Atoms of a Group Valued Measures

Atoms of Group Valued Measures

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Any real valued measure may be written (uniquely) as a sum of an atomic and of an atomless measure. This result was extended first time by J. Hoffmann-Jørgensen ([1] Theorem 6) to absolutely continuous measures with values in locally convex spaces and then by K. Musiał ([2] Theorem 1) to the more general case of group valued measures satisfying ccc. The aim of this paper is to extend this result once more for group valued measures satisfying asc (Proposition 3.2 and Theorem 3.6). Any measure satisfying ccc (even locally) satisfies asc (Proposition 3.4). If the group G is complete any G-valued measure satisfies asc (Propositions 3.5 and 2.1). The nature of atoms for measures not satisfying ccc becomes by far more complicated and so a great part of this paper is dedicated to their study.

Throughout this paper we shall denote by $\Re$ *a* δ-ring (i.e. $\Re \neq \varnothing$ *and for any sequence* $(A_n)_{n \in \mathbf{N}}$ *in* $\Re$ *we have* $\bigcap_{n \in \mathbf{N}} A_n \in \Re$ *and* $A_0 \triangle A_1 \in \Re$) *by* $\Re$ *a subset of* $\Re$ *such that the union of any finite family in* $\Re$ *belongs to* $\Re$, *and by* G *a Hausdorff topological commutative group. We consider* $\Re$ *ordered by the inclusion relation and denote by* Λ *the set of lower directed nonempty subsets of* $\Re \setminus \{\varnothing\}$. *For any* $\mathfrak{A} \in \Lambda$ *we denote by* $\mathfrak{F}(\mathfrak{A})$ *the filter on* $\Re$ *generated by the filter base*

$$\{\{B \in \mathfrak{A} \mid B \subset A\} \mid A \in \mathfrak{A}\}.$$

A *system of null sets of* $\Re$ is a nonempty subset $\mathfrak{N}$ of $\Re$ such that: (a) any set of $\Re$ belongs to $\mathfrak{N}$ if it is contained in a set of $\mathfrak{N}$; (b) the union of any countable family in $\mathfrak{N}$ belongs to $\mathfrak{N}$ if it belongs to $\Re$.

1. Atoms

Throughout this section we shall denote by $\mathfrak{N}$ *a system of null sets of* $\Re$. We denote by $\Lambda(\mathfrak{N})$ the set of subsets $\mathfrak{A} \neq \varnothing$ of $\Re \setminus \mathfrak{N}$ such that the intersection of any countable family in $\mathfrak{A}$ belongs to $\mathfrak{A}$. It is obvious that $\Lambda(\mathfrak{N}) \subset \Lambda$. The maximal elements of $\Lambda(\mathfrak{N})$ (for the inclusion relation) will be called *atoms* (with respect to

$\mathfrak{N}$). A *set-atom* (with respect to $\mathfrak{N}$) is a set $A \in \mathfrak{M} \backslash \mathfrak{N}$ such that for any subset B of A belonging to $\mathfrak{M}$ we have either $B \in \mathfrak{N}$ or $A \backslash B \in \mathfrak{N}$. We say that $\mathfrak{N}$ satisfies the *countable chain condition* (abreviated *ccc*) if any disjoint family $(A_\iota)_{\iota \in I}$ in $\mathfrak{M} \backslash \mathfrak{N}$ (i.e. $A_\iota \cap A_{\iota'} = \varnothing$ for different elements ι, ι' of I) is countable. We say that $\mathfrak{N}$ satisfies *locally ccc* if any disjoint family $(A_\iota)_{\iota \in I}$ in $\mathfrak{M} \backslash \mathfrak{N}$ is countable if $\bigcup_{\iota \in I} A_\iota$ is contained in a set of $\mathfrak{M}$.

PROPOSITION 1.1. *Assume $\mathfrak{N}$ satisfies locally ccc and let $\mathfrak{A} \in \Lambda$. Then there exists a decreasing sequence $(A_n)_{n \in \mathbf{N}}$ in $\mathfrak{A}$ such that we have for any $A \in \mathfrak{A}$*

$$\bigcap_{n \in N} A_n \backslash A \in \mathfrak{N}.$$

Assume that for any decreasing sequence $(A_n)_{n \in \mathbf{N}}$ in $\mathfrak{A}$ there exists $A \in \mathfrak{A}$ such that

$$\bigcap_{n \in \mathbf{N}} A_n \backslash A \notin \mathfrak{N}.$$

Let ω_1 be the first uncountable ordinal number. We shall construct inductively a family $(B_\xi)_{\xi < \omega_1}$ in $\mathfrak{A}$ such that

$$\bigcap_{\xi < \mu} B_\xi \backslash B_\mu \notin \mathfrak{N}$$

for any $\mu < \omega_1$. Let $\mu < \omega_1$ and assume the family $(B_\xi)_{\xi < \mu}$ with the required property is constructed. Since this family is countable and since $\mathfrak{A}$ is lower directed there exists a decreasing sequence $(A_n)_{n \in \mathbf{N}}$ in $\mathfrak{A}$ such that

$$\bigcap_{n \in \mathbf{N}} A_n \subset \bigcap_{\xi < \mu} B_\xi.$$

By the hypothesis there exists $B_\mu \in \mathfrak{A}$ with

$$\bigcap_{n \in \mathbf{N}} A_n \backslash B_\mu \notin \mathfrak{N}.$$

Hence

$$\bigcap_{\xi < \mu} B_\xi \backslash B_\mu \notin \mathfrak{N}.$$

The existence of the family $(\bigcap_{\xi < \mu} B_\xi \backslash B_\mu)_{\mu < \omega_1}$ contradicts the hypothesis that $\mathfrak{N}$ satisfies locally *ccc*. ∎

PROPOSITION 1.2. *For any* $A \in \mathfrak{R} \backslash \mathfrak{N}$ *we set*

$$\mathfrak{A}_A := \{ B \in \mathfrak{R} \mid A \backslash B \in \mathfrak{N} \}$$

Then:

(a) $\mathfrak{A}_A \in \Lambda(\mathfrak{N})$;

(b) $\mathfrak{A}_A$ *is an atom* $\Leftrightarrow A$ *is a set-atom;*

(c) *if* $\mathfrak{N}$ *satisfies locally ccc then for any atom* $\mathfrak{A}$ *there exists a set-atom* $A \in \mathfrak{R} \backslash \mathfrak{N}$ *with* $\mathfrak{A}_A = \mathfrak{A}$; *in particular for any* $B \in \mathfrak{R}$ *the set of atoms* $\mathfrak{A}$ *such that* $B \in \mathfrak{A}$ *is countable.*

(a) The intersection of any sequence in $\mathfrak{A}_A$ belongs to $\mathfrak{A}_A$.

(b) Let A be a set-atom. Let $\mathfrak{A} \in \Lambda(\mathfrak{N})$ with $\mathfrak{A}_A \subset \mathfrak{A}$ and let $B \in \mathfrak{A}$. Then $A \cap B \in \mathfrak{A}$ and therefore $A \cap B \notin \mathfrak{N}$. We get $A \backslash B \in \mathfrak{N}$ and therefore $B \in \mathfrak{A}_A$. Hence $\mathfrak{A}_A = \mathfrak{A}$. Thus $\mathfrak{A}_A$ is an atom.

Assume now $\mathfrak{A}_A$ is an atom. If A is not a set-atom then there exists $B \in \mathfrak{R} \backslash \mathfrak{N}$ such that $B \subset A$ and $A \backslash B \notin \mathfrak{N}$. We get $\mathfrak{A}_B \in \Lambda(\mathfrak{N})$, $\mathfrak{A}_A \subset \mathfrak{A}_B$, and $B \in \mathfrak{A}_B \backslash \mathfrak{A}_A$. Hence $\mathfrak{A}_A$ is not a maximal element of $\Lambda(\mathfrak{N})$ and this is a contradiction.

(c) Assume $\mathfrak{N}$ satisfies locally *ccc* and let $\mathfrak{A}$ be an atom. By Proposition 1.1 there exists $A \in \mathfrak{A}$ such that $\mathfrak{A} \subset \mathfrak{A}_A$. Since $\mathfrak{A}$ is maximal in $\Lambda(\mathfrak{N})$ we deduce by (a) $\mathfrak{A} = \mathfrak{A}_A$. By (b) A is a set-atom.

Let $B \in \mathfrak{R}$ and assume that the set of atoms $\mathfrak{A}$ such that $B \in \mathfrak{A}$ is uncountable. Let ω_1 be the first uncountable ordinal number. There exists a family $(\mathfrak{A}_\xi)_{\xi < \omega_1}$ of atoms such that $B \in \mathfrak{A}_\xi$ for any $\xi < \omega_1$ and such that $\mathfrak{A}_\xi \neq \mathfrak{A}_\eta$ for any $\xi < \eta < \omega_1$. For any $\xi < \omega_1$ let A_ξ be a set-atom such that $\mathfrak{A}_\xi = \mathfrak{A}_{A_\xi}$. Let $\xi < \eta$. If $A_\xi \cap A_\eta \notin \mathfrak{N}$ then $A_\eta \backslash A_\xi \in \mathfrak{N}$ and we get for any $C \in \mathfrak{A}_\xi$

$$A_\xi \backslash C \in \mathfrak{N}, \qquad A_\eta \backslash C \in \mathfrak{N}, \qquad C \in \mathfrak{A}_\eta.$$

Hence $\mathfrak{A}_\xi \subset \mathfrak{A}_\eta$ and this leads to the contradiction $\mathfrak{A}_\xi = \mathfrak{A}_\eta$. Hence $A_\xi \cap A_\eta \in \mathfrak{N}$. We get for any $\eta < \omega_1$

$$A_\eta \cap B \backslash \bigcup_{\xi < \eta} A_\xi \in \mathfrak{A}_\eta$$

Hence $(A_\eta \cap B \backslash \bigcup_{\xi < \eta} A_\xi)_{\eta < \omega_1}$ is an uncountable disjoint family of subsets of B in $\mathfrak{R} \backslash \mathfrak{N}$ and this contradicts the hypothesis that $\mathfrak{N}$ satisfies locally *ccc*. ∎

Remark. From (c) it follows that if $\mathfrak{N}$ satisfies locally *ccc* the atoms and the set-atoms may be identified.

PROPOSITION 1.3. *Let* $\mathfrak{A}$ *be an atom. Then*

(a) *If* $B \in \mathfrak{R}$ *and if there exists* $A \in \mathfrak{A}$ *such that* $A \setminus B \in \mathfrak{R}$ *then* $B \in \mathfrak{A}$.

(b) *If* $B \in \mathfrak{R}$ *and if* $A \cap B \notin \mathfrak{R}$ *for any* $A \in \mathfrak{A}$ *then* $B \in \mathfrak{A}$.

(c) *If* $(A_\iota)_{\iota \in I}$ *is a countable family in* $\mathfrak{R}$ *whose union belongs to* $\mathfrak{A}$ *then there exists* $\iota \in I$ *with* $A_\iota \in \mathfrak{A}$.

(d) $\mathfrak{A}$ *is a maximal element of* Λ.

(a) The set $\{C \in \mathfrak{R} \mid \exists A \in \mathfrak{A}, A \setminus C \in \mathfrak{R}\}$ belongs to $\Lambda(\mathfrak{R})$ and contains $\mathfrak{A}$. Since $\mathfrak{A}$ is a maximal element of $\Lambda(\mathfrak{R})$ this set coincides with $\mathfrak{A}$. Hence $B \in \mathfrak{A}$.

(b) The set $\{C \in \mathfrak{R} \mid \exists A \in \mathfrak{A}, C \supset A \cap B\}$ belongs to $\Lambda(\mathfrak{R})$ and contains $\mathfrak{A}$. Since $\mathfrak{A}$ is a maximal element of $\Lambda(\mathfrak{R})$ this set coincides with $\mathfrak{A}$. Hence $B \in \mathfrak{A}$.

(c) Assume $A_\iota \notin \mathfrak{A}$ for any $\iota \in I$. By (b) there exists for any $\iota \in I$ a $B_\iota \in \mathfrak{A}$ such that $A_\iota \cap B_\iota \in \mathfrak{R}$. From

$$\left(\bigcup_{\iota \in I} A_\iota \right) \cap \left(\bigcap_{\iota \in I} B_\iota \right) \subset \bigcup_{\iota \in I} (A_\iota \cap B_\iota)$$

it follows $(\bigcup_{\iota \in I} A_\iota) \cap (\bigcap_{\iota \in I} B_\iota) \in \mathfrak{R} \cap \mathfrak{A}$ and this is a contradiction.

(d) Let $\mathfrak{B} \in \Lambda$ with $\mathfrak{A} \subset \mathfrak{B}$. Let $B \in \mathfrak{B} \setminus \mathfrak{A}$. By (b) there exists $A \in \mathfrak{A}$ with $A \cap B \in \mathfrak{R}$. By (a) we get $A \setminus B \in \mathfrak{A}$ and therefore

$$\varnothing = (A \setminus B) \cap B \in \mathfrak{B}$$

which is a contradiction. Hence $\mathfrak{B} = \mathfrak{A}$ and $\mathfrak{A}$ is a maximal element of Λ. ∎

COROLLARY 1.4. *Let* $\mathfrak{R}'$, $\mathfrak{R}''$ *be two systems of null sets on* $\mathfrak{R}$ *and let* $\mathfrak{A}$ *be an atom with respect to* $\mathfrak{R}'$. *Then either* $\mathfrak{A}$ *is an atom with respect to* $\mathfrak{R}''$ *or* $\mathfrak{A} \cap \mathfrak{R}'' \neq \varnothing$.

If $\mathfrak{A} \cap \mathfrak{R}'' = \varnothing$ then $\mathfrak{A} \in \Lambda(\mathfrak{R}'')$. By the Proposition 1.3(d) $\mathfrak{A}$ is a maximal element of Λ and therefore a fortiori it is a maximal element of $\Lambda(\mathfrak{R}'')$. ∎

PROPOSITION 1.5. *Let* Φ *be a countable set of atoms. Then there exists a disjoint family* $(A_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$ *in* $\mathfrak{R}$ *such that* $A_{\mathfrak{A}} \in \mathfrak{A}$ *for any* $\mathfrak{A} \in \Phi$.

Let $\mathfrak{A}$, $\mathfrak{B}$ be two different atoms of Φ and let $A_{\mathfrak{A},\mathfrak{B}} \in \mathfrak{A} \setminus \mathfrak{B}$. By Proposition 1.3(b) there exists $B_{\mathfrak{A},\mathfrak{B}} \in \mathfrak{B}$ such that

$$A_{\mathfrak{A},\mathfrak{B}} \cap B_{\mathfrak{A},\mathfrak{B}} \in \mathfrak{R}.$$

By Proposition 1.3(a)

$$A_{\mathfrak{A},\mathfrak{B}} \setminus B_{\mathfrak{A},\mathfrak{B}} \in \mathfrak{A}, \qquad B_{\mathfrak{A},\mathfrak{B}} \setminus A_{\mathfrak{A},\mathfrak{B}} \in \mathfrak{B}.$$

We set for any $\mathfrak{A} \in \Phi$

$$A_{\mathfrak{A}} := \bigcap_{\substack{\mathfrak{B} \in \Phi \\ \mathfrak{B} = \mathfrak{A}}} ((A_{\mathfrak{A},\mathfrak{B}} \setminus B_{\mathfrak{A},\mathfrak{B}}) \cap (B_{\mathfrak{B},\mathfrak{A}} \setminus A_{\mathfrak{B},\mathfrak{A}})).$$

It is obvious that $(\mathfrak{A}_A)_{\mathfrak{A} \in \Phi}$ possesses the required properties. ∎

PROPOSITION 1.6. *Let Φ be a set of atoms such that for any $A \in \mathfrak{R}$ the set $\{\mathfrak{A} \in \Phi \mid A \in \mathfrak{A}\}$ is countable. Then there exists a family $(A_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$ in $\mathfrak{R}$ such that $A_{\mathfrak{A}} \in \mathfrak{A} \setminus \mathfrak{A}'$ for any $\mathfrak{A}, \mathfrak{A}' \in \Phi, \mathfrak{A} \neq \mathfrak{A}'$.*

Let $(B_{\mathfrak{A}})_{\mathfrak{A}}$ be a family in $\mathfrak{R}$ such that $B_{\mathfrak{A}} \in \mathfrak{A}$ for any $\mathfrak{A} \in \Phi$. Let $\mathfrak{A} \in \Phi$ and let

$$\Psi(\mathfrak{A}) := \{\mathfrak{A}' \in \Phi \mid B_{\mathfrak{A}} \in \mathfrak{A}'\}.$$

By the hypothesis $\Psi(\mathfrak{A})$ is countable. By Proposition 1.5 there exists a disjoint family $(C_{\mathfrak{A}'})_{\mathfrak{A}' \in \Psi(\mathfrak{A})}$ such that $C_{\mathfrak{A}'} \in \mathfrak{A}'$ for any $\mathfrak{A}' \in \Psi(\mathfrak{A})$. We set

$$A_{\mathfrak{A}} := B_{\mathfrak{A}} \cap C_{\mathfrak{A}}.$$

Let $\mathfrak{A}, \mathfrak{A}'$ be two different atoms of Φ. Then $A_{\mathfrak{A}} \in \mathfrak{A}$. If $\mathfrak{A}' \notin \Psi(\mathfrak{A})$ then $B_{\mathfrak{A}} \notin \mathfrak{A}'$ and therefore $A_{\mathfrak{A}} \notin \mathfrak{A}'$ (Proposition 1.3(a)). If $\mathfrak{A}' \in \Psi(\mathfrak{A})$ then

$$A_{\mathfrak{A}} \cap A_{\mathfrak{A}'} \subset C_{\mathfrak{A}} \cap C_{\mathfrak{A}'} = \varnothing$$

and this shows that $A_{\mathfrak{A}} \notin \mathfrak{A}'$. ∎

2. Measures

A *measure on $\mathfrak{R}$* is a map μ of $\mathfrak{R}$ into a Hausdorff topological commutative group such that for any countable disjoint family $(A_\iota)_{\iota \in I}$ in $\mathfrak{R}$ whose union belongs to $\mathfrak{R}$ the family $(\mu(A_\iota))_{\iota \in I}$ is summable and its sum is $\mu(\bigcup_{\iota \in I} A_\iota)$. A measure μ on $\mathfrak{R}$ is called *$\mathfrak{R}$-regular* if for any $A \in \mathfrak{R}$ and for any neighbourhood V of $\mu(A)$ there exists $K \in \mathfrak{R}$ contained in A such that

$$\{\mu(B) \mid B \in \mathfrak{R}, K \subset B \subset A\} \subset V.$$

For any measure μ on $\mathfrak{R}$ we set

$$\mathfrak{N}(\mu) := \{A \in \mathfrak{R} \mid \forall B \in \mathfrak{R}, B \subset A \Rightarrow \mu(B) = 0\}.$$

$\mathfrak{N}(\mu)$ is a system of null sets of $\mathfrak{R}$. We say that μ *satisfies ccc* (resp. that μ *satisfies locally ccc*) if $\mathfrak{N}(\mu)$ satisfies *ccc* (resp. satisfies locally *ccc*). It is obvious that the set of G-valued measures on $\mathfrak{R}$, the set of $\mathfrak{R}$-regular G-valued measures on $\mathfrak{R}$, and the set of G-valued measures on $\mathfrak{R}$ satisfying *ccc* or satisfying locally *ccc* are subgroups of $G^{\mathfrak{R}}$. For any measure μ on $\mathfrak{R}$ and for any $\mathfrak{A} \in \Lambda$ we denote by $\mu(\mathfrak{F}(\mathfrak{A}))$ the image of the filter $\mathfrak{F}(\mathfrak{A})$ through μ (i.e. the filter generated by the filter base $\{\mu(\mathfrak{B}) \mid \mathfrak{B} \in \mathfrak{F}(\mathfrak{A})\}$); if this filter converges we denote by $\mu_{\mathfrak{A}}$ its limit. By Proposition 1.1 $\mu(\mathfrak{F}(\mathfrak{A}))$ converges for any measure μ satisfying locally *ccc* and for any $\mathfrak{A} \in \Lambda$. We call *atom of* μ any atom with respect to $\mathfrak{N}(\mu)$. An atom $\mathfrak{A}$ of μ is called *improper* if $\mu(\mathfrak{F}(\mathfrak{A}))$ converges to 0; otherwise we call it *proper*. Let $\mathfrak{A}$ be an improper atom of μ. If for any $A \in \mathfrak{A}$ there exists a proper atom $\mathfrak{A}'$ of μ such that $A \in \mathfrak{A}'$ we say that $\mathfrak{A}$ is *of the first kind*. An improper atom which is not of the first kind will be called *of the second kind*. We call *set-atom of* μ any set-atom with respect to $\mathfrak{N}(\mu)$.

A *preorder relation* on a set I is a binary relation $\le$ on I such that:

(a) $\iota \in I \Rightarrow \iota \le \iota$;

(b) $\iota, \iota', \iota'' \in I, \iota \le \iota', \iota' \le \iota'' \Rightarrow \iota \le \iota''$.

An *upper directed preordered set* is a set I endowed with a preorder relation $\le$ such that for any $\iota', \iota'' \in I$ there exists $\iota \in I$ with $\iota' \le \iota, \iota'' \le \iota$. The section filter of an upper directed nonempty set $(I, \le)$ is the filter on I generated by the filter base

$$\{\{\iota \in I \mid \iota \ge \lambda\} \mid \lambda \in I\}.$$

A *net* in a set X is a pair (I, f) such that I is an upper directed preordered set and f is a map of I into X.

Let X be a topological space. An *ω-net in X* is a net (I, f) in X such that for any increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in X the sequence $(f(\iota_n))_{n \in \mathbf{N}}$ is convergent. An *ω-filter on X* is a filter $\mathfrak{F}$ on X such that there exists an ω-net (I, f) in X such that $f(\mathfrak{G}) \subset \mathfrak{F}$, where $\mathfrak{G}$ denotes the section filter of I. An *ω space* is a topological space for which any ω-filter converges.

PROPOSITION 2.1. *Any ω-filter on a uniform space is a Cauchy filter. Hence any complete uniform space is an ω-space.*

Let X be a uniform space, let (I, f) be an ω-net in X, and let $\mathfrak{F}$ be the section filter of I. Let further U be an arbitrary entourage ($=$ vicinity) of X and let V be an entourage of X such that $V \circ V^{-1} \subset U$. Assume that for any $\iota \in I$ there exists $\lambda \in I$ such that $\lambda \geq \iota$ and $(f(\iota), f(\lambda)) \notin V$. Then we may construct inductively an increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in I such that $(f(\iota_n), f(\iota_{n+1})) \notin V$ for any $n \in N$. The sequence $(f(\iota_n))_{n \in \mathbf{N}}$ being convergent this is a contradiction. Hence there exists $\iota \in I$ with $(f(\iota), f(\lambda)) \in V$ for any $\lambda \in I$, $\lambda \geq \iota$. We get $(f(\iota'), f(\iota'')) \in U$ for any $\iota', \iota'' \in I$ with $\iota' \geq \iota$, $\iota'' \geq \iota$. Hence $f(\mathfrak{F})$ is a Cauchy filter. ∎

PROPOSITION 2.2. *For any measure μ on $\mathfrak{R}$ and for any $\mathfrak{A} \in \Lambda$, $\mu(\mathfrak{F}(\mathfrak{A}))$ is an ω-filter and therefore a Cauchy filter.*

Let us order $\mathfrak{A}$ by the converse inclusion relation, let $\mathfrak{G}$ be the section filter of $\mathfrak{A}$, and let $\mu \mid \mathfrak{A}$ be the restriction of μ to $\mathfrak{A}$. Then $(\mathfrak{A}, \mu \mid \mathfrak{A})$ is an ω-net and

$$\mu(\mathfrak{F}(\mathfrak{A})) = \mu \mid \mathfrak{A}(\mathfrak{G}).$$

Hence $\mu(\mathfrak{F}(\mathfrak{A}))$ is an ω-filter. By Proposition 2.1 it is a Cauchy filter. ∎

PROPOSITION 2.3. *Let μ be a measure on $\mathfrak{R}$ and let $\mathfrak{A}$ be a maximal element of Λ. Then either $\mathfrak{A}$ is an atom of μ or $\mu(\mathfrak{F}(\mathfrak{A}))$ converges to 0.*

Assume that $\mu(\mathfrak{F}(\mathfrak{A}))$ does not converge to 0. By Proposition 2.2 there exist a 0-neighbourhood V and an $A \in \mathfrak{A}$ such that

$$\{\mu(B) \mid B \in \mathfrak{A}, B \subset A\} \cap V = \varnothing.$$

Let $(A_n)_{n \in \mathbf{N}}$ be a sequence in $\mathfrak{A}$. If $\bigcap_{n \in \mathbf{N}} A_n = \varnothing$ then there exists a decreasing sequence $(B_n)_{n \in \mathbf{N}}$ in $\mathfrak{A}$ with empty intersection and such that $B_0 \subset A$. It follows that $(\mu(B_n))_{n \in \mathbf{N}}$ converges to 0 and this contradicts the above relation. Hence $\bigcap_{n \in \mathbf{N}} A_n \neq \varnothing$. The set

$$\left\{ B \in \mathfrak{R} \mid \exists C \in \mathfrak{A}, \left(\bigcap_{n \in \mathbf{N}} A_n \right) \cap C \subset B \right\}$$

belongs to Λ and contains $\mathfrak{A}$. Since $\mathfrak{A}$ is maximal it coincides with $\mathfrak{A}$. We deduce $\bigcap_{n \in \mathbf{N}} A_n \in \mathfrak{A}$. Since $\mathfrak{A} \cap \mathfrak{N}(\mu) = \varnothing$ we deduce $\mathfrak{A} \in \Lambda(\mathfrak{N}(\mu))$. It is obvious that $\mathfrak{A}$ is a maximal element of $\Lambda(\mathfrak{N}(\mu))$. Hence $\mathfrak{A}$ is an atom of μ. ∎

COROLLARY 2.4. *Let μ, ν be measures on $\mathfrak{R}$ and let $\mathfrak{A}$ be an atom of μ. Then either $\mathfrak{A}$ is an atom of ν or $\nu(\mathfrak{F}(\mathfrak{A}))$ converges to* 0.

By Proposition 1.3(d) $\mathfrak{A}$ is a maximal element of Λ and the assertion follows from the proposition. ∎

PROPOSITION 2.5. *Let μ be a $\mathfrak{R}$-regular measure on $\mathfrak{R}$ and let $\mathfrak{A}$ be a proper atom of μ. Then for any $A \in \mathfrak{A}$ there exists $K \in \mathfrak{R} \cap \mathfrak{A}$ with $K \subset A$.*

By Proposition 2.3 $\mu(\mathfrak{F}(\mathfrak{A}))$ is a Cauchy filter. Since $\mathfrak{A}$ is a proper atom of μ, 0 is not an adherent point of this filter. Hence there exist a neighbourhood V of 0 and a set $B \in \mathfrak{A}$ such that

$$\{\mu(C) \mid C \in \mathfrak{A}, C \subset B\} \cap V = \varnothing.$$

Since μ is $\mathfrak{R}$-regular there exists $K \in \mathfrak{R}$ such that $K \subset A \cap B$ and

$$\{\mu(C) \mid C \in \mathfrak{R}, C \subset A \cap B \setminus K\} \subset V.$$

Let $C \in \mathfrak{A}$. If $C \cap K \in \mathfrak{R}(\mu)$ then by Proposition 1.3(a) $C \setminus K \in \mathfrak{A}$ and therefore

$$A \cap B \cap (C \setminus K) \in \mathfrak{A}, \qquad \mu(A \cap B \cap (C \setminus K)) \in V$$

which is a contradiction. Hence $C \cap K \notin \mathfrak{R}(\mu)$ for any $C \in \mathfrak{A}$. By Proposition 1.3(b) we get $K \in \mathfrak{A}$. ∎

3. Atomic Measures

A measure possessing no proper atom is called *atomless*. If it possesses no atoms at all it is called *strictly atomless*. Any improper atom of an atomless measure is of the second kind.

PROPOSITION 3.1. *The set of atomless (resp. strictly atomless) G-valued measures on $\mathfrak{R}$ is a subgroup of $G^{\mathfrak{R}}$.*

Let μ, ν be two G-valued measures on $\mathfrak{R}$ and let $\mathfrak{A}$ be an atom of $\mu - \nu$. Since

$$\mathfrak{R}(\mu) \cap \mathfrak{R}(\nu) \subset \mathfrak{R}(\mu - \nu)$$

it follows that

$$\mathfrak{A} \cap \mathfrak{M}(\mu) = \varnothing \qquad \text{or} \qquad \mathfrak{A} \cap \mathfrak{M}(\nu) = \varnothing$$

By Corollary 1.4 we deduce that $\mathfrak{A}$ is an atom of either μ or ν. This shows that the set of strictly non-atomic G-valued measures is a subgroup of $G^{\mathfrak{M}}$.

Assume now that $\mathfrak{A}$ is a proper atom of $\mu - \nu$. By Proposition 1.3(d) $\mathfrak{A}$ is a maximal element of Λ. By Proposition 2.3, $\mathfrak{A}$ is a proper atom of either μ or ν. Hence the set of G-valued atomless measures on is a subgroup of G . ∎

We say that a measure μ on $\mathfrak{M}$ satisfies the *atom condition* (abbreviated *ac*) if for any atom $\mathfrak{A}$ of μ, $\mu(\mathfrak{F}(\mathfrak{A}))$ is convergent; according to the general convention made above we denote by $\mu_{\mathfrak{A}}$ the limit of $\mu(\mathfrak{F}(\mathfrak{A}))$ which may be interpreted as the value of μ at $\mathfrak{A}$. By Proposition 1.1 any measure satisfying locally *ccc* satisfies *ac*. By Proposition 2.2 any measures with values in an ω-topological group and a fortiori in a complete topological group (Proposition 2.1) satisfies *ac*. The set of G-valued measures on $\mathfrak{M}$ satisfying *ac* is a subgroup of $G^{\mathfrak{M}}$ (Corollary 2.4).

A measure μ on $\mathfrak{M}$ satisfying *ac* is called *atomic* if for any $A \in \mathfrak{M}$, $\mu(A)$ is the sum of the family $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$, where Φ denotes the set of atoms $\mathfrak{A}$ of μ such that $A \in \mathfrak{A}$. By Propositions 1.3(d) and 2.3 we may replace Φ in the above definition by the set of maximal elements $\mathfrak{A}$ of Λ such that $A \in \mathfrak{A}$. From this remark it follows immediately that the set of G-valued atomic measures on $\mathfrak{M}$ is a subgroup of $G^{\mathfrak{M}}$. Any improper atom of an atomic measure is of the first kind. A measure which is at the same time atomic and atomless vanishes identically.

PROPOSITION 3.2. *Let* μ, μ' *be two atomless* G-*valued measures on* $\mathfrak{M}$ *and let* ν, ν' *be two atomic* G-*valued measures on* $\mathfrak{M}$. *If*

$$\mu + \nu = \mu' + \nu'$$

then $\mu = \mu'$ *and* $\nu = \nu'$.

By Proposition 3.1, $\mu - \mu'$ is an atomless measure on $\mathfrak{M}$. Since it is at the same time an atomic measure it vanishes identically. ∎

We say that a measure μ on $\mathfrak{M}$ satisfying *ac* satisfies the *atomical summability condition* (abbreviated *asc*) if for any $A \in \mathfrak{M}$ the family $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$ is summable, where Φ denotes the set of atome $\mathfrak{A}$ of μ such that $A \in \mathfrak{A}$. By Propositions 1.3(d) and 2.3 we may replace Φ in the above definition by the set of maximal elements $\mathfrak{A}$ for Λ for which $A \in \mathfrak{A}$. From this remark it follows immediately that the set of G-valued measures on $\mathfrak{M}$ satisfying *asc* is a subgroup of $G^{\mathfrak{M}}$. Any atomic measure satisfies *asc*.

PROPOSITION 3.3. *Let μ be a G-valued measure on $\mathfrak{R}$ satisfying ac and let Φ be a countable set of atoms of μ such that $\bigcap_{\mathfrak{A} \in \Phi} \mathfrak{A} \neq \varnothing$ and such that $\mu(\mathfrak{F}(\bigcap_{\mathfrak{A} \in \Phi} \mathfrak{A}))$ converges. Then the family $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$ is summable and its sum is the limit of $\mu(\mathfrak{F}(\bigcap_{\mathfrak{A} \in \Phi} \mathfrak{A}))$.*

Let M be a subset of $\mathbf{N} \smallsetminus \{\varnothing\}$ and let $(\mathfrak{A}_n)_{n \in M}$ be a family of atoms of μ such that $\mathfrak{A}_m \neq \mathfrak{A}_n$ for any different $m, n \in M$ and $\Phi = \{\mathfrak{A}_n \mid n \in M\}$. We set $\mathfrak{A} := \bigcap_{n \in M} \mathfrak{A}_n$. Let V be an arbitrary 0-neighbourhood in G and let $(V_n)_{n \in \mathbf{N}}$ be a sequence of 0-neighbourhoods in G such that $V_0 + V_0 - V_0 \subset V$ and such that $V_{n+1} + V_{n+1} \subset V_n$ for any $n \in \mathbf{N}$. For any $n \in M$ there exists $A_n \in \mathfrak{A}_n$ such that

$$\{\mu(B) \mid B \in \mathfrak{A}_n, B \subset A_n\} \subset \mu_{\mathfrak{A}_n} + V_n.$$

There exists $A \in \bigcap_{n \in M} \mathfrak{A}_n$ such that

$$\{\mu(B) \mid B \in \mathfrak{A}, B \subset A\} \subset \mu_{\mathfrak{A}} + V_0.$$

By Proposition 1.5 there exists a disjoint family $(B_n)_{n \in M}$ in $\mathfrak{R}$ such that $B_n \in \mathfrak{A}_n$ for any $n \in M$. Then $(A_n \cap B_n \cap A)_{n \in M}$ is a disjoint family in $\mathfrak{R}$ whose union belongs to $\mathfrak{R}$ and therefore

$$\mu\left(\bigcup_{n \in M} (A_n \cap B_n \cap A)\right) = \sum_{n \in M} \mu(A_n \cap B_n \cap A)$$

Since $\bigcup_{n \in M} (A_n \cap B_n \cap A) \in \mathfrak{A}$ (Proposition 1.3(a)) we have

$$\mu\left(\bigcup_{n \in M} (A_n \cap B_n \cap A)\right) \in \mu_{\mathfrak{A}} + V_0.$$

For any $n \in M$ we get

$$\mu(A_n \cap B_n \cap A) \in \mu_{\mathfrak{A}_n} + V_n.$$

Let M_0 be a finite subset of M such that

$$\sum_{n \in M'} \mu(A_n \cap B_n \cap A) - \sum_{n \in M} \mu(A_n \cap B_n \cap A) \in V_0$$

for any finite subset M' of M containing M_0. We deduce for any finite subset M' of M containing M_0.

$$\sum_{n \in M'} \mu_{\mathfrak{A}_n} - \mu_{\mathfrak{A}} \in V_0 + V_0 - V_0 \subset V.$$

Since V is arbitrary it follows that $(\mu_{\mathfrak{A}_n})_{n \in M}$ is summable and its sum is $\mu_{\mathfrak{A}_0}$. ∎

PROPOSITION 3.4. *Any measure satisfying locally ccc satisfies asc.*

Let μ be a measure on $\mathfrak{R}$ satisfying locally *ccc*. By Proposition 1.1 $\mu(\mathfrak{F}(\mathfrak{A}))$ converges for any $\mathfrak{A} \in \Lambda$; in particular μ satisfies *ac*. Let $A \in \mathfrak{R}$ and let Φ be the set of atoms $\mathfrak{A}$ of μ such that $A \in \mathfrak{A}$. By Proposition 1.2(c) Φ is countable. By the preceding proposition $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$ is summable. Hence μ satisfies *asc*. ∎

PROPOSITION 3.5. *If G is an ω-space then any G-valued measure satisfies asc.*

Let μ be a G-valued measure on $\mathfrak{R}$. By Proposition 2.2 for any $\mathfrak{A} \in \Lambda$ the filter $\mu(\mathfrak{F}(\mathfrak{A}))$ converges; in particular μ satisfies *ac*. Let $A \in \mathfrak{R}$, let Φ be the set of atoms $\mathfrak{A}$ of μ such that $A \in \mathfrak{A}$, and let $\mathfrak{P}_c(\Phi)$ be the set of countable subsets of Φ ordered by the inclusion relation. By Proposition 3.3 for any $\Psi \in \mathfrak{P}_c(\Phi)$ the family $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Psi}$ is summable; let us denote by f the map

$$\Psi \mapsto \sum_{\mathfrak{A} \in \Psi} \mu_{\mathfrak{A}} : \mathfrak{P}_c(\Phi) \to G.$$

Then $(\mathfrak{P}_c(\Phi), f)$ is an ω-net in G. Hence if $\mathfrak{F}$ denotes the section filter of $\mathfrak{P}_c(\Phi)$ then $f(\mathfrak{F})$ converges. We deduce that the family $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$ is summable. Hence μ satisfies *asc*. ∎

THEOREM 3.6. *Let μ be a $\mathfrak{R}$-regular G-valued measure on $\mathfrak{R}$ satisfying asc. We denote for any $A \in \mathfrak{R}$ by $\Phi(A)$ the set of atoms $\mathfrak{A}$ of μ such that $A \in \mathfrak{A}$ and by μ' the map*

$$A \mapsto \sum_{\mathfrak{A} \in \Phi(A)} \mu_{\mathfrak{A}} : \mathfrak{R} \to G.$$

Then μ' (resp. $\mu - \mu'$) is an atomic (resp. atomless) $\mathfrak{R}$-regular measure on $\mathfrak{R}$ absolutely continuous with respect to μ (i.e. $\mathfrak{N}(\mu) \subset \mathfrak{N}(\mu') \cap \mathfrak{N}(\mu - \mu')$). The proper atoms and the improper atoms of the first kind of μ and μ' coincide and we have

$\mu_{\mathfrak{A}} = \mu_{\mathfrak{A}}$ *for any atom* $\mathfrak{A}$ *of* μ. *Any improper atom of* μ *of the second kind is an improper atom of* $\mu - \mu'$ *of the second kind.*

Let $(A_\iota)_{\iota \in I}$ be a countable disjoint family in $\mathfrak{R}$ whose union belongs to $\mathfrak{R}$. Then $(\Phi(A_\iota))_{\iota \in I}$ is a disjoint family and by Proposition 1.3(c) its union is $\Phi(\bigcup_{\iota \in I} A_\iota)$. We get

$$\mu'\left(\bigcup_{\iota \in I} A_\iota\right) = \sum_{\mathfrak{A} \in \Phi(\bigcup_{\iota \in I} A_\iota)} \mu_{\mathfrak{A}} = \sum_{\iota \in I} \sum_{\mathfrak{A} \in \Phi(A_\iota)} \mu_{\mathfrak{A}} = \sum_{\iota \in I} \mu'(A_\iota).$$

Hence μ' is a measure.

Let $A \in \mathfrak{R}$ and let U be a closed 0-neighbourhood in G. Then there exists a finite subset Ψ_0 of $\Phi(A)$ such that

$$\mu'(A) - \sum_{\mathfrak{A} \in \Psi} \mu_{\mathfrak{A}} \in U$$

for any finite subset Ψ of $\Phi(A)$ containing Ψ_0. By Proposition 2.5 there exists for any $\mathfrak{A} \in \Phi(A)$ a set $K_{\mathfrak{A}} \in \mathfrak{K} \cap \mathfrak{A}$ with $K_{\mathfrak{A}} \subset A$. We set

$$K := \bigcup_{\mathfrak{A} \in \Psi_0} K_{\mathfrak{A}} \in \mathfrak{K}.$$

Let B be a set of $\mathfrak{R}$ such that $K \subset B \subset A$. Then $\Phi(B)$ is a subset of $\Phi(A)$ containing Ψ_0 (Proposition 1.3(a)) and therefore

$$\mu'(A) - \mu'(B) = \mu'(A) - \sum_{\mathfrak{A} \in \Phi(B)} \mu_{\mathfrak{A}} \in U.$$

This shows that μ' is $\mathfrak{K}$-regular. We deduce that $\mu - \mu'$ is a $\mathfrak{K}$-regular measure on $\mathfrak{R}$. It is obvious that μ' and $\mu - \mu'$ are absolutely continuous with respect to μ.

Let $\mathfrak{A}$ be an atom of μ. Let U be a closed 0-neighbourhood in G and let V be a 0-neighbourhood in G such that $V - V - V \subset U$. There exists $A \in \mathfrak{A}$ such that

$$\{\mu(B) \mid B \in \mathfrak{A}, B \subset A\} \subset \mu_{\mathfrak{A}} + V.$$

Let Ψ be a finite nonempty subset of $\Phi(A) \setminus \{\mathfrak{A}\}$. Then there exists

$$B \in \bigcap_{\mathfrak{A}' \in \Psi} \mathfrak{A}' \setminus \mathfrak{A}$$

such that $B \subset A$ and

$$\left\{ \mu(C) \mid C \in \bigcap_{\mathfrak{A}' \in \Psi} \mathfrak{A}', C \subset B \right\} \subset \sum_{\mathfrak{A}' \in \Psi} \mu_{\mathfrak{A}'} + V.$$

Then $A \setminus B \in \mathfrak{A}$ (Proposition 1.3(c)) and therefore

$$\mu(A \setminus B) \in \mu_{\mathfrak{A}} + V,$$

$$\sum_{\mathfrak{A}' \in \Psi} \mu_{\mathfrak{A}'} \in \mu(B) - V = \mu(A) - \mu(A \setminus B) - V \subset V - V - V \subset U.$$

Since Ψ is arbitrary we get

$$\sum_{\mathfrak{A}' \in \Phi(A) \setminus \{\mathfrak{A}\}} \mu_{\mathfrak{A}'} \in U, \quad \mu'(A) \in \mu_{\mathfrak{A}} + U.$$

Since U is arbitrary we deduce $\mu_{\mathfrak{A}}^{\zeta} = \mu_{\mathfrak{A}}$. Hence the proper atoms of μ and μ' coincide (Corollary 2.4). We deduce further that the improper atoms of μ and μ' of the first kind coincide (Corollary 1.4). Moreover for any $A \in \mathfrak{R}$ we get

$$\mu'(A) = \sum_{\mathfrak{A} \in \Phi} \mu_{\mathfrak{A}} = \sum_{\mathfrak{A} \in \Phi} \mu_{\mathfrak{A}}^{\zeta},$$

where Φ denotes the set of maximal elements $\mathfrak{A}$ of Λ such that $A \in \mathfrak{A}$ (Propositions 1.3(d) and 2.3). Hence μ' is an atomic measure.

Let $\mathfrak{A}$ be an atom of $\mu - \mu'$. Then $\mathfrak{A}$ is an atom of μ (Corollary 1.4) and by the above considerations it follows that $\mathfrak{A}$ is an improper atom of $\mu - \mu'$. Hence $\mu - \mu'$ is atomless.

Let $\mathfrak{A}$ be an improper atom of μ of the second kind. Then there exists $A \in \mathfrak{A}$ such that $A \in \mathfrak{R}(\mu')$ and therefore $\mathfrak{A}$ is an atom of $\mu - \mu'$ (Corollary 1.7)). Since $\mu - \mu'$ is atomless it is an improper atom of $\mu - \mu'$ of the second kind. ∎

Example. We want to give an example of a locally convex space E and of an E-valued measure on a σ-algebra of sets, possessing an improper atom of the second kind. Let X be a set. For any $A \subset X \times [0, 1]$ and for any $x \in X$ we set

$$A(x) := \{ y \in [0, 1] \mid (x, y) \in A \}.$$

We denote by $\mathfrak{R}$ the set of $A \subset X \times [0, 1]$ such that: (a) $A(x)$ is a Borel set for any $x \in X$; (b) the set $\{ x \in X \mid A(x) \neq \varnothing$ and $A(x) \neq [0, 1] \}$ is countable. It is obvious

that $\mathfrak{R}$ is a σ-algebra of subsets of $X \times [0, 1]$. For any $A \in \mathfrak{R}$ we denote by $\mu(A)$ the map

$$x \mapsto \lambda(A(x)) : X \to \mathbf{R},$$

where λ denotes the Lebesque measure on $[0, 1]$. It is easy to see that μ is an atomless $\mathbf{R}^X$-valued measure on $\mathfrak{R}$. Let $\mathfrak{F}$ be a non-trivial ultrafilter on X such that the intersection of any countable family in $\mathfrak{F}$ belongs to $\mathfrak{F}$ (we assume that such an ultrafilter exists). We set

$$\mathfrak{A} := \{A \in \mathfrak{R} \mid \{x \in X \mid A(x) = [0, 1]\} \in \mathfrak{F}\}.$$

Then $\mathfrak{A} \in \Lambda(\mu)$. Let $\mathfrak{A}' \in \Lambda(\mu)$ with $\mathfrak{A} \subset \mathfrak{A}'$, let $A \in \mathfrak{A}'$, and let

$$X_0 := \{x \in X \mid A(x) \neq \varnothing\}$$

If $X_0 \notin \mathfrak{F}$ then $X \setminus X_0 \in \mathfrak{F}$ and therefore $(X \setminus X_0) \times [0, 1] \in \mathfrak{A}$ and this leads to the contradictory relation

$$\varnothing = A \cap ((X \setminus X_0) \times [0, 1]) \in \mathfrak{A}'.$$

Hence $X_0 \in \mathfrak{F}$. Since $A \in \mathfrak{R}$ the set

$$\cdot \{x \in X \mid A(x) \neq \varnothing \quad \text{and} \quad A(x) \neq [0, 1]\}$$

is countable and therefore it does not belong to $\mathfrak{F}$. Hence

$$\{x \in X \mid A(x) = [0, 1]\} \in \mathfrak{F}$$

and we deduce successively $A \in \mathfrak{A}$, $\mathfrak{A} = \mathfrak{A}'$ and $\mathfrak{A}$ is an atom of μ. The measure μ being atomless $\mathfrak{A}$ is an improper atom of the second kind.

Example. Let X be an uncountable set, let $\mathfrak{P}(X)$ be the set of subsets of X, and for any $A \in \mathfrak{P}(X)$ let l_A be the characteristic function of A. Then

$$A \mapsto l_A : \mathfrak{P}(X) \to R^X$$

is an example of a measure satisfying *asc* and not satisfying *ccc*.

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