A New Bound for the Genus of a Nilpotent Group

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Introduction

Gr is the category of groups with the usual morphisms; [Gr] the class of isomorphism classes of Gr; η_0 the full subcategory of Gr consisting of all finitely generated infinite nilpotent groups with finite commutator subgroups; N_p and $\hat{N}_p$ are the p-localization and the p-completion of the nilpotent group N.

Assume that N is a finitely generated nilpotent group. In the sense of Pickel ([1]) the genus set of N, denoted by $G_p(N)$, is the set of isomorphism classes of the finitely generated nilpotent groups M with $\hat{N}_p \simeq \hat{M}_p$ for each prime p, and $N_0 \simeq M_0$ (N_0 is the rationalization of N). In the sense of Mislin, the genus set of N, $G(N)$, is the set of the finitely generated nilpotent groups M with $M_p \simeq N_p$ for each prime p (see [2]).

In this note, we are only concerned with groups N in η_0; in this case, it has just been proved by Warfield in [3] that $G_p(N) = G(N)$.[(1)] This result is also an easy consequence of our proof.

This paper provides a bound for $G_p(N)$, when N is in η_0, thus a bound for $G(N)$ (see the theorem); examples in section 3 show that this bound can be an improvement of Mislin's one ([2]).

1. Preliminaries

For a nilpotent group N, we denote its maximal torsion subgroup by TN, the p-components by T_pN and the quotient N/TN by SN.

LEMMA 1. *Let N be in η_0, M a finitely generated nilpotent group with $\hat{M}_p \simeq \hat{N}_p$ for each prime p. Then*

(*a*) $TN \simeq TM$ (*b*) $SN \simeq SM$

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(1) I am very grateful to Professor Mislin for having reported me this fact before the publication of [3].

(*c*) *M belongs to* η_0 (*d*) $N_0 \simeq M_0$

(*e*) *the class of M belongs to* $G_p(N)$.

Proof. (a) is proved by Pickel ([1], proposition 3.5).

(b) SN is abelian (since $[N, N]$ is included in TN) thus free abelian; then SM is also abelian as a subgroup of $\widehat{SM}_p$ which is isomorphic to $\widehat{SN}_p$ and SM is free abelian with the same rank since SN and SM have isomorphic finite quotients ([1], proposition 3.5).

(c) Since SM is abelian, $[M, M]$ is included in TM which is finite.

(d) $N_0 \simeq (SN)_0 \simeq (SM)_0 \simeq M_0$.

(e) obvious, by hypothesis and *d*.

Among the invariants in $G_p(N)$, we have thus
—the groups TN and SN, which we shall denote simply by T and S.
—the rank k of S.
—the set Q of primes p for which T_p is non-trivial.

The lemma shows us that the class of M belongs to $G_p(N)$ if and only if $\hat{N}_p \simeq \hat{M}_p$ for each prime p and that each element in $G_p(N)$ can be described as the class of the central term M of an extension:

$$T \rightarrowtail M \twoheadrightarrow S$$

With this description, we have only to consider primes in Q, since, for $p \notin Q$, $\hat{M}_p \simeq \hat{S}_p \simeq \hat{N}_p$. This is the basis of the proof of the theorem; before stating it, we need two further notions.

From [4] we recall the definition of Blackburn's function for a prime p.

It is the function defined recursively by

$$d_p(0) = 0$$

$$d_p(n) = d_p(n-1) + m$$

where p^m is the highest power of p that does not exceed n.

With the aid of d_p, we introduce

$$u(p) = t(p) + d_p(\text{nil } N) \quad \text{where} \quad \exp T = \prod_{p \in Q} p^{t(p)}$$

and

$$u = \prod_{p \in Q} p^{u(p)}$$

2. The Main Result

THEOREM. *Let N be in η_0. Then* $|G_p(N)| \leq \phi(u)/2$ (ϕ is Euler's function and we assume $u > 2$).

Proof. Since all the products have Q as their index set, we shall omit the subscripts. Define $E(X, Y)$ to be the class of equivalence classes of all the extensions

$$Y \rightarrowtail K \twoheadrightarrow X$$

θ is the map from $E(S, T)$ to [Gr] defined by "take the isomorphism class of the central term."

We know that $G_p(N)$ is included in Im θ. In particular, we can choose x_0 in $E(S, T)$ such that $\theta(x_0)$ is the class of N.

We intend to build a diagram Δ:

$$\begin{tikzcd}
\hat{\Omega} = \prod \mathrm{GL}(k, \hat{\mathbb{Z}}_p) \arrow[d, "\delta"] \arrow[dr, "\lambda"] & & \\
\mathrm{GL}(k, \mathbb{Z}/u) \arrow[r, "\mu"] \arrow[d, "\gamma"] & E(S, T) \arrow[r, "\theta"] & {[\mathrm{Gr}]} \\
(\mathbb{Z}/u)^*/\{1, -1\} \arrow[urr, "\nu"] & &
\end{tikzcd}$$

where $(\mathbb{Z}/u)^*/\{1, -1\}$ is the cokernel of $(\mathrm{GL}(k, \mathbb{Z}) \twoheadrightarrow \mathrm{GL}(k, \mathbb{Z}/u))$; γ, δ are the canonical epimorphism; the two triangles are commutative; $G_p(N) = \mathrm{Im}\,\nu = \mathrm{Im}\,(\theta \circ \mu)$.

The result follows from $|(\mathbb{Z}/u)^*/\{1, -1\}| = \phi(u)/2$.

First step: Definition of λ

For each p in Q, there exists a map C_p from $E(S, T)$ to $E(\hat{S}_p, T_p)$ defined by completion (we know that the completion is exact). Let us denote by $\hat{C}$ the product map. C_p can be factorized into a map τ_p from $E(S, T)$ to $E(S, T_p)$:

$$\tau_p(T \rightarrowtail E \twoheadrightarrow S) = (T_p \rightarrowtail E/T_{p'}, E \twoheadrightarrow S)$$

(p' is the complement of p in the set of prime numbers) and the p-completion $\bar{C}_p$, applied now to $E(S, T_p)$ see [ ] p. 339). We denote by τ and $\bar{C}$ the product maps.

On the other hand, we can define:

a map pb from $\prod E(\hat{S}_p, T_p)$ to $\prod E(S, T_p)$ by pull-back along the canonical injections of S in $\hat{S}_p$

and a map PB from $\prod E(S, T_p)$ to $E(S, T)$ defined by

$$\begin{pmatrix} T_p \rightarrowtail E_p \overset{\sigma_p}{\twoheadrightarrow} S \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ T_q \rightarrowtail E_q \overset{\sigma_q}{\twoheadrightarrow} S \end{pmatrix} \longmapsto (T \rightarrowtail E \twoheadrightarrow S)$$

where E is the pull-back of the groups E_p along the morphisms σ_p.

It is easy to check that $\bar{C}$ and pb, τ and PB, are inverses of one another. Thus:

$$E(S, T) \simeq \prod E(\hat{S}_p, T_p)$$

Now, each ω_p belonging to $\mathrm{GL}(k, \hat{\mathbb{Z}}_p)$ acts on $E(\hat{S}_p, T_p)$ by pull-back along ω_p^{-1}. We define $\hat{\Omega} = \prod GL(k, \hat{\mathbb{Z}}_p)$ and

$$\lambda : \hat{\Omega} \rightarrow E(S, T) : (\omega_p) \mapsto \bar{C}^{-1}(\omega_p C_p(x_0))$$

Second step: Definition of μ

Consider δ to be the canonical epimorphism from $\hat{\Omega}$ onto $\mathrm{GL}(k, \mathbb{Z}/u) \simeq \prod GL(k, \mathbb{Z}/p^{u(p)})$.

The existence of μ and the commutativity of the upper triangle of Δ are proved by the

LEMMA 2. *If ω belongs to ker δ, then $\lambda(\omega) = x_0$.*

Proof. For simplicity, if $\hat{X}_p$ is a p-complete nilpotent group, we denote by $\hat{X}_p^{(u)}$ the subgroup of $\hat{X}_p$ generated by the $u(p)$-powers and by $\hat{X}_p^{/u}$ the quotient of $\hat{X}_p$ modulo $\hat{X}_p^{(u)}$. If ω belongs to ker δ, the map induced by ω_p on $\hat{S}_p^{/u}$ is the identity. On the other hand, by our choice of $u(p)$, each element of $\hat{N}_p^{(u)}$ is a $p^{t(p)}$-power ([4], Lemma 2), thus Im $T_p \cap \hat{N}_p^{(u)} = \{1\}$. We can consider the commutative diagrams

$$\begin{tikzcd} T_p \arrow[r, tail] \arrow[d, equal] & \hat{N}_p \arrow[r, two heads, dashed] \arrow[d] & \hat{S}_p \arrow[d] \\ T_p \arrow[r, tail] & \hat{N}_p^{/u} \arrow[r, two heads] & \hat{S}_p^{/u} \end{tikzcd}$$

where the right squares are pull-backs. The action of the ω_p's does not modify the bottoms, neither, consequently, the tops, and thus $\lambda(\omega) = x_0$.

Third Step: $\mathrm{Im}\,(\mu \circ \theta) = G_p(N)$

If the class of M belongs to $\mathrm{Im}\,(\mu \circ \theta)$, M is clearly finitely generated and nilpotent, and $\hat{M}_p$ is isomorphic to $\hat{N}_p$ for each p; Lemma 1 shows that the class of M belongs to $G_p(N)$.

Suppose now that the class of M belongs to $G_p(N)$. We know that there exists

$$x : T \rightarrowtail M \twoheadrightarrow S$$

and, for each p, a commutative diagram

$$\begin{tikzcd} \hat{C}_p x_0 : T_p \arrow[r, tail] \arrow[d, "\xi_p"', "\wr"] & \hat{N}_p \arrow[r, two heads] \arrow[d, "\epsilon_p"', "\wr"] & \hat{S}_p \arrow[d, "\omega_p"', "\wr"] \\ \hat{C}_p x : T_p \arrow[r, tail] & \hat{M}_p \arrow[r, two heads] & \hat{S}_p \end{tikzcd}$$

We state that M is isomorphic to the central term of $\mu(\delta(\omega)) = \lambda(\omega)$, where $\omega = (\omega_p)$. Indeed, there exists, for each p, a commutative diagram

$$\begin{tikzcd} T_p \arrow[r, tail] \arrow[d, "\xi_p"', "\wr"] & N_{\omega_p} \arrow[r, two heads] \arrow[d, "\epsilon_p"', "\wr"] & \hat{S}_p \arrow[d, equal] \\ T_p \arrow[r, tail] & \hat{M}_p & \hat{S}_p \end{tikzcd}$$

the first row being $\omega_p C_p x_0$. Applying *pb* and then *PB*, we obtain the required isomorphism.

Fourth step: Definition of ν

We can identify $\mathrm{GL}(k, \mathbb{Z})$ with a subgroup of $\hat{\Omega}$. Now we prove:

LEMMA 3. *Let* ω_1, ω_2 *be in* $\hat{\Omega}$ *and suppose there exists* η *in* $\mathrm{GL}(k, \mathbb{Z})$ *such that* $\omega_2 = \eta\omega_1$. *Then* $(\theta \circ \lambda)\omega_1 = (\theta \circ \lambda)\omega_2$.

Proof. For each p, there exists a commutative diagram

$$\begin{tikzcd} T_p \arrow[r, tail] \arrow[d, equal] & N_{\omega_{1,p}} \arrow[r, two heads] \arrow[d, "\wr"] & \hat{S}_p \arrow[d, "\wr"] \\ T_p \arrow[r, tail] & N_{\omega_{2,p}} \arrow[r, two heads] & \hat{S}_p \end{tikzcd}$$

where the rows are $\omega_{1,p}C_p x_0$ and $\omega_{2,p}C_p x_0$ and η_p is the extension of η to $\hat{S}_p$. Such a diagram still exists after applying pb, with the same η for each p and PB produces isomorphic central terms.

In order to define ν and assure the commutativity of the lower triangle of Δ, we have to prove that $(\theta\circ\mu)\,\delta(\omega)$ only depends on the class of $\delta(\omega)$ modulo $\delta(\mathrm{GL}(k,\mathbb{Z}))$. Suppose $\delta(\omega_2)=\delta(\eta)\cdot\delta(\omega_1)$. Then: $(\theta\circ\mu)\,\delta(\omega_2)=(\theta\circ\mu)\,\delta(\eta\omega_1)=(\theta\circ\lambda)(\eta\omega_1)=(\theta\circ\lambda)\omega_1$ (by Lemma 3) $=(\theta\circ\mu)\,\delta(\omega_1)$. The proof of the theorem is completed.

Remark. The proof can be rewritten with p-localization instead of p-completions. All differences disappear after the second step, and the third implies the equality, in η_0, of $G_p(N)$ and $G(N)$ (see [3] for another proof.)

3. Comparison with Mislin's Result

Since the theorem gives a bound for $|G(N)|$, it is interesting to compare this bound to that established by Mislin in [2]. We adopt the improvement of Mislin's bound introduced at the end of [5] and we follow the notation of this paper as far as possible. We have to compare $u(p)=t(p)+d_p(\mathrm{nil}\,N)$ to the exponents $v(p)$ of $\exp(QN)_{ab}$ where $QN=N/(ZN)^{\exp TZN}$. First, two remarks.

If N is abelian, $N=ZN$, $TN=TZN$, $\exp(QN)_{ab}=\exp TN$ and $d_p(1)=0$ for any p: the two bounds are always equal. More generally, if we assume that TN is central (and $2\notin Q$), then $\mathrm{nil}\,N\leq 2$ and $d_p(\mathrm{nil}\,N)=0$; $(QB)_{ab}$ contains at least a copy of $\mathbb{Z}/\exp TN$, and $u(p)\leq v(p)$. In this case, we can have $u(p)<v(p)$ (see below).

Our bound can give new information, as it is shown by the following class of examples (a modified version of examples elaborated by Professor G. Frei, whom I thank here). The proof is standard and we omit it.

Consider groups N with generators $a_1, a_2, a_3, \ldots, a_{2n+1}$, relations $[a_i, a_j]=1$ except if i is odd and $j=i+1$; in this case, $[a_i, a_{i+1}]=a_{i+2}$; we assume also that the a_i's are of finite order for $i\in J=\{3,5,\ldots,2n+1\}$. From $[a_i^m, a_{i+1}]=a_{i+2}^m$ $(i=3,5,\ldots,2n-1)$, we conclude that the orders r_i of $a_i (i\in J)$ are related by $r_{2n+1}\,|\,r_{2n-1}\cdots|\,r_3$. We assume $r_{2n+1}\neq 1$, and we write s_i for $r_i/r_{i+2} (i\in J)$ with $r_{2n+3}=1$. For $i\geq 2$, $\Gamma_i(N)$ is generated by $\{a_j\,|\,j\in J, j\geq 2i-1\}$. Thus

N is nilpotent of class $n+1$ and its commutator subgroup is finite. ZN is generated by $a_1^{r_3}$, $a_2^{r_3}$, $a_3^{r_5}, \ldots,$ a_{2n+1} and $TZN \simeq \prod_{i\in J} \mathbb{Z}/s_i$; its exponent is the LCM of the s_i's, denoted by s. r_{2n+1} divides s such that s is not 1. $ZN^{\exp TZN}$ is the free abelian subgroup generated by $a_1^{sr_3}$, $a_2^{sr_3}$, $a_4^{sr_5}, \ldots.$ Finally, $(QN)_{ab}$ is the abelian group with $a_1, a_2, a_4, \ldots, a_{2n}$ as generators and relations $a_1^{sr_3}=1, a_2^{sr_3}=1, \ldots, a_{2n}^{sr_{2n+1}}=1$; its exponent is sr_3. On the other hand, TN is generated by $\{a_i \mid i \in J\}$. It is abelian and its exponent is r_3.

If $x(p)$ represents the exponent of the p-component of the natural x,

$$v(p)-u(p)=s(p)-d_p(n+1)$$

where

$$s(p)=\max\,(r_i(p)-r_{i+2}(p))$$

$$s(p)\geq r_{2n+1}(p);$$

$$s(p)\geq 1 \quad \text{for each} \quad p \quad \text{in} \quad Q$$

We can conclude:

—for any choice of the nilpotency class except 1 (n fixed) and any choice of $Q(2\notin Q)$ there exists a group in η_0 for which our bound is strictly smaller than Mislin's one: it is sufficient to choose r_{2n+1} adequately.

—on the other hand, if we fix a bound for the exponent of TN, it is always possible (now choosing n) to find a group in η_0 for which Mislin's bound is better.

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