A Generalization of Milnor's Inequality Concerning Affine Foliations and Affine Manifolds

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Consider a bundle or continuous family of 2n-dimensional vector spaces over a compact 2n-dimensional manifold. With choices of orientation there is an integer invariant of the bundle, the Euler number, which measures the precise obstruction to a non-zero section. As we vary the bundle keeping the manifold fixed the Euler number takes on *infinitely* many values.

In dimension two the Euler number is a complete invariant of the isomorphism class of the bundle. In higher dimensions, the Euler class[1] and the characteristic classes of Pontryagin determine the isomorphism class up to finite number of possibilities.

Now suppose the family is endowed with a continuous system of isomorphisms between the vector spaces V_x and $V_{x'}$ over nearby points x and x' in the manifold M. We assume the natural compatibility of isomorphisms for three nearby points x, x', and x''. This system of isomorphisms can be pictured as a foliation of the total space of the bundle transverse to the fibres with the zero section a leaf and the holonomy linear.

We will construct *a finite upper bound k_M depending only on the topology of the manifold M for the absolute value of the Euler number of a bundle admitting one of these affine (or linear) foliations.* If M is surface of genus g, we recover the inequality of Milnor [M],

Euler number of a 2-plane bundle over M with affine foliation $< g$.

In dimension 2 Milnor showed that this condition is also sufficient for a 2-plane bundle to admit a linear foliation transverse to the fibres. In higher dimensions we are far from such a sufficient condition.

Our argument which is quite direct and geometric, clarifies John Wood's generalization of Milnor's work to foliated S^1-bundles over surfaces, [W]. This aspect of the paper was worked out with John Morgan.

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[1] The Euler characteristic class is determined by the Euler numbers of the bundle over submanifolds representing a Q-homology basis, [T].

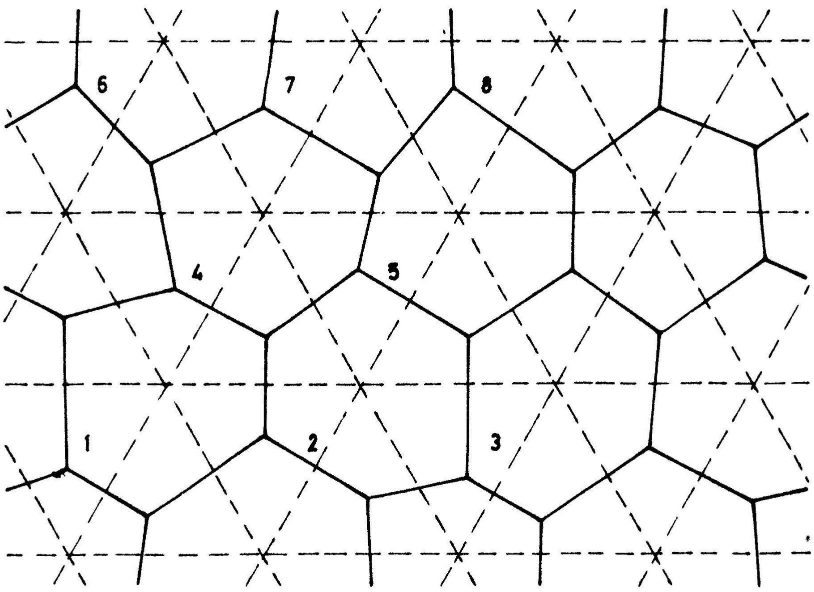
Our motivation beyond curiosity about Milnor's and Wood's work arising from conversations with Bill Thurston was to achieve the following corollary:

The collection of n-dimensional vector bundles over a compact triangulable space (n fixed) which admit affine foliations fall into finitely many isomorphism classes.[(1)]

The Chern–Weil description of the Pontryagin classes shows that they vanish for a bundle with a linear foliation. So the corollary follows from the inequality on the Euler number, the representability of a rational basis of homology by manifolds [T], and the remark above about finite determination of bundles. This latter fact in turn follows by obstruction theory and rational localization as described in [S p. 26].

Now we turn to the geometric argument concerning the Euler number. Actually, we will make use of a foliation transverse to the fibres of the $2n$-1 sphere bundle of oriented directions associated to the vector bundle over M^{2n}. We assume the isomorphisms induced by the foliation between nearby spheres are *affine* i.e. they are induced by linear maps of associated vector spaces. Such a foliation in the sphere bundle can be induced by the linear foliation in the vector bundle by dividing by the action of the multiplicative group of positive real numbers acting by homothety.

An upper bound for the Euler numbers can be calculated using any nice decomposition of M into simply connected pieces. For example consider the decomposition dual to some triangulation of M:



Our tactic will be to make a choice in the sphere of directions over a point in one piece of the decomposition and then push that choice around over the piece

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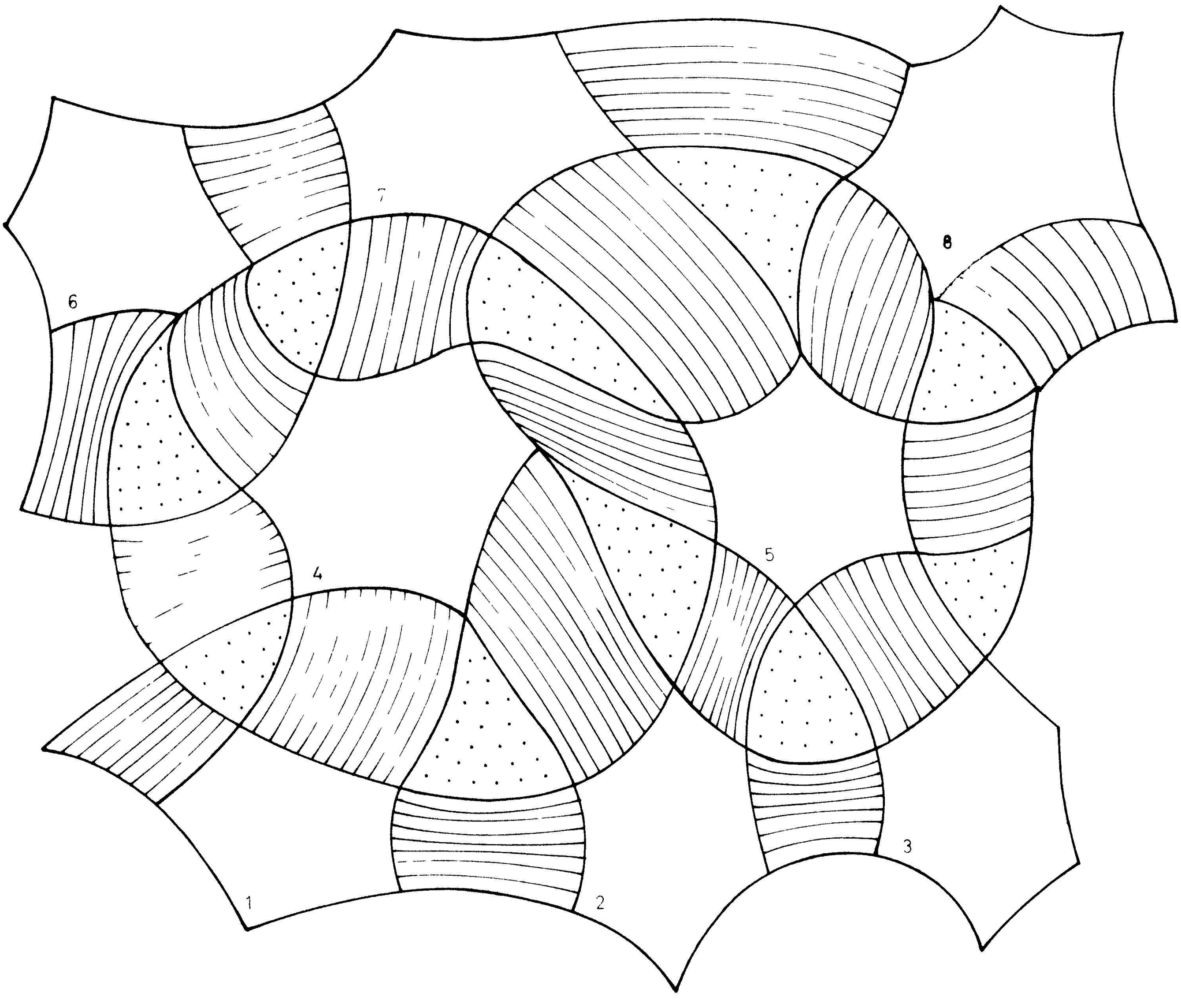
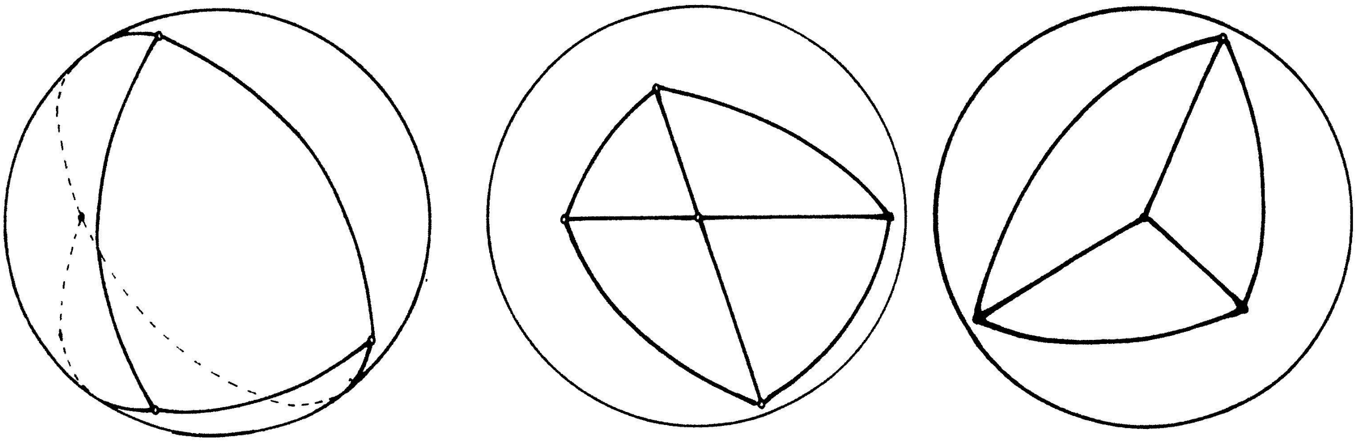
[1] This also follows from the finiteness of the number of components of the real algebraic variety of representations of π_1 into Gl (n, R). (From a discussion with George Lusztig).

using the canonical isomorphisms. The strategy will be try to build a homological cross section of the sphere bundle by starting over the top dual cells and proceeding down through the lower dimensional cells until we meet "estimable" obstructions.

So we start by choosing an oriented direction i.e. a point of S^{2n-1}, over one point in each top cell and employ the spreading by canonical isomorphisms to obtain cross sections over the various top cells which probably disagree at the codimension 1 cells. From each codimension 1 cell choose a point and in the sphere of directions above this point construct an arc between the two points determined by the previous choices in the two adjacent top cells. Now spread these arcs over the codimension 1 cells to partially fill in the discontinuity of our preliminary cross section. Now over a point in a codimension 2 cell we find the boundary of a triangle. We fill in this boundary with a triangle as before (see figure) and proceed in this way down to the zero dimensional cells.

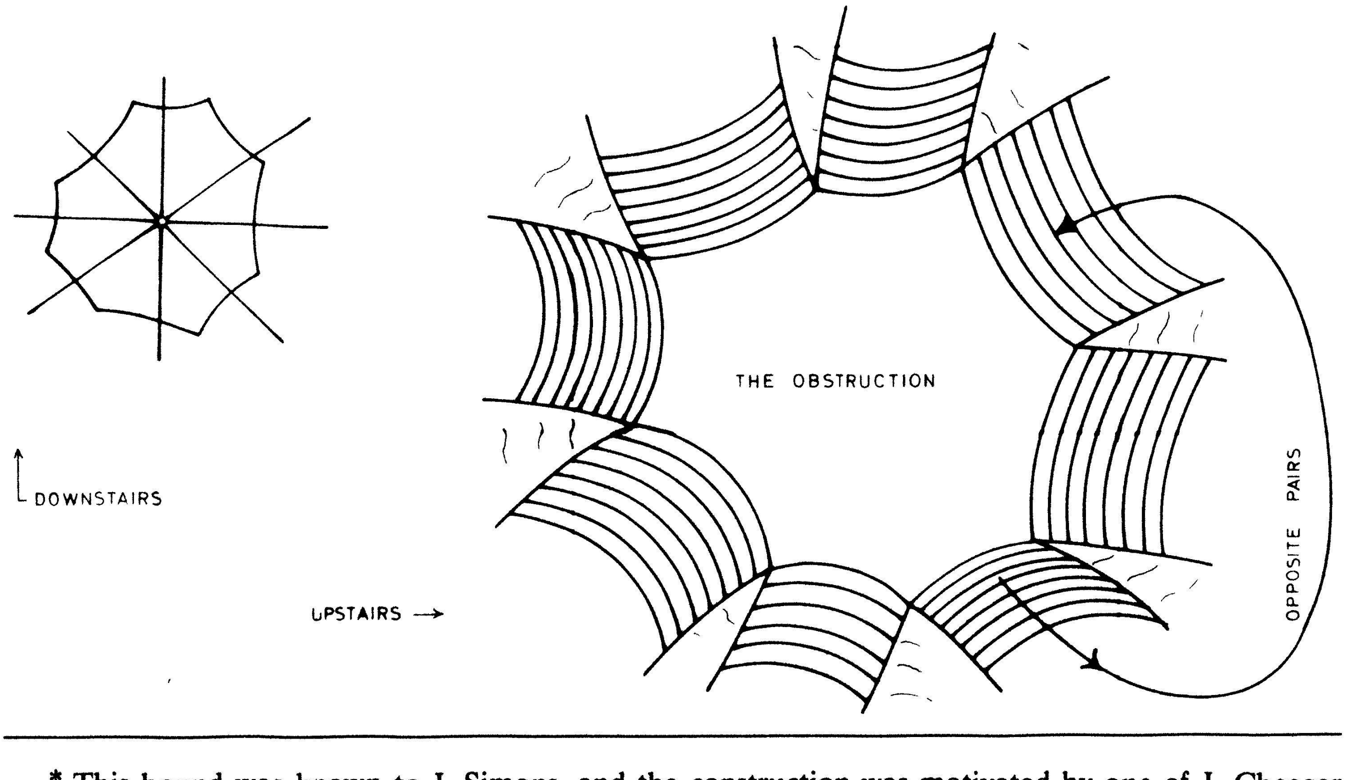


Now a linear map induces on the sphere of directions an *affine* map which preserves antipodal points and the class of geodesic simplices. So if we inductively choose geodesic arcs, geodesic triangles, geodesic tetrahedra, etc. we would have for each zero cell constructed a map of the boundary of the $(2n)$ simplex into S^{2n-1} where each face is carried into a geodesic simplex. Furthermore if we oriented the hypothesis and the constructions we would have a geodesic cycle and could define an integral degree. The degree of such a cycle is easily seen to be 0 or ± 1 (in any case for our purposes it is bounded by area considerations)

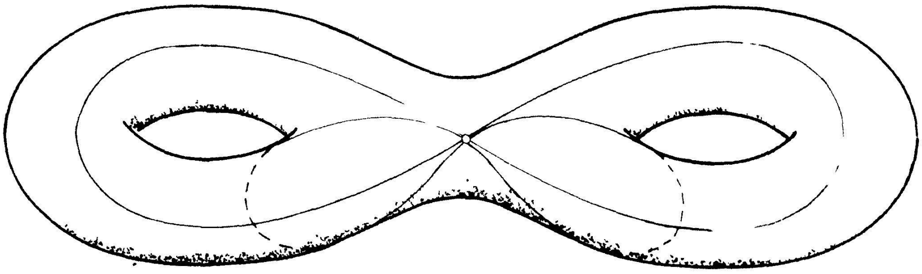


DEGREE ± 1 DEGREE 0

Since the sum of these degrees is the Euler number of our sphere bundle we see we *can take k_M to be the number of $2n$-simplices in any triangulation of M^{2n}.** Other decompositions lead to other inequalities perhaps better. For example, in dimension two the familiar decomposition of a surface of genus g into one vertex, $2g$ edges, and one 2-cell is more convenient and leads by the above argument to one geodesic cycle on S^1 with $4g$ edges whose degree is the Euler number. Now



* This bound was known to J. Simons, and the construction was motivated by one of J. Cheeger and J. Simons.



the two edges corresponding to the same one cell are counted with opposite orientation. So to compute the Euler number we must add up $2g$ pairs of terms of opposite sign where each term is strictly less than $\frac{1}{2}$ or there is exact cancellation. Thus the Euler number must be strictly less than the genus.

This is Milnor's inequality for the case of affine connections and also Wood's generalization to foliated S^1 bundles where holonomy preserves antipodal points. We see Wood's generalization is possible in our terms because any homeomorphism of S^1 preserves geodesic simplices. Without the antipodal condition, the above argument yields the condition Euler number $<2g$. Now a simple covering space argument shows that if we have a bound of the form—Euler characteristic plus constant—we can take the constant to be zero. Thus the Euler number is $\leq 2g-2$. This is Wood's further result from foliated S^1-bundles without the antipodal condition. It is best possible because the unit tangent bundle with Euler number $2g-2$ carries the famous Anosov foliations related to the geodesic flow. All other examples subject to the inequalities are achieved from these tangent bundle examples by pulling back to surfaces of higher genus by degree one maps. Thus we have a complete recapitulation of the Euler results in [M] and [W].

Affine Manifolds.

There is one very geometric situation where one might hope to apply the above inequality. Suppose M is a compact affine manifold, i.e. M is the union of a finite number of open sets in R^n where the attaching maps are affine, $x \to Ax+b$, A a matrix, b a vector.

This is presumably quite a large class of manifolds[1] including flat manifolds, certain nilmanifolds, and the product of any manifold of constant curvature with a circle.[2] One interesting question see [M] *is whether the Euler characteristic of a compact affine manifold is zero.*

Now the tangent bundle of an affine manifold has a linear (or affine) foliation and the above inequalities hold. For example, in dimension two we see the

[1] See remark (c) p. 124 in Whitney "Geometric Integration", Princeton, 1956.

[2] Communicated by Deligne and Thurston.

theorem of (Benzecri-Milnor) that *a surface of higher genus is not affine* because it is false that $2g-2<g$. In particular the Euler characteristic of an affine 2-manifold is zero.

One might hope to extend this argument to higher dimensions and find a geometric decomposition of the affine manifold so that the inequality achieved above leads to an impossible condition on the Euler characteristic forcing it to vanish. The fact that the pieces of the decomposition need only be simply connected (and not contractible) provides the possible leverage.

If this is correct, one of the worst cases for this argument occurs when M is a $K(\pi, 1)$. For example if M is obtained from R^n by dividing by a discrete group Γ of affine transformations.

We have not carried the above program very far but this last class of affine manifolds can be independently taken care of.

The Euler characteristic of R^n/Γ is zero, where Γ is a discrete group of affine transformation of R^n.

This is proved in two steps:

(i) if $x \to Ax+b$ has no fixed points then 1 is an eigenvalue of A. (If $x \neq Ax+b$ for all x, then $(A-I)x \neq b$ implies $A-I$ is not invertible.) Thus the tangent bundle of R^n/Γ has as structure group a subgroup of matrices each one of which has 1 as an eigenvalue. (This was pointed out to me by Mo Hirach.)

(ii) *A R^n-bundle with structure group whose elements satisfy the equation* $\det(X-I)=0$ *always has a trivial Euler class with real coefficients.* By taking the real algebraic closure of the structure group, passing to the component of the identity, and then to a maximal compact subgroup we can reduce the question to the case where the structure group is a connected compact subgroup of $SO(n)$. But then the characteristic classes of our bundle are computable by restricting invariant polynomials on the Lie algebra of skew symmetric matrices to the Lie algebra of the subgroup. The Pfaffian polynomial whose square is the determinant gives the Euler class. In our case the restriction of the Pfaffian is identically zero because each skew symmetric matrix in the sub-Lie-algebra has zero as an eigenvalue.

Note that this proves the Euler characteristic is zero in case each affine transformation in the "developing" representation $\pi_1 M \to$ Affine group has no fixed points. This is potentially more general than the R^n/Γ case.

The Pfaffian argument was worked out with Bert Kostant. (See [KS]).

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