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The Construction of a Module of Finite Projective Dimension from a Finitely Generated Module of Finite Injective Dimension[1])

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§0. Introduction

In this article, the word "ring" will mean "commutative, Noetherian ring with non-zero multiplicative identity", and it is to be understood that ring homomorphisms respect identity elements. If A denotes a ring, then we shall use $\mathscr{P}(A)$ (resp. $\mathscr{T}(A)$) to denote the category of all finitely generated A-modules of finite projective (resp. injective) dimension, and all homomorphisms between them.

Let R denote a local ring. Bass conjectured ([2], p. 14) that for there to exist a non-zero module in $\mathscr{T}(R)$ it is necessary that R be a Cohen-Macaulay ring. A great deal of progress on this question was made by Peskine and Szpiro: in [7], they showed that, provided the ring R belongs to a certain very large class of local rings (which includes all the local rings of algebraic geometry), then the assertion of Bass's conjecture is true.

Actually, much of Peskine's and Szpiro's work in [7] is concerned with modules in $\mathscr{P}(R)$, and their results about Bass's conjecture are consequences of their "Intersection theorem" about modules in $\mathscr{P}(R)$. In order to apply their Intersection theorem to Bass's conjecture, they show that, provided the local ring R concerned is sufficiently well behaved, one may construct, from a module T in $\mathscr{T}(R)$, a module in $\mathscr{P}(R)$ which has the same support as T. It is this transformation from a module in $\mathscr{T}(R)$ to one in $\mathscr{P}(R)$ that is the concern of the present paper: Peskine and Szpiro used a roundabout method involving passage to the completion, and this paper presents a more direct construction.

Peskine's and Szpiro's work enables one to make such a construction provided that the local ring R concerned possesses a dualizing complex. (This condition would be satisfied if R were a homomorphic image of a Gorenstein local ring: see Chapter V of Hartshorne [4].) Assuming R satisfies this condition, they manufacture, from a module in $\mathscr{T}(R)$, a module in $\mathscr{P}(\hat{R})$ (where $\hat{R}$ denotes the completion of R), and then they use dualizing complex techniques to pass back to $\mathscr{P}(R)$. (See [7], Chapter I, §§4, 5.)

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In the present paper, we shall study modules over a local ring A which can be expressed as a homomorphic image of a Gorenstein local ring: in fact, we shall produce a natural functor $\Delta:\mathscr{T}(A)\rightarrow\mathscr{P}(A)$ which has the property that, if T is a module in $\mathscr{T}(A)$, then T and $\Delta(T)$ have the same annihilator. The construction given below is rather different from Peskine's and Szpiro's: it does not involve passage to the completion, and the theory of dualizing complexes is avoided in the presentation, although a reader who is familiar with Hartshorne's work will recognize that some of the ideas used have been inspired by [4].

§1. General Results About Bounded Injective Complexes and Bounded Flat Complexes

(1.1) TERMINOLOGY. Throughout §1, A denotes a ring, and $\mathscr{C}(A)$ denotes the category of all A-modules and all A-homomorphisms between them. The additive A-category ([6], 3.3) of all complexes of A-modules and morphisms (i.e. translations) of complexes will be denoted by $\mathscr{Y}(A)$. Capital letters followed by a dot, as in $X^{\cdot}$, will generally be used to denote objects of $\mathscr{Y}(A)$; if $X^{\cdot}\in\mathscr{Y}(A)$, the n-th term of $X^{\cdot}$ will be denoted by X^n, and the n-th differentiation of $X^{\cdot}$ by $d^n_{X^{\cdot}}:X^n\rightarrow X^{n+1}$. If M is an A-module, we shall also denote by M the complex of A-modules which has M as 0-th term and all its other terms zero. A morphism of complexes will generally be denoted by a small letter followed by a dot. If $u^{\cdot}:X^{\cdot}\rightarrow Y^{\cdot}$ is a morphism in $\mathscr{Y}(A)$, then $u^{\cdot}$ is a family of A-homomorphisms, the n-th member of which will be denoted by $u^n:X^n\rightarrow Y^n$. (Also $d^n_{Y^{\cdot}}\cdot u^n=u^{n+1}\cdot d^n_{X^{\cdot}}$ for all integers n, of course.)

A complex $X^{\cdot}\in\mathscr{Y}(A)$ will be said to be *bounded* if there is only a finite number of integers n for which $X^n\neq 0$. The full subcategory of $\mathscr{Y}(A)$ whose objects are the bounded complexes will be denoted by $\mathscr{Y}^b(A)$, while $\mathscr{Y}^b_c(A)$ will denote the full subcategory of $\mathscr{Y}^b(A)$ whose objects are the bounded complexes all of whose cohomology modules are finitely generated.

The functors $\otimes_A$ and Hom_A (from $\mathscr{C}(A)\times\mathscr{C}(A)$ to $\mathscr{C}(A)$) induce functors (also denoted by $\otimes_A$ and Hom_A respectively) from $\mathscr{Y}(A)\times\mathscr{Y}(A)$ to $\mathscr{Y}(A)$. (See Chapter IV, §5 of Cartan-Eilenberg [3] and §6.2 of Northcott [6].)

A complex $L^{\cdot}\in\mathscr{Y}(A)$ will be called *injective* (resp. *flat*) if, for all integers n, L^n is injective (resp. flat).

(1.2) LEMMA. *Suppose X, L, M are A-modules.*

(i) *There exists a (unique) A-homomorphism $\xi(X, L, M)$: $X\otimes_A\mathrm{Hom}_A(L, M)\rightarrow\mathrm{Hom}_A(\mathrm{Hom}_A(X, L), M)$ such that, for $x\in X$, $f\in\mathrm{Hom}_A(L, M)$ and $g\in\mathrm{Hom}_A(X, L)$, we have $[\xi(X, L, M)(x\otimes f)](g)=f(g(x))$. Furthermore, the $\xi(X, L, M)$ constitute a morphism of functors (from $\mathscr{C}(A)\times\mathscr{C}(A)\times\mathscr{C}(A)$ to $\mathscr{C}(A)$):*

$$(\ )\otimes_A\mathrm{Hom}_A(\ ,\ )\rightarrow\mathrm{Hom}_A(\mathrm{Hom}_A(\ ,\ ),\ ).$$

(ii) *If M is injective, $\xi(X, L, M)$ is an isomorphism whenever X is finitely generated.*

(iii) *If L and M are both injective, then* $\mathrm{Hom}_A(L, M)$ *is a flat A-module.*

Proof. This is straightforward, and left to the reader.

(1.3) The functors $(\ )\otimes_A \mathrm{Hom}_A(\ ,\ )$ and $\mathrm{Hom}_A(\mathrm{Hom}_A(\ ,\ ),\ )$ discussed in (1.2) induce additive functors, also denoted by $(\ )\otimes_A \mathrm{Hom}_A(\ ,\ )$ and $\mathrm{Hom}_A(\mathrm{Hom}_A(\ ,\ ),\ )$ respectively, from $\mathscr{Y}(A)\times\mathscr{Y}(A)\times\mathscr{Y}(A)$ to $\mathscr{Y}(A)$. (See §5 of Chapter IV of Cartan-Eilenberg [3].) Moreover, the morphism of functors ξ of (1.2) (i) induces a morphism of functors (from $\mathscr{Y}(A)\times\mathscr{Y}(A)\times\mathscr{Y}(A)$ to $\mathscr{Y}(A)$)

$$\eta_1(\ ,\ ,\ )^{\cdot}:(\ )\otimes_A \mathrm{Hom}_A(\ ,\ )\to \mathrm{Hom}_A(\mathrm{Hom}_A(\ ,\ ),\ ).$$

Now let $I^{\cdot}$ and $J^{\cdot}$ be injective, bounded complexes in $\mathscr{Y}(A)$. We may use the above morphism of functors to define a morphism $\eta(\ )^{\cdot}$ of functors (from $\mathscr{Y}(A)$ to $\mathscr{Y}(A)$) by

$$\eta(\ )^{\cdot}=\eta_1(\ ,I^{\cdot},J^{\cdot})^{\cdot}:(\ )\otimes_A \mathrm{Hom}_A(I^{\cdot},J^{\cdot})\to \mathrm{Hom}_A(\mathrm{Hom}_A(\ ,I^{\cdot}),J^{\cdot}).$$

It follows from (1.2) (ii) that $\eta(M)^{\cdot}$ is an isomorphism of complexes whenever M is a finitely generated A-module.

(1.4) DEFINITION. If n is an integer, then $H^n:\mathscr{Y}(A)\to\mathscr{C}(A)$ will denote the n-th cohomology functor. So, if $X^{\cdot}\in\mathscr{Y}(A)$, then $H^n(X^{\cdot})=\ker d^n_{X^{\cdot}}/\mathrm{im}\, d^{n-1}_{X^{\cdot}}$. A morphism of complexes $u^{\cdot}:X^{\cdot}\to Y^{\cdot}$ in $\mathscr{Y}(A)$ is said to be a *quasi-isomorphism* if, for all integers n, $H^n(u^{\cdot}):H^n(X^{\cdot})\to H^n(Y^{\cdot})$ is an isomorphism.

(1.5) THEOREM. *Let $I^{\cdot}$ and $J^{\cdot}$ be injective, bounded complexes in $\mathscr{Y}(A)$. Then, whenever $X^{\cdot}$ is a bounded complex in $\mathscr{Y}(A)$ with the property that all its cohomology modules are finitely generated, the morphism of complexes*

$$\eta(X^{\cdot})^{\cdot}:X^{\cdot}\otimes_A \mathrm{Hom}_A(I^{\cdot},J^{\cdot})\to \mathrm{Hom}_A(\mathrm{Hom}_A(X^{\cdot},I^{\cdot}),J^{\cdot})$$

of (1.3) *is a quasi-isomorphism.*

Proof. $X^{\cdot}\otimes_A \mathrm{Hom}_A(I^{\cdot},J^{\cdot})$ is the single complex $B^{\cdot}$ associated with a certain triple complex which has (i,j,k)-th component given by $B^{i,j,k}=X^i\otimes_A \mathrm{Hom}_A(I^{-j},J^k)$ for all integers i,j,k. (See §§4, 5 of Chapter IV of Cartan-Eilenberg [3].) Similarly, $\mathrm{Hom}_A(\mathrm{Hom}_A(X^{\cdot},I^{\cdot}),J^{\cdot})$ is the single complex $C^{\cdot}$ associated with a triple complex which has (i,j,k)-th component given by $C^{i,j,k}=\mathrm{Hom}_A(\mathrm{Hom}_A(X^i,I^{-j}),J^k)$ for all integers i,j,k. There are filtrations $\{(F^pB)^{\cdot}\}_p$ and $\{(F^pC)^{\cdot}\}_p$ of the complexes $B^{\cdot}$ and $C^{\cdot}$ respectively for which

$$(F^pB)^n=\sum_{\substack{i+j+k=n\\ j+k\geqslant p}} B^{i,j,k}\ \text{(direct sum)}\quad\text{and}\quad (F^pC)^n=\sum_{\substack{i+j+k=n\\ j+k\geqslant p}} C^{i,j,k}\ \text{(direct sum)}$$

for all integers p and n. Both these filtrations are regular, and so strongly convergent. Let the associated spectral sequences be denoted by

$$E_2^{p,q} \underset{p}{\Rightarrow} H^n(B^{\cdot}), \quad G_2^{p,q} \underset{p}{\Rightarrow} H^n(C^{\cdot})$$

The morphism of complexes $\eta(X^{\cdot})^{\cdot}: B^{\cdot} \to C^{\cdot}$ is compatible with these two filtrations ([3], Chapter XV, §3), and so induces homomorphisms $\eta_r^{p,q}: E_r^{p,q} \to G_r^{p,q}$ for all integers $r \geq 1$, p, q. It is not difficult to see that, for all p and q, there is a commutative diagram of A-homomorphisms

$$\begin{array}{ccc} E_1^{p,q} & \xrightarrow{\cong} & [H^q(X^{\cdot}) \otimes_A \operatorname{Hom}_A(I^{\cdot}, J^{\cdot})]^p \\ \downarrow{\scriptstyle \eta_1^{p,q}} & & \downarrow \\ G_1^{p,q} & \xrightarrow{\cong} & [\operatorname{Hom}_A(\operatorname{Hom}_A(H^q(X^{\cdot}), I^{\cdot}), J^{\cdot})]^p, \end{array}$$

in which the right hand vertical homomorphism is the p-th component of the morphism of complexes

$$\eta(H^q(X^{\cdot}))^{\cdot}: H^q(X^{\cdot}) \otimes_A \operatorname{Hom}_A(I^{\cdot}, J^{\cdot}) \to \operatorname{Hom}_A(\operatorname{Hom}_A(H^q(X^{\cdot}), I^{\cdot}), J^{\cdot})$$

of (1.3). But $H^q(X^{\cdot})$ is a finitely generated A-module, so that $\eta(H^q(X^{\cdot}))^{\cdot}$ is an isomorphism of complexes. Hence $\eta_1^{p,q}$ is an isomorphism for all p and q. That $\eta(X^{\cdot})^{\cdot}$ is a quasi-isomorphism now follows immediately from the strong convergence of the two filtrations. ([3], Chapter XV, theorem 3.2.)

Note. I am grateful to the referee for suggesting the above proof; my original was considerably longer.

The following spectral sequence arguments will be of considerable help when discussing bounded flat complexes or bounded injective complexes.

(1.6) (i) Let $X^{\cdot}$, $L^{\cdot}$ be complexes in $\mathscr{Y}(A)$, and suppose that $L^{\cdot}$ is flat and bounded. Consideration of the second spectral sequence of the appropriate double complex (which has regular second filtration) yields a spectral sequence

$$E_2^{p,q} = H^p(H^q(X^{\cdot}) \otimes_A L^{\cdot}) \underset{p}{\Rightarrow} H^n(X^{\cdot} \otimes_A L^{\cdot}).$$

(ii) Let $Y^{\cdot}$, $K^{\cdot}$ be complexes in $\mathscr{Y}(A)$ and suppose that $K^{\cdot}$ is injective and bounded. An examination (similar to that used above in (i)) of the second spectral sequence of the appropriate double complex (whose second filtration is regular) yields a spectral sequence

$$E_2^{p,q} = H^p(\operatorname{Hom}_A(H^{-q}(Y^{\cdot}), K^{\cdot})) \underset{p}{\Rightarrow} H^n(\operatorname{Hom}_A(Y^{\cdot}, K^{\cdot})).$$

The following technical corollary, which will be used in §2, gives some indication of the relevance of the above spectral sequences: the corollary can be proved by straightforward use of the technique of (1.6) (i) in conjunction with corollary 5.4 of Chapter XV of Cartan-Eilenberg [3].

(1.7) COROLLARY. *Suppose* $X^{\cdot}$, $L^{\cdot}$ *are complexes in* $\mathscr{Y}^b(A)$, *and that* $L^{\cdot}$ *is flat. If either* $X^{\cdot}$ *or* $L^{\cdot}$ *is an exact complex, then* $X^{\cdot}\otimes_A L^{\cdot}$ *is exact. However, if neither* $X^{\cdot}$ *nor* $L^{\cdot}$ *is exact, and* r *(resp.* s*) is the largest integer* i *for which* $H^i(X^{\cdot})$ *(resp.* $H^i(L^{\cdot})$*)* $\neq 0$, *then* $H^i(X^{\cdot}\otimes_A L^{\cdot})=0$ *for all* $i>r+s$, *while*

$$H^{r+s}(X^{\cdot}\otimes_A L^{\cdot})\cong H^r(X^{\cdot})\otimes_A H^s(L^{\cdot}).$$

§2. Application to Finitely Generated Modules of Finite Injective Dimension

(2.1) NOTATION. If C is a ring and X is a C-module, then the *support* of X is the set $\{\mathfrak{p}\in\operatorname{Spec}C: X_{\mathfrak{p}}\neq 0\}$; this will be denoted by $\operatorname{Supp}X$ (or $\operatorname{Supp}_C X$ if it is desirable to emphasize the ring concerned). If C is local, and X is non-zero and finitely generated, then the length of every maximal X-sequence is called the *depth* of X and denoted $\operatorname{depth}_C X$ (or depth X). In particular, depth C denotes the length of every maximal C-sequence. The reader is referred to Bass's paper [2] for details of the concepts of minimal injective resolution and Gorenstein ring.

Throughout this section, A will denote a local ring having depth t, and we shall assume that there exists a Gorenstein local ring B and a surjective ring homomorphism $\phi: B\to A$; we shall use m to denote the (Krull) dimension of B. (This restriction of attention to a local ring which is expressible as a homomorphic image of a Gorenstein local ring is only mild, as most local rings occurring in algebraic geometry and all complete local rings are expressible as homomorphic images of regular local rings, and a regular local ring is a Gorenstein local ring.) T will denote a non-zero finitely generated A-module of finite injective dimension: the purpose of this section is to construct a finitely generated A-module of finite projective dimension which has the same support as T.

We shall maintain the following further notation. Recalling ([2], (3.3)) that the injective dimension of T is equal to t, let

$$0\to T\to J^0\to J^1\to\cdots\to J^t\to 0$$

be a minimal injective resolution for T, and let $J^{\cdot}$ denote the complex

$$\cdots\to 0\to J^0\to J^1\to\cdots\to J^t\to 0\to\cdots$$

of A-modules and homomorphisms. So, in the notation of §1, $J^{\cdot}$ is an injective bounded complex in $\mathscr{Y}(A)$.

Also B, when considered as a module over itself, has finite injective dimension (equal to m); let

$$0 \to B \to K^0 \to K^1 \to \cdots \to K^m \to 0$$

be a minimal injective resolution for the B-module B. We shall use $K^{\cdot}$ to denote the bounded injective complex in $\mathscr{Y}(B)$ given by:

$$\ldots 0 \to K^0 \to K^1 \to \cdots \to K^m \to 0 \to \cdots.$$

Consider also the complex

$$\ldots 0 \to \mathrm{Hom}_B(A, K^0) \to \mathrm{Hom}_B(A, K^1) \to \cdots \to \mathrm{Hom}_B(A, K^m) \to 0 \ldots.$$

The terms of this complex have natural structures as A-modules, and when they are regarded as such the differentiations become A-homomorphisms. We shall use $I^{\cdot}$ to denote this complex when it is regarded as a member of $\mathscr{Y}(A)$. Note that, by proposition 6.1a of Chapter II of Cartan-Eilenberg [3], the terms of $I^{\cdot}$ are all injective A-modules; furthermore, for each integer i we have $H^i(I^{\cdot}) \cong \mathrm{Ext}^i_B(A, B)$ (where the latter is regarded as an A-module in the natural way), so that $H^i(I^{\cdot})$ is a finitely generated A-module. It follows that $I^{\cdot}$ is an injective bounded complex which actually belongs to $\mathscr{Y}^b_c(A)$.

We may now apply theorem (1.5) to deduce the existence of a quasi-isomorphism

$$\eta'(I^{\cdot})^{\cdot} : I^{\cdot} \otimes_A [\mathrm{Hom}_A(I^{\cdot}, J^{\cdot})] \to \mathrm{Hom}_A([\mathrm{Hom}_A(I^{\cdot}, I^{\cdot})], J^{\cdot}).$$

(Note that the complexes $I^{\cdot} \otimes_A [\mathrm{Hom}_A(I^{\cdot}, J^{\cdot})]$ and $I^{\cdot} \otimes_A \mathrm{Hom}_A(I^{\cdot}, J^{\cdot})$ are isomorphic; also the complexes $\mathrm{Hom}_A([\mathrm{Hom}_A(I^{\cdot}, I^{\cdot})], J^{\cdot})$ and $\mathrm{Hom}_A(\mathrm{Hom}_A(I^{\cdot}, I^{\cdot}), J^{\cdot})$ are isomorphic.) We shall use $F^{\cdot}$ and $E^{\cdot}$ to denote $\mathrm{Hom}_A(I^{\cdot}, J^{\cdot})$ and $\mathrm{Hom}_A(I^{\cdot}, I^{\cdot})$ respectively. In order to exploit the above quasi-isomorphism, we next make some calculations about the cohomology modules of the complexes $I^{\cdot}$, $F^{\cdot}$, $E^{\cdot}$, $I^{\cdot} \otimes_A F^{\cdot}$, and $\mathrm{Hom}_A(E^{\cdot}, J^{\cdot})$.

(2.2) LEMMA. *$(m-t)$ is the greatest integer i such that $H^i(I^{\cdot}) \neq 0$.*

Proof. We have $t = \mathrm{depth}\, A = \mathrm{depth}_B A$ (when A is regarded as a B-module by means of ϕ). Since B, when considered as a module over itself, has finite injective dimension, it follows from §2 of [5] that $m-t$ is the greatest integer i such that $\mathrm{Ext}^i_B(A, B) \neq 0$. Since, for all integers i, the A-modules $H^i(I^{\cdot})$ and $\mathrm{Ext}^i_B(A, B)$ are isomorphic, the result follows.

Henceforth, we shall use W to denote the non-zero, finitely generated A-module $H^{m-t}(I^{\cdot})$.

Next, we examine the cohomology modules of the complex $E^{\cdot}=\mathrm{Hom}_A(I^{\cdot}, I^{\cdot})$. However, we need one preliminary lemma.

(2.3) LEMMA. *Let* $L^{\cdot}=\mathrm{Hom}_B(K^{\cdot}, K^{\cdot})$, *and let* Y *be a* B*-module. Then*

$$H^i(Y\otimes_B L^{\cdot})=0 \quad \textit{if} \quad i\neq 0;$$

and

$$H^0(Y\otimes_B L^{\cdot})\cong Y.$$

Proof. It follows easily from (1.6) (ii) that

$$H^i(L^{\cdot})=0 \quad \text{if} \quad i\neq 0 \quad \text{and} \quad H^0(L^{\cdot})\cong B. \tag{$*$}$$

Next, using (1.2) (iii), $L^{\cdot}$ is a flat bounded complex in $\mathscr{Y}(B)$. (In fact, $L^i=0$ whenever $|i|>m$.) Let $Z^0=\ker d^0_{L^{\cdot}}$. It follows from ($*$) that there is an exact sequence

$$0\to Z^0\to L^0\to L^1\to\cdots\to L^m\to 0.$$

Since $L^0, L^1,\ldots, L^m$ are all flat B-modules, we can break this long exact sequence up into short exact sequences to deduce that Z^0 is flat, and that

$$0\to Y\otimes_B Z^0\to Y\otimes_B L^0\to\cdots\to Y\otimes_B L^m\to 0$$

is exact also. Now consider the exact sequence

$$0\to L^{-m}\to L^{-m+1}\to\cdots\to L^{-1}\to Z^0\to B\to 0,$$

obtainable from ($*$). Since Z^0 and B are flat B-modules, a similar argument to that used above shows that

$$0\to Y\otimes_B L^{-m}\to\cdots\to Y\otimes_B L^{-1}\to Y\otimes_B Z^0\to Y\otimes_B B\to 0$$

is exact also. The result follows.

Note. Using the theory of injective modules, one can show that $L^i=0$ whenever $i<0$. However, as this fact is not relevant to our discussion, its proof has been omitted.

We are now in a position to calculate the cohomology modules of $E^{\cdot}$.

(2.4) PROPOSITION. *Let* $E^{\cdot}$ *denote* $\mathrm{Hom}_A(I^{\cdot}, I^{\cdot})$. *Then*

$$H^i(E^{\cdot})=0 \quad \textit{if} \quad i\neq 0;$$

and

$$H^0(E^{\cdot})\cong A.$$

Proof. $\mathrm{Hom}_A(I^\cdot, I^\cdot)=\mathrm{Hom}_A(I^\cdot, [\mathrm{Hom}_B(A, K^\cdot)])$, where $\mathrm{Hom}_B(A, K^\cdot)$ is regarded as a member of $\mathscr{Y}(A)$ in the natural way. The ideas of §8.5 of [6] enable us to construct an isomorphism of functors from $\mathscr{C}(A)\times\mathscr{C}(B)$ to $\mathscr{C}(A)$:

$$\mathrm{Hom}_A(\ ,\mathrm{Hom}_B(A,\ ))\to\mathrm{Hom}_B(\ ,\ ).$$

This gives rise to an isomorphism of complexes in $\mathscr{Y}(A)$:

$$\mathrm{Hom}_A(I^\cdot, [\mathrm{Hom}_B(A, K^\cdot)])\to\mathrm{Hom}_B(I^\cdot, K^\cdot).$$

(The second of these two complexes has a natural structure as a member of $\mathscr{Y}(A)$, of course.)

Next,

$$\mathrm{Hom}_B(I^\cdot, K^\cdot)=\mathrm{Hom}_B([\mathrm{Hom}_B(A, K^\cdot)], K^\cdot).$$

But $K^\cdot$ is a bounded injective complex in $\mathscr{Y}(B)$, so we may adapt the ideas of (1.3) to produce an isomorphism of complexes in $\mathscr{Y}(A)$:

$$\mathrm{Hom}_B([\mathrm{Hom}_B(A, K^\cdot)], K^\cdot)\to A\otimes_B[\mathrm{Hom}_B(K^\cdot, K^\cdot)].$$

The result now follows from (2.3).

(2.5) COROLLARY. *The cohomology modules of the complex* $\mathrm{Hom}_A([\mathrm{Hom}_A(I^\cdot, I^\cdot)], J^\cdot)=\mathrm{Hom}_A(E^\cdot, J^\cdot)$ *are given by:*

$$H^i(\mathrm{Hom}_A(E^\cdot, J^\cdot))=0 \quad \textit{if} \quad i\neq 0;$$

and

$$H^0(\mathrm{Hom}_A(E^\cdot, J^\cdot))\cong T.$$

Proof. The technique of (1.6) (ii) provides us with a spectral sequence

$$E_2^{p,q}=\mathrm{Ext}_A^p(H^{-q}(E^\cdot), T)\underset{p}{\Rightarrow}H^n(\mathrm{Hom}_A(E^\cdot, J^\cdot)).$$

The information provided by (2.4) (together with the fact that A is a projective A-module) shows that $E_2^{p,q}=0$ if either $p\neq 0$ or $q\neq 0$; the result follows easily.

We now turn our attention to the complex $F^\cdot=\mathrm{Hom}_A(I^\cdot, J^\cdot)$, which (using (1.2) (iii)) is a flat, bounded complex in $\mathscr{Y}(A)$.

(2.6) PROPOSITION. *For each integer i, the A-module $H^i(F^\cdot)$ is finitely generated. Furthermore*

$$H^i(F^\cdot)=0 \quad \textit{whenever} \quad i<t-m;$$

and

$$H^{t-m}(F^{\cdot})\cong \mathrm{Hom}_A(W, T).$$

Proof. We use the technique from (1.6) (ii) of employing the second spectral sequence of an appropriate double complex to obtain a spectral sequence

$$E_2^{p,q}=\mathrm{Ext}_A^p(H^{-q}(I^{\cdot}), T)\underset{p}{\Rightarrow} H^n(\mathrm{Hom}_A(I^{\cdot}, J^{\cdot}))=H^n(F^{\cdot}).$$

All the cohomology modules of $I^{\cdot}$ are finitely generated; hence, for all integers p and q, $E_2^{p,q}$ is finitely generated, and so $E_\infty^{p,q}$ is finitely generated. Since the complexes $I^{\cdot}$ and $J^{\cdot}$ are both bounded, it follows that $H^n(F^{\cdot})$ is finitely generated for all integers n.

Also, $E_2^{p,q}=0$ if either $p<0$ or $q<t-m$ (since (by (2.2)) $m-t$ is the greatest integer i such that $H^i(I^{\cdot})\neq 0$). We can now apply corollary 5.4 of Chapter XV of [3] to see that $H^i(F^{\cdot})=0$ whenever $i<t-m$, while $H^{t-m}(F^{\cdot})\cong E_\infty^{0,t-m}=E_2^{0,t-m}$. But $W=H^{m-t}(I^{\cdot})$; hence $H^{t-m}(F^{\cdot})\cong \mathrm{Hom}_A(W, T)$.

(2.7) COROLLARY. *$(t-m)$ is the unique integer i for which $H^i(F^{\cdot})\neq 0$. Furthermore*

$$W\otimes_A \mathrm{Hom}_A(W, T)\cong T.$$

Proof. Using (1.5), there is a quasi-isomorphism

$$\eta'(I^{\cdot})^{\cdot}: I^{\cdot}\otimes_A F^{\cdot}\rightarrow \mathrm{Hom}_A([\mathrm{Hom}_A(I^{\cdot}, I^{\cdot})], J^{\cdot})=\mathrm{Hom}_A(E^{\cdot}, J^{\cdot}).$$

But we calculated (in (2.5)) the cohomology modules of the second of these two complexes: it follows that 0 is the unique integer i for which $H^i(I^{\cdot}\otimes_A F^{\cdot})\neq 0$, and $H^0(I^{\cdot}\otimes_A F^{\cdot})\cong T$.

Next we observe that $I^{\cdot}$ and $F^{\cdot}$ are in $\mathscr{Y}^b(A)$, and that $F^{\cdot}$ is flat; hence we may apply corollary (1.7), which immediately shows that $F^{\cdot}$ is not exact. So let s be the greatest integer i such that $H^i(F^{\cdot})\neq 0$; we know already from (2.2) that $m-t$ is the greatest integer i such that $H^i(I^{\cdot})\neq 0$, and that $W=H^{m-t}(I^{\cdot})$ is a finitely generated A-module. But (2.6) shows that $H^s(F^{\cdot})$ is finitely generated too; since A is a local ring, it follows that $W\otimes_A H^s(F^{\cdot})\neq 0$. We can now apply (1.7) again to deduce that

$$(m-t)+s=0,$$

and

$$H^{m-t}(I^{\cdot})\otimes_A H^s(F^{\cdot})\cong H^0(I^{\cdot}\otimes_A F^{\cdot}).$$

Thus $s=(t-m)$, so that (by (2.6)) $H^s(F^{\cdot})\cong \mathrm{Hom}_A(W, T)$. It follows that

$$W\otimes_A \mathrm{Hom}_A(W, T)\cong T.$$

Finally, (2.6) shows that $H^i(F^{\cdot})=0$ whenever $i<t-m=s$. It follows that $(t-m)$ is the unique integer i for which $H^i(F^{\cdot})\neq 0$.

(2.8) COROLLARY. *$\mathrm{Hom}_A(W, T)$ is a non-zero, finitely generated A-module which has finite projective dimension and has the same support as T.*

Proof. Let $Z^{t-m}=\ker d_{F^{\cdot}}^{t-m}$. Now $F^{\cdot}$ is bounded: suppose $F^i=0$ for all $i>u$. Then we can split the long exact sequence (which is obtainable from (2.7))

$$0\to Z^{t-m}\to F^{t-m}\to F^{t-m+1}\to \cdots \to F^u\to 0$$

into short exact sequences and use the fact that $F^{\cdot}$ is a flat complex to see that Z^{t-m} is a flat A-module. Next, we can use (2.7) again to see that there is an exact sequence

$$0\to F^v\to F^{v+1}\to \cdots \to F^{t-m-1}\to Z^{t-m}\to \mathrm{Hom}_A(W, T)\to 0,$$

where v is chosen so that $F^i=0$ for all $i<v$. Since $F^{\cdot}$ is a flat complex and Z^{t-m} is a flat A-module, we see that $\mathrm{Hom}_A(W, T)$ has finite weak homological dimension. (See §7.9 of [6].) But $\mathrm{Hom}_A(W, T)$ is a finitely generated A-module, so that ([6], §7.9, theorem 19) $\mathrm{Hom}_A(W, T)$ has finite projective dimension.

Finally, the isomorphism $W\otimes_A \mathrm{Hom}_A(W, T)\cong T$ of (2.8) ensures that T and $\mathrm{Hom}_A(W, T)$ have the same annihilator, and hence the same support.

We have now virtually proved the main theorem of the paper; however, before we state this theorem, note that there is a natural homomorphism $\psi(T)$: $W\otimes_A \mathrm{Hom}_A(W, T)\to T$ for which, if $w\in W$ and $f\in \mathrm{Hom}_A(W, T)$ we have $\psi(T)(w\otimes f)=f(w)$. Now as it stands, (2.7) shows only that $W\otimes_A \mathrm{Hom}_A(W, T)$ and T are isomorphic A-modules, and gives no information at all about the mapping (or mappings) which provide this isomorphism; moreover, the task of attempting to extract such information from the proof of (2.7) is daunting, since that proof involved several spectral sequence arguments. However, we shall see below that a direct and simple argument is available for showing that $\psi(T)$ is an isomorphism.

(2.9) THEOREM. *Suppose A is a local ring which has depth t and can be expressed as a homomorphic image of a Gorenstein local ring B of Krull dimension m. Let $W=\mathrm{Ext}_B^{m-t}(A, B)$, regarded as an A-module in the natural way. Let $\mathscr{P}(A)$ (resp. $\mathscr{T}(A)$) denote the category of all finitely generated A-modules of finite projective (resp. injective) dimension, and all homomorphisms between them.*

Then, whenever T' is a module in $\mathscr{T}(A)$, we have that $\mathrm{Hom}_A(W, T')$ is in $\mathscr{P}(A)$.

In other words, we have a functor

$$\mathrm{Hom}_A(W,\ ): \mathscr{T}(A) \to \mathscr{P}(A).$$

Furthermore, if T' is a module in $\mathscr{T}(A)$, then the natural homomorphism $\psi(T')$: $W \otimes_A \mathrm{Hom}_A(W, T') \to T'$ is an isomorphism, so that T' and $\mathrm{Hom}_A(W, T')$ have the same annihilator, and hence the same support.

Proof. Although our previous work in §2 has concerned a non-zero module T in $\mathscr{T}(A)$, the zero module (which is a member of $\mathscr{T}(A)$) presents no difficulties. In view of (2.8), it remains only to show that whenever T is a non-zero module in $\mathscr{T}(A)$, then $\psi(T)$ is an isomorphism.

To achieve this, first note that $\mathrm{Hom}_A(W, T)$ is finitely generated: let F be a finitely generated free module for which there exists an exact sequence

$$F \to \mathrm{Hom}_A(W, T) \to 0.$$

Let p denote the rank of F; tensoring this exact sequence with W and using (2.7) shows that there is an exact sequence

$$\oplus p W \xrightarrow{\gamma} T \to 0$$

(where $\oplus p W$ denotes the direct sum of p copies of W).

Now $W \otimes_A \mathrm{Hom}_A(W, T)$ and T are isomorphic A-modules. In order to show that $\psi(T)$ is an isomorphism, it is sufficient (by [1], Chapter 6, exercise 1 (i)) to show that $\psi(T)$ is surjective. So suppose $0 \neq x \in T$. Now $x \in \mathrm{im}\,\gamma$, so if we represent γ by the matrix $[f_1\ f_2\ \dots\ f_p]$ with the f_i in $\mathrm{Hom}_A(W, T)$, then there exist $w_1, w_2, \dots, w_p \in W$ such that $\sum_{i=1}^{p} f_i(w_i) = x$. Therefore $x \in \mathrm{im}(\psi(T))$; consequently $\psi(T)$ is surjective.

(2.10) *Remark.* In order to make the account independent of the theory of dualizing complexes, we proved the main theorem ((2.9)) under the hypothesis that the local ring A concerned could be expressed as a homomorphic image of a Gorenstein local ring; however, in [7] (Chapitre I, théorème 5.7) Peskine and Szpiro proved a theorem of a similar type on the assumption that the local ring concerned possessed a dualizing complex. While it is known that any homomorphic image of a Gorenstein local ring possesses a dualizing complex, it is not known (as far as the present author is aware) whether a local ring which possesses a dualizing complex must be expressible as a homomorphic image of a Gorenstein local ring. (§10 of Chapter V of Hartshorne [4] seems to indicate that distinguishing between these two classes of local rings will be rather difficult.)

Consequently, it is appropriate to end this paper by indicating how the above argument may be adapted to deal with the case of a finitely generated module T' of finite injective dimension over a local ring A' which possesses a dualizing complex.

However, this is easy, since the existence of a dualizing complex for A' implies the existence of a bounded injective complex $I'^{\cdot}$ in $\mathscr{Y}(A')$ all of whose cohomology modules are finitely generated, and which has the further property that

$$H^i(\mathrm{Hom}_{A'}(I'^{\cdot}, I'^{\cdot}))=0 \quad \text{if} \quad i\neq 0, \text{ while}$$
$$H^0(\mathrm{Hom}_{A'}(I'^{\cdot}, I'^{\cdot}))\cong A'.$$

If we let k be the greatest integer i such that $H^i(I'^{\cdot})\neq 0$, and write $W'=H^k(I'^{\cdot})$, then an argument almost identical to that employed in (2.5)–(2.9) above will show that $\mathrm{Hom}_{A'}(W', T')$ is a finitely generated A'-module of finite projective dimension having the same support as T', and moreover the natural A'-homomorphism

$$\psi'(T'): W'\otimes_{A'}\mathrm{Hom}_{A'}(W', T')\to T'$$

is an isomorphism.

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