \xrightarrow[\varrho]{} & J(X) \end{array}$$

Of course (3.1) determines each of the compositions $\varrho\tau$. Now suppose that V is itself a G-vector bundle over X. In that case the action $g^i_{\#}: V \to V$ determines an isomorphism $h^i: V \to V^i$ $(i=1,\dots,n)$. To make the isomorphism

$$h=h^1 \oplus \cdots \oplus h^n: V \oplus \cdots \oplus V \to V^1 \oplus \cdots \oplus V^n$$

a G-isomorphism it is of course necessary for G to permute the summands of the domain of h, according to the regular representation, as well as act on the individual summands. Using the homotopy (2.2), however, we conclude that this "twisted" direct sum of G-vector bundles is homotopy-G isomorphic to $V \otimes ((n-1)R \oplus L)$, where G acts on Z_2 and hence on L through the sign representation. Hence we obtain

THEOREM (3.2). *Let* $\alpha \in K/G(X)$. *Then* $\tau\varrho\alpha = n\alpha$ *if either* (i) $n = |G|$ *is odd or* (ii) $\varrho\alpha \in \Phi(X)$.

A similar argument goes through for the associated sphere-bundles, using the multiple join instead of the direct sum, and we obtain

THEOREM (3.3). *Let* $\beta \in J/G(X)$. *Then* $\tau\varrho\beta = n\beta$ *if either* (i) $n = |G|$ *is odd or* (ii) $\varrho\beta \in \Psi(X)$.

The effect of these two results, combined with (3.1), is to give an upper bound for the exponents of the kernels and cokernels of $\varrho: K/G(X) \to K(X)$ and $\varrho: J/G(X) \to J(X)$, for G finite. Henceforth we denote the kernels of these homomorphisms by $K'/G(X)$ and $J'/G(X)$, respectively.

4. The Key Monomorphisms

Consider the cohomology of G with coefficients in the G-group $\tilde{K}(SX)$, where G operates through the induced automorphisms $(Sg_{\#})^*$ $(g \in G)$. A monomorphism

$$k: K'/G(X) \rightarrowtail H^1(G; \tilde{K}(SX))$$

can be defined as follows. Let V be a G-vector bundle over X which is trivial as a vector bundle. Choose a trivialization $\lambda: V \to X \times M$, where M is a trivial G-module, and transfer the G-structure of V to $X \times M$ through λ. Then we obtain for each element $g \in G$ a homomorphism $g_{\#}: X \times M \to X \times M$ and hence a vector bundle V_g over SX, by using $g_{\#}$ as a clutching function. It is easy to check that

$$[V_{gh}] = (Sh_{\#})^* [V_g] + [V_h] \qquad (g, h \in G)$$

in $\tilde{K}(SX)$. Hence a cocyle $c \in Z^1(G; \tilde{K}(SX))$ is defined by $c(g) = [V_g]$. If λ is replaced by $\lambda\xi$, where ξ is an automorphism of the vector bundle $\mathbf{M} = X \times M$, then $[V_g]$ is replaced by $-(Sg_{\#})^*\theta + [V_g] + \theta$, where $\theta \in \tilde{K}(SX)$ is the element obtained by treating ξ as a clutching function. Hence the cohomology class $[c] \in H^1(G; \tilde{K}(SX))$ of c is independent of the choice of trivialization. Now λ is a homotopy-G isomorphism if and only if V_g is trivial for all $g \in G$. Hence $k[V] = [c]$ defines a monomorphism

$$k: K'/G(X) \to H^1(G; \tilde{K}(SX)).$$

We refer to k as the *key monomorphism*. Notice, incidentally, that $K'/G(X)$ is finite (finitely generated) if G is finite (finitely generated), since $\tilde{K}(SX)$ is finitely generated.

Now consider the group $\sigma(X)$ of homotopy classes of maps of X into the H-space of homotopy equivalences $S^m \to S^m$ (m large) with G acting through $(g_{\#})^*$. Regarding $\tilde{J}(SX)$ as a subgroup of $\sigma(X)$, in the usual way, let $I: \tilde{K}(SX) \to \sigma(X)$ denote the homomorphism defined by J. We shall now define a monomorphism j which makes the following diagram commutative, where I_* denotes the coefficient homomorphism determined by I.

$$\begin{array}{ccc} K'/G(X) & \xrightarrow{k} & H^1(G; \tilde{K}(SX)) \\ {\scriptstyle J/G}\downarrow & & \downarrow{\scriptstyle I_*} \\ J'/G(X) & \xrightarrow{j} & H^1(G; \sigma(X)) \end{array}$$

Let V be a G-vector bundle over X such that $S(V)$ is trivial, in the sense of fibre homotopy type. Choose a fibre homotopy equivalence $\mu: S(V) \to X \times S$, where S is a trivial G-sphere, and let $\nu: X \times S \to S(V)$ be a fibre homotopy inverse of μ. For each element $g \in G$ the composition

$$\mu g_{\#} \nu: X \times S \to X \times S$$

determines, through the Hopf construction, an element $c(g) \in \sigma(X)$. Proceeding as before we find that $c \in Z^1(G; \sigma(X))$ and that the cohomology class $[c] \in H^1(G; \sigma(X))$ of c is independent of the choice of μ and ν. Hence $j[V] = [c]$ defines a homomorphism, as in (4.1), and just as in the case of k we see that j is injective. Finally, to show that $j(J/G) = I_* k$ we take V to be trivial as a vector bundle and choose $\mu = S(\lambda)$, $\nu = S(\lambda^{-1})$, where λ is a trivialization of V.

Recall from (2.2′) in Chapter V of [16] that $\sigma(X)$ is finite, and from (1.5) of [3] that $\tilde{J}(X)$ is finite. Since j is injective we obtain

PROPOSITION (4.2). *If G is finitely generated then $\tilde{J}/G(X)$ is finite.*

When the action of G on the coefficient group A is trivial we identify $H^1(G; A) = \mathrm{Hom}(G, A)$ in the usual way. When $G = Z_2$, in particular, we further identify $\mathrm{Hom}(Z_2, A)$ with the kernel ${}_2A$ of $2: A \to A$. When $G = Z_2$ and the action on A is by sign reversal we identify $H^1(G; A) = A/2A$ in the usual way.

For example, consider the Z_2-space $S(nL \oplus R) = S^n$. I assert that

$$k: K'/Z_2(S^n) \approx H^1(Z_2; \tilde{K}(S^{n+1})). \tag{4.3}$$

Consider the Clifford algebra $C_{n+1} = C(nL \oplus R)$. The action of Z_2 on $S^n = S(nL \oplus R)$ is given by $x \mapsto -exe$, where $e = e_{n+1} \in R$ is a generator such that $e^2 = -1$. Given a graded C_{n+1}-module (M^0, M^1) we construct a vector bundle U over S^{n+1} by using $\theta: S^n \times M^0 \to M^0$ as clutching function, where $\theta(x, y) = exy$ $(x \in S^n,\ y \in M^0)$. Over the Z_2-space S^n, a Z_2-structure on $S^n \times M^0$ is given by $(x, y) \mapsto (-exe, exy)$, and U is related to this Z_2-vector bundle as in the definition of the key homomorphism. By (5.1), (5.4) and (11.5) of [5], however, we can choose (M^0, M^1) so that $[U]$ generates $\tilde{K}(S^{n+1})$. Hence k is surjective and thus an isomorphism, as asserted.

Now $H^1(Z_2; \tilde{K}(S^{n+1}))=Z_2$ when $n\equiv 0, 1, 3$ or $7 \bmod 8$, and is zero otherwise. If $n\not\equiv 0 \bmod 4$ then $\tilde{K}_{Z_2}(S^n)$ is cyclic, by (3.3) of [6], and so $\tilde{K}/Z_2(S^n)$ is cyclic. Hence and from (4.3) we obtain the following table

$n(\bmod 8)$	0	1	2	3	4	5	6	7
$\tilde{K}/Z_2(S^n)$	$Z\oplus Z_2$	Z_4	Z_2	Z_2	Z	0	0	Z_2

Since $\tilde{J}(S^{n+1})$ is a direct summand of $\sigma(S^n)$ the coefficient homomorphism

$$I_*: H^1(Z_2; \tilde{K}(S^{n+1}))\to H^1(Z_2; \sigma(S^n))$$

is injective. When $n\not\equiv 0 \bmod 4$ it follows from (4.1) that $\tilde{J}/Z_2: \tilde{K}/Z_2(S^n)\approx \tilde{J}/Z_2(S^n)$ since $\tilde{J}: \tilde{K}(S^n)\approx \tilde{J}(S^n)$. The determination of $\tilde{J}/Z_2(S^n)$ when $n\equiv 0 \bmod 4$ appears to be difficult and I have only been able to obtain fragmentary results.

As a second example, with applications in §6 below, consider the Hopf Z_2-line bundle H over the real projective space $P^n=P(L\oplus nR)$, where $n\geqslant 1$. For $n=1$ it follows from what we have just proved that $[H]\in\tilde{J}/Z_2(S^1)$ is of order 4 (in fact this can easily be deduced from first principles). Let $\phi(n)$ denote the number of integers s in the range $0<s<n$ such that $s\equiv 0, 1, 2$ or $4 \bmod 8$. Recall (see [2]) that $\tilde{K}(P^n)$ is cyclic of order $2^{\phi(n)}$ with generator $[H]$ and $J: \tilde{K}(P^n)\approx\tilde{J}(P^n)$. Let r_n denote the order of $[H]$ in $\tilde{K}/Z_2(P^n)$. We shall prove that

$$\begin{aligned} r_n &= 2^{\phi(n)} \quad \text{if } n>1 \text{ and } n\not\equiv 3 \bmod 4,\\ &= 2^{\phi(n+1)} \quad \text{if } n=1 \text{ or } n\equiv 3 \bmod 4. \end{aligned} \tag{4.4}$$

Given a trivial Z_2-module M of dimension m we have that $mH\approx(S^n\times M)/Z_2$, with the Z_2-action which sends $\pm(x, y)$ into $\pm(-exe, y)$ $(x\in S^n, y\in M)$. Suppose that $M=M^0$, where (M^0, M^1) is a graded C_{n+1}-module. Then a vector bundle trivialization

$$\lambda: (S^n\times M^0)/Z_2\to P^n\times M^1$$

is given by $\lambda(\pm(x, y))=(\pm x, x\cdot y)$. The Z_2-structure on $P^n\times M^1$ thus obtained transforms (x, z) into $(x, \psi(x, z))$, where $\psi: P^n\times M^1\to M^1$ is given by $\psi(\pm x, z)=xexez$ $(x\in S^n, z\in M^1)$. Now ψ is equal to the composition

$$P^n\times M^1 \xrightarrow{\pi\times\sigma} S^n\times M^0 \xrightarrow{\mu} M^1,$$

where $\pi(\pm x)=xex$, $\sigma z=ez$ and $\mu(x, y)=xy$. Therefore the vector bundle over SP^n obtained from ψ by the clutching construction is isomorphic to $(S\pi)^*W$, where W is the vector bundle over S^{n+1} obtained from μ by the clutching construction. If (M^0, M^1) is irreducible, so that $\dim M^0=2^{\phi(n)}$, then $[W]$ generates $\tilde{K}(S^{n+1})$, by (11.5) of [5], and so we obtain

LEMMA (4.5). *The image of the coefficient homomorphism*

$$((S\pi)^*)_*: H^1(Z_2; \tilde{K}(S^{n+1})) \to H^1(Z_2; \tilde{K}(SP^n))$$

is generated by $2^{\phi(n)}k[H]$, *where*

$$k: K'/Z_2(P^n) \to H^1(Z_2; \tilde{K}(SP^n)).$$

When $n \equiv 2$, 4, 5 or 6 mod 8 this proves (4.4) immediately since $\tilde{K}(S^{n+1})=0$. When $n \equiv 0$ or 1 mod 4 the results of Karoubi [13] show that $(S\pi)^*=0$ and again (4.4) follows at once. Let $n \equiv 3$ mod 4, therefore, and consider the exact sequence shown below, where $u: P^{n-1} \subset P^n$.

$$\tilde{K}(S^{n+1}) \xrightarrow{(S\pi)^*} \tilde{K}(SP^n) \xrightarrow{(Su)^*} \tilde{K}(SP^{n-1}).$$

We have that $\tilde{K}(S^{n+1})=Z$ and $\tilde{K}(SP^n)=Z \oplus Z_2$, $\tilde{K}(SP^{n-1})=Z_2$, as shown in [13]. It follows that $(S\pi)^*$ admits a left inverse as a homomorphism, hence admits a left inverse as a Z_2-homomorphism. Hence the coefficient homomorphism in (4.5) is injective, therefore non-trivial, and so the remainder of (4.4) follows at once.

5. The Auxiliary Space

We regard the join $E=G*G$ as a principal G-bundle over the suspension SG, in the usual way. Given a G-space X let $\hat{X}=E\prod_G X$ denote the associated bundle with fibre X. Regarding $G*G$ as a 1-complex, on which G operates by permuting vertices, we see that the homotopy type of this auxiliary space $\hat{X}$ depends only on the homotopy-G type of the G-space X. For example, take $G=Z_2$. Then $E=G*G$ is Z_2-equivalent to the circle S^1 with the antipodal action of Z_2. In this case, therefore, we can construct $\hat{X}$ from $X \times [0, 1]$ by identifying $(x, 0)$ with $(g_\# x, 1)$ for all $x \in X$, where g generates Z_2.

Returning to the general case we observe that if V is a G-vector bundle over X then $\hat{V}$ can be regarded as a vector bundle over $\hat{X}$. Thus a functor is defined from the category of G-vector bundles over $\hat{X}$ to the category of vector bundles over X. Let U, V be G-vector bundles over X and let $f: U \to V$ be a homotopy-G isomorphism. Then for each element $g \in G$ there exist a homotopy H_t^g of f into $g_\# f g_\#^{-1}$ which is an isomorphism for all values of t. Hence an isomorphism $\hat{f}: \hat{U} \to \hat{V}$ is defined by

$$\hat{f}((g, t, e), x) = ((g, t, e), H_t^g x), \tag{5.1}$$

where $x \in U$ and e denotes the neutral element of G. Conversely, let $\hat{f}: \hat{U} \to \hat{V}$ be an isomorphism. Then $\hat{f}$ determines an isomorphism $f: U \to V$ by restriction to the subspace $X \subset \hat{X}$, and using $\hat{f}$ in the reverse direction we see that f is a homotopy-G isomorphism. Therefore U and V are homotopy-G isomorphic if and only if $\hat{U}$ and $\hat{V}$ are isomorphic. Passing to equivalence classes we obtain a monomorphism $\xi: K/G(X) \to K(\hat{X})$.

A similar argument shows that $S(U)$ and $S(V)$ have the same fibre homotopy-G type if and only if $S(\hat{U})$ and $S(\hat{V})$ have the same fibre homotopy type. Hence it follows that there exists a monomorphism η such that $\eta(J/G) = J\xi$, as shown in the following diagram

$$\begin{tikzcd} K/G(X) \arrow[r, tail, "\xi"] \arrow[d, "J/G"'] & K(\hat{X}) \arrow[d, "J"] \\ J/G(X) \arrow[r, tail, "\eta"'] & J(\hat{X}) \end{tikzcd} \tag{5.2}$$

Thus instead of setting up further homotopy-G theory for computing J/G we can pass across to the auxiliary space and use the classical theory of Adams [2]

For example, let us again consider the Z_2-space $P^n = P(L \oplus nR)$. We are now ready to prove

THEOREM (5.3). *The order of* $[H]$ *in* $\tilde{J}/Z_2(P^n)$ *is precisely* r_n, *where* r_n *is as in* (4.4).

The case $n=1$ is disposed of in §4. The case $n \not\equiv 3 \bmod 4$, with $n>1$, follows at once from (4.4) since $r_n = 2^{\phi(n)}$. There remains the case $n \equiv 3 \bmod 4$. To deal with this we regard P^n as a Z_2-subspace of $P^{n+1} = P(L \oplus (n+1)R)$, so that $\hat{P}^n \subset \hat{P}^{n+1}$ with inclusion map v, say. We prove

LEMMA (5.4). *Let* $n \equiv 3 \bmod 4$. *Then*

$$v^*: \tilde{K}(\hat{P}^{n+1}) \to \tilde{K}(\hat{P}^n)$$

is surjective.

It is easy to check that $\tilde{K}(\hat{S}^n)$ is finite, where $S^n = S(L \oplus nR)$, and hence $\tilde{K}(\hat{P}^n)$ is finite, since n is odd. Since $\hat{P}^{n+1}/\hat{P}^n$ has the homotopy type of $S^{n+2} \vee S^{n+1}$, we have an exact sequence

$$\tilde{K}(\hat{P}^{n+1}) \xrightarrow{v^*} \tilde{K}(\hat{P}^n) \to \tilde{K}(S^{n+1} \vee S^n) = Z,$$

and so v^* is surjective, as asserted.

When n is odd the homotopy-Z_2 type of P^{n+1} is trivial, and so $\hat{P}^{n+1}$ has the homotopy type of $P^{n+1} \times S^1$. Consider the Adams operator ψ^k, where k is odd. Recall (see [1]) that ψ^k acts trivially on $\tilde{K}(P^{n+1})$ and $\tilde{K}(S^1)$, hence acts trivially on $\tilde{K}(P^{n+1} \times S^1)$. Thus in our case ψ^k acts trivially on $\tilde{K}(\hat{P}^n)$, by (5.4). Since ξ is injective r_n is the order of $\xi[H] = [\hat{H}]$ in $\tilde{K}(\hat{P}^n)$, and $r_n = 2^{\phi(n+1)}$, by (4.4). Applying

(5.16) of [2] we calculate the "cannibalistic" characteristic classes of even multiples of $[\hat{H}]$ and deduce from (6.1) of [2] that $\eta[H]=[\hat{H}]$ is of order $2^{\phi(n+1)}$ in $\tilde{J}(\hat{P}^n)$. Since η is injective this completes the proof of (5.3).

6. Homotopy-equivariant S-theory

The homotopy-equivariant version of Spanier-Whitehead S-theory presents no difficulty. The suspension of a homotopy-G map is a homotopy-G map, and the converse holds in the stable range. Thus the notions of stable homotopy-G type, etc; are defined.

Only the treatment of duality perhaps requires comment. Consider the sphere S^n with homotopy-G trivial G-structure. If X and Y are G-spaces with join $X*Y$ then a homotopy-G map $u:X*Y\to S^n$ which is a duality map in the ordinary sense will be described as a *homotopy-G duality map*. In that case if $g\in G$ then the dual of the stable homotopy class of $g_{\#}:X\to X$ is the stable homotopy class of $g_{\#}^{-1}: Y\to Y$. Note that X is homotopy-G S-trivial if Y is.

We say that X is *homotopy-G reducible* if there exists a homotopy-G map $f:S^n\to X$ such that $f_*:H_r(S^n)\approx H_r(X)$ for $r\geqslant n$. We say that X is *homotopy-G coreducible* if there exists a homotopy-G map $f:X\to S^n$ such that $f^*:H^r(S^n)\approx H^r(X)$ for $r\leqslant n$. The corresponding stable notions are defined in the obvious way. If X is homotopy-G S-reducible then the dual of X is homotopy-G S-coreducible, and conversely.

Now suppose that X is a smooth G-manifold. Under certain conditions (see [14]) there exist equivariant embeddings of X in S^n where G acts on S^n through rotations. Given such an embedding take $Y=S^n-X$. The corresponding duality map $X*Y\to S^n$ is equivariant and it follows easily that the stable homotopy-G type of Y depends only on the stable homotopy-G type of X and not on the choice of embedding, etc.

Returning to the general situation, observe that the Thom space of a G-vector bundle V over X is a (pointed) G-space X^V. If $U=V\oplus\mathbf{R}^n$, where $n\geqslant 1$, then X^U is G-equivalent to S^nX^V. It follows that the stable homotopy-G type of X^V depends only on the class α of V in $J/G(X)$, and can therefore be denoted by X^α. We prove

THEOREM (6.1). *If $X^\alpha=X^0$, where $\alpha\in J/G(X)$, then $\alpha=0$.*

For suppose that X^V and X^T have the same fibre homotopy-G type, where V is a G-vector bundle and $T=X\times R^n$ is trivial. Proceeding as in the ordinary case we construct a homotopy-G retraction $S(V\oplus\mathbf{R})\to S^n$. Combining this with the projection $S(V\oplus\mathbf{R})\to X$ we obtain a fibre-preserving homotopy-G map

$$h:S(V\oplus\mathbf{R})\to S(T\oplus\mathbf{R})$$

which is a homotopy equivalence. Since h is a fibre homotopy-G equivalence this proves (6.1). Note that $X^\alpha=X^0$ if and only if X^α is homotopy-G S-coreducible.

Now let X be a smooth Riemannian G-manifold, without boundary. Let $T(X)$ denote the tangent G-vector bundle of X. By a straight-forward G-version of the proof of the corresponding result for ordinary manifolds, as given in §3 of [3], we obtain

THEOREM (6.2). *Let τ denote the class of $T(X)$ in $J/G(X)$. Then X^{α} and $X^{-\alpha-\tau}$ are dual, in the sense of stable homotopy-G type, for all $\alpha \in J/G(X)$.*

For general X there is an interesting subset $N/G(X) \subset J/G(X)$, consisting of those elements α such that X^{α} is homotopy-G S-trivial. Clearly $0 \in N/G(X)$ if and only if X-itself is homotopy-G S-trivial. When X is a G-manifold, as above, we obtain

COROLLARY (6.3). *If $\alpha \in N/G(X)$ then $-\alpha-\tau \in N/G(X)$.*

For example, consider once more the Z_2-space $P^n = P(L \oplus nR)$. The Thom spaces of the multiples of H are the stunted projective spaces studied in [12]. By (5.3) the homotopy-Z_2 S-type of the Thom space of rH $(r=1, 2, \ldots)$ depends only on the residue class of $r \bmod r_n$, where r_n is as in (4.4). The class of the tangent bundle is given by

$$T(P^n) \oplus \mathbf{R} \approx H \otimes (L \oplus nR). \tag{6.4}$$

Hence the Thom space of rH is homotopy-Z_2 S-trivial if and only if the (virtual) Thom space of $-[H \otimes (L \oplus (n+r)R)]$ is homotopy-Z_2 S-trivial. These results contain (1.4) and (1.5) of [12], with improvements when $n \equiv 1 \bmod 4$. Using (6.1) and (6.2) moreover we obtain

PROPOSITION (6.5). *The Thom space of the Z_2-vector bundle $H \otimes (L \oplus (m-n-2) \times R)$ over P^n $(m=0, \pm 1, \ldots)$ is homotopy-Z_2 S-reducible if and only if $m \equiv 0 \bmod r_n$.*

Write $n+1=k$ and consider the pair $(V_{m,k}, P_{m,k})$ where $P_{m,k}$ denotes the stunted projective space embedded in $V_{m,k}$ as described in §4 of [12]. Points of $V_{m,k}$ are represented, in the usual way, by matrices with k rows and m columns. We regard $V_{m,k}$ as a Z_2-space under the involution $T': V_{m,k} \to V_{m,k}$ which changes the sign of the first row, the first column and the last column. Then $P_{m,k}$ is Z_2-stable and can be identified with the Thom space of the Z_2-vector bundle $H \otimes (L \oplus (m-k-1)R)$ over $P(L \oplus (k-1)R)$ as described in (4.3) of [3]. Therefore (1.1) and (1.2) will follow from (6.5) and

LEMMA (6.6). *The Stiefel manifold $V_{m,k}$ admits a simple cross-section if and only if the Z_2-space $P_{m,k}$ is homotopy-Z_2 S-reducible.*

Since $T \simeq T'$, we can replace T by T' in the definition of simple cross-section. Now $V_{m,k}$ admits a simple cross-section if and only if $V_{3m,k}$ admits a simple cross-section, by (4.7) of [12]. Also $V_{3m,k}$ admits a simple cross-section if and only if $P_{3m,k}$ is homotopy-Z_2 S-reducible, since $(V_{3m,k}, P_{3m,k})$ is $(6m-2k)$-connected. Since $P_{3m,k}$ has the same homotopy-Z_2 S-type as $P_{m,k}$ this proves (6.6).

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