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1. Introduction

Let π be a group. A (π, m)-*complex* X is a finite connected m-dimensional CW complex having fundamental group π and trivial homotopy modules $\pi_i(X) = 0$ in dimensions $i = 2, \dots, m-1$. A π-module π_m is said to be *topologically realizable* if $\pi_m \approx \pi_m(X)$ for some (π, m)-complex X. The classification problem for (π, m)-complexes is the problem of describing the set HT (π, m) of homotopy types of (π, m)-complexes.

For π a finite group of order n, $H^{m+1}(\pi; \pi_m) \cong Z_n$ as a ring. An important aspect in this classification is the boundary operator $\partial: Z_n^* = \text{Units}\ (H^{m+1}(\pi; \pi_m)) \to \tilde{K}_0 Z\pi$, the (reduced) projective class group of the integral group ring $Z\pi$, associated with the Milnor Mayer-Vietoris sequence in algebraic K-theory [10].

This arises as follows. The cellular chain complex $C_*(\tilde{X})$ of the universal cover $\tilde{X}$ is a truncated resolution of the trivial π-module Z:

$$0 \longrightarrow \pi_m \longrightarrow C_m(\tilde{X}) \xrightarrow{\partial_m} \cdots \xrightarrow{\partial_1} C_0(\tilde{X}) \xrightarrow{\epsilon} Z \longrightarrow 0.$$

The algebraic m-type $T(X)$ of X is the triple $(\pi, \pi_m(X), k(X))$ where $k(X) \in H^{m+1}(\pi, \pi_m)$ is the k-invariant which arises by comparing the truncated resolution above with a standard resolution (see section 6; also [5], [6]). One can show that $k(X) \in \text{Units}\ (H^{m+1}(\pi; \pi_m))$; furthermore any $k \in Z_n^*$ can be the k-invariant of a finitely generated truncated projective resolution

$$(*) \quad \mathcal{P}_k: 0 \to \pi_m \to P_m \to P_{m-1} \to \cdots \to P_0 \to Z \to 0.$$

Also the assignment $(\pi, \pi_m, k) \to$ Euler characteristic $\chi(\mathcal{P}_k) = \sum_{i=0}^{m} (-1)^i [P_i]$ ($[P]$ is the class of the projective P in $\tilde{K}_0 Z\pi$) is the negative of the Milnor boundary ∂. Then (π, π_m, k) ($k \in Z_n^*$, $m \geq 3$) is the m-type of a (π, m)-complex iff $k \in \ker \partial$ [4].

The purpose of this paper is to generalize the above to groups other than finite groups.

1.1. THEOREM. *Let π be a group and m be an integer $m \geq 0$ such that $H^{m+1}(\pi; Z\pi) = 0$. Let π_m be any finitely generated topologically realizable π-module. Then*

(a) *$H^{m+1}(\pi; \pi_m)$ has the structure of a ring with identity such that the units $U(H^{m+1}(\pi, \pi_m))$ are the projective k-invariants,* i.e., *those* k*-invariants realizable by a resolution of the form (*).*

(b) *The function $\chi_m : U(H^{m+1}(\pi; \pi_m)) \to \tilde{K}_0 Z\pi$ which assigns to each $k \in U$ the Euler characteristic of a truncated resolution $\mathcal{P}_k$ realizing the m-type (π, π_m, k) is a homomorphism.*

We say that an m-type (π, π_m, k) comes from a (π, m)-complex if there exists a (π, m)-complex X such that $T(X) \cong (\pi, \pi_m, k)$ in the appropriate sense (see [4], [6] for a definition).

1.2. COROLLARY. *If $m \geq 3$ and $H^{m+1}(\pi; Z\pi) = 0$, then* ker χ_m *is the set of k-invariants which come from (π, m)-complexes.*

The corollary follows from a theorem of J. Milnor [11, theorem 3.1] concerning the realizability of a resolution by a (π, m)-complex.

DEFINITION. The subgroup im $\chi_m \subset \tilde{K}_0 Z\pi$ is called the *Swan subgroup of $\tilde{K}_0 Z\pi$ in dimension m.*

If π is a finite group of order n, let $N = \sum_{x \in \pi} x \in Z\pi$ be the norm element. The left ideal (p, N) of $Z\pi$ is projective provided p is prime to n. For π finite, im $\chi_m =$ im $\partial = \{[(p, N)] \in \tilde{K}_0 Z\pi \mid 1 \leq p < n, (p, n) = 1\}$. If π is a (Poincaré) duality group of cohomological dimension m, then im $\chi_{m-i} = 0$ $(2 \leq i \leq m)$.

The Swan subgroup im χ_m is important because the *Wall obstruction* of any CW complex having fundamental group π and realizable π_m, which is dominated by a (π', m)-complex lies in im χ_m [12].

The organization of the paper is as follows. Let R be a ring. Section 2 gives certain constructions associated with the exact sequence of R-modules $0 \to K \to P \to C \to 0$. We say that P is K-projective if ∂: End $(K) \to$ Ext (C, K) is *surjective.* Section 3 gives conditions under which Ext (C, K) inherits a ring structure from End (K), provided P is K-projective. Section 4 shows that elements in End (K) which determine K-projective extensions are right units in Ext (C, K). Section 5 studies conditions under which each K-projective element in End (K) is a unit in Ext (C, K). Theorem 1 is proved in section 6. In an appendix we study conditions under which $H^i(\pi; Z\pi) = 0$.

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2. Extensions as Pushouts and Pull-backs.

Let R be a ring. All modules are left R-modules. Let C be a given R-module and $\xi: 0 \longrightarrow K \xrightarrow{i} P \xrightarrow{j} C \longrightarrow 0$ be an exact sequence of R-modules.

It is shown in [9, page 66] that given any module homomorphism $k: K \to K'$ there exists a module kP and a homomorphism $k\beta: P \to kP$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i} & P & \xrightarrow{j} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle k\beta} & & \| & & \\ 0 & \longrightarrow & K' & \xrightarrow[i_k]{} & kP & \xrightarrow[j_k]{} & C & \longrightarrow & 0 \end{array} \qquad (2.1)$$

Here the bottom row is exact also. kP is defined as the pushout of i and k.

Furthermore, given any module homomorphism $s: C \to C$, there exists a module Ps and a homomorphism $\beta s: Ps \to P$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i^s} & Ps & \xrightarrow{j^s} & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \beta s} & & \downarrow{\scriptstyle s} & & \\ 0 & \longrightarrow & K & \xrightarrow{i} & P & \xrightarrow{j} & C & \longrightarrow & 0 \end{array} \qquad (2.2)$$

Ps is defined to be the pullback of j and s.

3. $\mathrm{Ext}_R(C, K)$ as a Ring.

Let R be a ring and

$$\xi: 0 \longrightarrow K \xrightarrow{i} P \xrightarrow{j} C \longrightarrow 0$$

be an exact sequence of (left) R-modules.

DEFINITION We say that P is *K-projective* if

$i^*: \mathrm{Ext}_R^1(P, K) \to \mathrm{Ext}_R^1(K, K)$

is a monomorphism.

Of course, it follows from the long exact sequence for $\mathrm{Ext}_R^i(-, K)$ [9, page 74] associated with ξ that P is K-projective iff the boundary operator $\partial: \mathrm{End}_R(K) \to \mathrm{Ext}_R^1(C, K)$ is surjective. Here $\partial(k)$ equals the equivalence class of the extension kP for any $k \in \mathrm{End}(K)$. If $\mathrm{Ext}_R(P, K) = 0$, then P is K-projective; in particular, any projective R-module is K-projective.

3.1. THEOREM. *If* $0 \longrightarrow K \xrightarrow{i} P \xrightarrow{j} C \longrightarrow 0$ *is an exact sequence of R-modules with P K-projective, then the boundary operator ∂ induces an isomorphism*

$$\bar{\partial}: \frac{\mathrm{End}_R(K)}{i^*(\mathrm{Hom}_R(P, K))} \to \mathrm{Ext}_R^1(C, K).$$

For each $k \in \mathrm{End}(K)$, let $\{k\}$ denote the element $\partial(k)$ in $\mathrm{Ext}_R^1(C, K)$.

$\mathrm{End}(K)$ has a ring structure under composition. The question is: when is $B = i^* \mathrm{Hom}(P, K)$ a two-sided ideal? If we denote the composition $K \xrightarrow{\alpha} K \xrightarrow{\beta} K$ by $\beta\alpha$, then

$$B = \{\alpha: K \to K \mid \alpha \text{ extends to a map } \alpha': P \to K\}$$

is always a left ideal. For, if $\alpha \in B$, $\beta \in \mathrm{End}(K)$ and $\alpha' \in \mathrm{Hom}(P, K)$ extends α, then $\beta\alpha'$ extends $\beta\alpha$. Thus B is a right ideal and $B \neq \mathrm{End}(K)$ implies that $\mathrm{Ext}(C, K)$ is a ring with identity.

We will now delineate a sequence of sufficient conditions that imply that B is a right ideal.

3.2. (*C*). *The composition in* $\mathrm{End}(K)$ *is commutative modulo B.*

3.3. (RE). *Each homomorphism in* End(K) *extends to a homomorphism in* End(P).

3.4. (E). *Each homomorphism in* Hom(K, P) *extends to a homomorphism in* End(P).

Note that the following implications hold:

$(E) \Rightarrow (RE) \Rightarrow B$ is a right ideal $\Leftarrow (C)$.

3.5. If $\operatorname{Ext}(C, P) = 0$, *then* ($E$) *is true.* This follows because $\operatorname{Ext}(C, P) = 0$ implies $i^*: \operatorname{End}(P) \to \operatorname{Hom}(K, P)$ is surjective. If $\operatorname{Ext}(P, P) = 0$, then ($E$) *is equivalent to* $\operatorname{Ext}(C, P) = 0$. In particular, this is true if P is projective.

3.6. Also, one can easily see that (RE) iff *the boundary homomorphism* $\partial: \operatorname{End}(C) \to \operatorname{Ext}(C, K)$ *is surjective* iff $j_*: \operatorname{Ext}(C, P) \to \operatorname{Ext}(C, C)$ *is a monomorphism.*

Note that $\operatorname{Ext}(C, K)$ is cyclic automatically implies (C).

We may call P *C-injective* if $j_*: \operatorname{Ext}(C, P) \to \operatorname{Ext}(C, C)$ is a monomorphism. Thus $\operatorname{Ext}(C, K)$ has a ring structure as above if P is C-injective *and* K-projective.

More generally, we may proceed as follows: *let P be K-projective.*

DEFINITION. Let $\operatorname{Ext}(C, K)_K$ denote the subset of $\operatorname{Ext}(C, K)$ such that $\{k\} \in \operatorname{Ext}(C, K)_K$ iff $Bk \subset B$.

It is clear that

(a) $\operatorname{Ext}(C, K)_K$ is a subgroup of $\operatorname{Ext}(C, K)$.
(b) $\operatorname{Ext}(C, K)_K$ is a ring with identity under composition.
(c) The image of the center of $\operatorname{End}(K)$ is contained in $\operatorname{Ext}(C, K)_K$.

$\operatorname{Ext}(C, K)_K$ is called the *maximal K-ring of* $\operatorname{Ext}(C, K)$.

Let $\partial_C: \operatorname{End}(C) \to \operatorname{Ext}(C, K)$ be the boundary operator in the exact sequence for $\operatorname{Ext}^i(C, -)$ associated with the extension $\xi: 0 \to K \to P \to C \to 0$. $\partial_C(r)$ is the equivalence class of the extension Pr (see 2.2).

3.7. PROPOSITION.

(a) End(C) *always induces a ring structure on the subgroup* $\operatorname{im} \partial_C = {}_C\operatorname{Ext}(C, K)$.

(b) ${}_C\operatorname{Ext}(C, K)$ *is a subring of* $\operatorname{Ext}(C, K)_K$

(c) *If ∂C is surjective, then* ${}_C\operatorname{Ext}(C, K) \cong \operatorname{Ext}(C, K)_K$ *as rings.*

Proof.

(a) P is K-projective implies that $\operatorname{im}\{j_*:\operatorname{Hom}(C,P)\to\operatorname{End}(C)\}$ is a two-sided ideal. This follows because each homomorphism in $\operatorname{End}(C)$ extends to a homomorphism in $\operatorname{End}(P)$. Consider $l\in\operatorname{End}(C)$ and the extension Pl. Then P is K-projective implies that there exists a $k\in\operatorname{End}(K)$ such kP and Pl are equivalent extensions. Thus there is an isomorphism $\alpha: kP\to Pl$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \to & K & \to & P & \to & C & \to & 0\\
 & & {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle 1\beta} & & \| & & \\
 & & & \nearrow & kP & \searrow & & & \\
0 & \to & K & & \downarrow & & C & \to & 0\\
 & & & \searrow & Pl & \nearrow & & & \\
 & & \| & & \downarrow{\scriptstyle \beta l} & & \downarrow{\scriptstyle l} & & \\
0 & \to & K & \to & P & \to & C & \to & 0
\end{array}$$

(b) Any $\{k\}\in\operatorname{Ext}(C,K)$ $(k\in\operatorname{End}(K))$ which is in the image of ∂_C clearly satisfies $Bk\subset B$. Let $\partial_C(l)=\{k\}$. Then we may choose an extension as in (a) so that the following commutes

$$\begin{array}{ccccccccc}
0 & \to & K & \to & P & \to & C & \to & 0\\
 & & {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle l} & & \\
0 & \to & K & \to & P & \to & C & \to & 0
\end{array}$$

Now $\alpha\in B$ iff *α extends the zero map* $0: C\to C$, i.e., the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \to & K & \to & P & \to & C & \to & 0\\
 & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta_\alpha} & & \downarrow{\scriptstyle 0} & & \\
0 & \to & K & \to & P & \to & C & \to & 0
\end{array}$$

But $\alpha\in B$ and $\{k\}\in\operatorname{im}\partial_C$ implies that $\alpha\circ k$ extends $0\circ l=0$. Thus (b) is proved.

(c) follows easily from (a) and (b). We only note that the ring isomorphism is given by the correspondence $\partial_C(l)\mapsto\{k\}$ where $k\in\operatorname{End}(K)$ extends $l\in\operatorname{End}(C)$. This completes 3.7.

Note that ∂_C is surjective iff condition (RE).

We now give a simple example to show that B is not always a right ideal. Let $R=Z$ and let the basic extension be given by

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z\oplus Z & \xrightarrow{\ i\ } & Z\oplus Z & \xrightarrow{\ j\ } & Z_3\oplus Z_2 & \longrightarrow & 0\\
 & & \| & & \| & & \| & & \\
 & & K & & P & & C & &
\end{array}$$

where i has matrix $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ with respect to the natural bases. Then $B \subset \operatorname{End}(Z \oplus Z)$ is the set of all 2×2 matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ over Z with the first column divisible by 3, the second by 2. $\operatorname{Ext}(C, K) \cong Z_3^2 \oplus Z_2^2$. Representatives of the cosets modulo B are given by

$$\mathscr{R} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| \begin{matrix} 0 \le a_{i1} \le 2 \\ 0 \le a_{i2} \le 1 \end{matrix}, \quad i = 1, 2 \right\}$$

It is easy to check that only the diagonal matrices in $\mathscr{R}$ have the property that $B \circ k \subset B$. Hence $\operatorname{Ext}(C, K)_K \cong Z_3 \oplus Z_2 \hookrightarrow \operatorname{Ext}(C, K)$ by embedding in the first and fourth coordinates.

4. *K*-Projective *k*-Invariants

Throughout this section we assume that $i^*: \operatorname{End}(K) \to \operatorname{Ext}(C, K)$ is surjective; i.e., that P is K-projective.

DEFINITION. The class $\{k\} \in \operatorname{Ext}(C, K)$ determined by $k \in \operatorname{End}(K)$ is called the *k-invariant of the extension kP*. A k-invariant $\{k\}$ is called *K-projective* if kP is a K-projective R-module. An element $k \in \operatorname{End}(K)$ is also called K-projective if $\{k\}$ is K-projective. Let $\mathscr{P}_K(\operatorname{Ext}(C, K))$ denote the set of K-projective k-invariants in $\operatorname{Ext}(C, K)$, $\mathscr{P}_K(\operatorname{End}(K))$ the set of K-projective elements $\operatorname{End}(K)$.

DEFINITION. Let E be a ring with identity. An element $\alpha \in E$ is a *right unit* if there exists $\beta \in E$ such that $\beta\alpha = 1$. The set of (right) units of E is denoted by $(R)U(E)$.

For each $\alpha \in E$, let α^* denote the abelian group homomorphism $E \to E$ given by right multiplication by α. α is a right unit iff α^* is surjective.

4.1. THEOREM. *Let* $\operatorname{Ext}(C, K)$ *inherit a ring structure from* $\operatorname{End}(K)$. $\{k\}$ *is a K-projective k-invariant iff $\{k\}$ is a right unit.*

Proof. Suppose that k is K-projective. Then $\partial_k: \operatorname{End}(K) \to \operatorname{Ext}(C, K)$ ($\partial_k(\alpha) = (\alpha \circ k)P$, $\alpha \in \operatorname{End}(K)$) is surjective. Thus there is a $k' \in \operatorname{End}(K)$ such that $(k' \circ k)P$ is equivalent to P as extensions. Hence $k' \circ k - 1 \in B$, and k is a right unit.

If $k' \circ k - 1 \in B$, we will show that kP is K-projective. P and $(k' \circ k)P$ are equivalent extensions, so there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & P & & & & \\
 & & & \overset{i}{\nearrow} & \uparrow & \overset{j}{\searrow} & & & \\
0 & \longrightarrow & K & & \Big|\cong & & C & \longrightarrow & 0 \\
 & & \uparrow & \searrow & & \nearrow & \Big\| & & \\
 & & k'\Big| & & (k' \circ k)P & & \Big\| & & \\
 & & \Big| & & \uparrow & & \Big\| & & \\
0 & \longrightarrow & K & \xrightarrow{i_k} & kP & \xrightarrow{j_k} & C & \longrightarrow & 0
\end{array}$$

Call the resulting map $\beta : kP \to P$. Apply $\operatorname{Ext}(-, K)$ to this diagram to obtain the commutative diagram:

$$\begin{array}{ccccc}
\operatorname{Ext}(C, K) & \xrightarrow{j^*} & \operatorname{Ext}(P, K) & \xrightarrow{i^*} & \operatorname{Ext}(K, K) \\
\Big\| & & \Big\downarrow \beta^* & & \Big\downarrow k'^* \\
\operatorname{Ext}(C, K) & \xrightarrow{j_k^*} & \operatorname{Ext}(kP, K) & \xrightarrow{i_k{}^*} & \operatorname{Ext}(K, K)
\end{array}$$

Thus $j_k^* = \beta^* j^* = 0$ because $j^* = 0$. Thus i_k^* is a monomorphism. This completes 4.1.

4.2. THEOREM. *If* $\{k \circ k'\} = \{k \circ k'\} = \{1\}$ *in* $\operatorname{Ext}(C, K)$, *then* $\operatorname{Ext}(kP, M) = 0$ *iff* $\operatorname{Ext}(P, M) = 0$, *where M is an R-module.*

If we were to define the "degree of projectivity" of k by the class of modules $\mathcal{M}_k$ such that $M \in \mathcal{M}_k$ iff $\operatorname{Ext}(kP, M) = 0$, then the above says that $\{k\}$ is a unit implies that $\mathcal{M}_k = \mathcal{M}_1$; i.e., kP is "just as projective" as P is.

Proof. Because $k' \circ k - 1 \in B$, the argument of (4.1) implies the existence of the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{i_k} & kP & \xrightarrow{j_k} & C & \longrightarrow & 0 \\
 & & k\Big\uparrow & & \beta\Big\uparrow & & \Big\| & & \\
0 & \longrightarrow & K & \xrightarrow{i} & P & \xrightarrow{j} & C & \longrightarrow & 0 \\
 & & k'\Big\uparrow & & \beta'\Big\uparrow & & \Big\| & & \\
0 & \longrightarrow & K & \xrightarrow{i_k} & kP & \xrightarrow{j_k} & C & \longrightarrow & 0
\end{array}$$

Now $k \circ k' = 1 + \alpha' \circ i$, where $\alpha' \in \operatorname{Hom}(P, K)$. Let M be any R-module such that $\operatorname{Ext}(P, M) = 0$. Apply the functor $\operatorname{Ext}(-, M)$ to the above diagram.

$$\begin{array}{ccccc}
\mathrm{Ext}(C, M) & \xrightarrow{j_k{}^*} & \mathrm{Ext}(kP, M) & \xrightarrow{i_k{}^*} & \mathrm{Ext}(K, M) \\
\| & & \downarrow{\scriptstyle \beta^*} & & \\
\| & & \mathrm{Ext}(P, M) & & \downarrow{\scriptstyle (k\circ k')^*} \\
\| & & \downarrow{\scriptstyle \beta'^*} & & \\
\mathrm{Ext}(C, M) & \xrightarrow[j_k{}^*]{} & \mathrm{Ext}(kP, M) & \xrightarrow[i_k{}^*]{} & \mathrm{Ext}(K, M)
\end{array}$$

The rows are exact at $\mathrm{Ext}(kP, M)$. $(k\circ k')^* = (1+\alpha'\circ i)^* = 1+(\alpha'\circ i)^* = 1$, since $(\alpha'\circ i)^* = 0$. Thus $\beta'^*\circ\beta^*$ is an isomorphism. Then $\mathrm{Ext}(P, M) = 0$ implies $\mathrm{Ext}(kP, M) = 0$. A similar argument shows the converse. This completes (4.2).

Since the set of right units is a semigroup under composition, the following is clear.

4.3. COROLLARY. *Let* $\mathrm{Ext}(C, K)$ *have a ring structure as above. Then the set* $\mathcal{P}_k(\mathrm{Ext}(C, K))$ *of* K*-projective* k*-invariants is a semigroup with identity under composition.* $\mathcal{P}_k$ *is a group iff each* K*-projective* k*-invariant is a unit.*

5. k-Invariants as Units.

In this section we will study conditions under which right units are units in the ring $\mathrm{Ext}(C, K)$. We continue our assumption that P is K-projective. We also assume in this section that B *is a right ideal.*

DEFINITION. For each $k\in\mathrm{End}(K)$, let $B_k = \mathrm{im}\{\mathrm{Hom}(kP, K)\to\mathrm{End}(K)\} = \ker\{\partial_k : \mathrm{End}(K)\to\mathrm{Ext}(C, K)\}$, where $\partial_k(\alpha) = (\alpha\circ k)P\ (\alpha\in\mathrm{End}(K))$.

5.1. LEMMA. $B = \mathrm{im}\{\mathrm{Hom}(P, K)\to\mathrm{End}(K)\}$ *is a right ideal iff* $B\subset B_k$ *for all* $k\in\mathrm{End}(K)$.

Proof. Let $\alpha\in B$. For any $k\in\mathrm{End}(K)$, $\alpha\circ k\in B$ since B is a right ideal. Thus $(\alpha\circ k)P\cong\alpha(kP)$ is trivial implies that $\alpha\in B_k$. Conversely, if $B\subset B_k$ for all $k\in\mathrm{End}(K)$, then let $\alpha\in B$, and consider $\alpha\circ k$ $(k\in\mathrm{End}(K))$. $\alpha\in B_k$ implies $\alpha(kP)\cong(\alpha\circ k)P\cong K\times C$ which in turn implies that $\alpha\circ k\in B$.

We say that $\{k\}\in\mathrm{Ext}(C, K)$ is a *right zero divisor* if there exists a $\{k'\}\neq 0$ such that $\{k'\circ k\} = 0$.

5.2. PROPOSITION. *$\{k\} \in \operatorname{Ext}(C, K)$ is not a right zero divisor iff $B = B_k$. If k is K-projective, then $\{k\}$ is a unit iff $B = B_k$.*

Proof. For each $k \in \operatorname{End}(K)$, let $k^*: \operatorname{Ext}(C, K) \to \operatorname{Ext}(C, K)$ be the function defined by right multiplication by $\{k\}$. It is a homomorphism of the underlying abelian group structure. Thus $\{k\}$ *is not a right zero divisor* iff k^* *is a monomorphism*. But k^* is a monomorphism iff $B = B_k$ follows from the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & B & \longrightarrow & \operatorname{End}(K) & \xrightarrow{\partial} & \operatorname{Ext}(C, K) & \longrightarrow & 0 & & \\
 & & \big\downarrow \cap & & \Vert & & \big\downarrow k^* & & & & \\
0 & \longrightarrow & B_k & \longrightarrow & \operatorname{End}(K) & \xrightarrow{\partial_k} & \operatorname{Ext}(C, K) & \longrightarrow & \operatorname{Ext}(kP, K) & \longrightarrow & \cdots
\end{array}$$

Here $\partial(\alpha) = \alpha P$, $\partial_k(\alpha) = \alpha(kP) = (\alpha \circ k)P$ and the horizontal sequences are exact. Furthermore, k^* is an isomorphism implies that ∂_k is surjective and hence $B = B_k$. $B = B_k$ together with ∂_k surjective implies k^* is an isomorphism.

5.3. LEMMA. *Let $k \in \operatorname{End}(K)$ and suppose there exists $k' \in \operatorname{End}(K)$ such that $k' \circ k - 1 \in B$. Then $B = B_{k'}$.*

Proof. Consider the homomorphisms k^*, k'^* as in the proof of (5.2). The composite $k^* \circ k'^* = (k' \circ k)^* = 1$. Thus k'^* is a monomorphism and, by (5.2), $B = B_{k'}$.

We will now give several conditions under which K-projective k-invariants are units. Clearly, if $\operatorname{Ext}(C, K)$ is *commutative* or has *no zero divisors*, then every right unit is a unit. Furthermore a theorem of N. Jacobson [7] shows that any ring having right units which are not units must be very large. The following is just a restatement of theorem 1 of [7].

5.4. THEOREM. *If $E = \operatorname{Ext}(C, K)$ has either the ascending or descending chain condition for principal right ideals generated by idempotent elements, then right units are units.*

Thus it follows that if E is finitely generated as a left (or right) E module, then right units are units in E. For example, if R is *commutative* and K is a *finitely generated* R-module, then $\operatorname{Ext}(C, K)$ is a finitely generated R-module and hence, by (5.4), right units are units.

Now let P be a *projective* R-module and consider any exact sequence

$$0 \longrightarrow K_1 \xrightarrow{\iota_1} P_1 \xrightarrow{j_1} K \longrightarrow 0$$

of R-modules where P_1 is projective. The boundary operator

$$\partial : \operatorname{Ext}^1(C, K) \to \operatorname{Ext}^2(C, K_1) = \operatorname{Ext}^1(K, K_1)$$

is given by $\partial(\{k\}) =$ class of the extension $P_1 k$ (see 2.2).

5.5. THEOREM. *If* $\partial : \operatorname{Ext}^1(C, K) \to \operatorname{Ext}^2(C, K_1)$ *is a monomorphism, then projective k-invariants are units in* $\operatorname{Ext}(C, K)$.

5.6. COROLLARY. *If* $\operatorname{Ext}(C, R) = 0$ *and K is finitely generated as an R-module, then projective k-invariants are units in* $\operatorname{Ext}(C, K)$.

The proof of (5.5) is postponed to (6.13). The corollary follows because K is finitely generated implies P_1 may be chosen to be finitely generated. $\operatorname{Ext}(C, R) = 0$ then yields $\operatorname{Ext}(C, P_1) = 0$ and this implies that ∂ is a monomorphism.

6. The k-Invariant of a Truncated Resolution.

Let M be an R-module. Choose a *projective* resolution

$$\mathcal{F}(M) : \cdots \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} C_{m-2} \longrightarrow \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} M \longrightarrow 0$$

of M, where each C_i is projective R-module. $\mathcal{F}(M)$ is called the *base resolution;* each $\pi_m = \ker \partial_m (m \geq 0)$ is called an *M-realizable* R-module. If $M = Z$, the trivial R-module, then π_m is *realizable* means it is *Z-realizable.* We say that a resolution $\mathcal{F}$ is of *finite type* if each C_i is a finitely generated R-module.

Let

$$\mathcal{G}(M) : \cdots \longrightarrow G_m \xrightarrow{g_m} G_{m-1} \xrightarrow{g_{m-1}} \cdots \xrightarrow{g_1} G_0 \xrightarrow{g_0} M \longrightarrow 0$$

be a (not necessarily projective) resolution of M. Let π'_m denote $\ker g_m$. The *k-invariant of $\mathcal{G}$ in dimension m* relative to $\mathcal{F}$ is the element $\{k\} \in \operatorname{Ext}_R^{m+1}(M, \pi'_m)$ determined by a chain map $f : \mathcal{F}(M) \to \mathcal{G}(M)$ covering the identity on M. Thus f is a sequence of maps making the following diagram commute:

$$\begin{array}{ccccccccccccc}
 & & C_{m+1} & \xrightarrow{\partial_{m+1}} & C_m & \xrightarrow{\partial_m} & C_{m-1} & \longrightarrow \cdots \longrightarrow & C_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \vdots k & & \downarrow f_m & & \downarrow f_{m-1} & & \downarrow f_0 & & \| & & \\
0 & \longrightarrow & \pi'_m & \longrightarrow & G_m & \xrightarrow{g_m} & G_{m-1} & \longrightarrow \cdots \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

The map $k = f_m \circ \partial_{m+1}: C_{m+1} \to \pi'_m$ determines an element $\{k\} \in \operatorname{Ext}_R^{m+1}(M, \pi'_m)$. This is well-defined by a standard argument [5].

6.1. LEMMA. *For each $m \geq 0$ and each element $\bar{k} \in \operatorname{Ext}_R^{m+1}(M, \pi'_m)$ $\exists$ a resolution $\mathcal{G}_{\bar{k}}$ of M realizing $\bar{k}$. If C_i $(i = 0, 1, \ldots m)$ and π'_m are finitely generated, then $\mathcal{G}_{\bar{k}}^{(m)}$ may be chosen to be of finite type.*

Proof. Consider $k: C_{m+1} \to \pi'_m$ realizing $\bar{k}$; $k \cdot \partial_{m+2} = 0$ implies that k defines a map $k': \pi_m \to \pi'_m$. Use the construction of section 2 to build

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_m & \longrightarrow & C_m & \longrightarrow & \pi_{m-1} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle k'} & & \Big\downarrow & & \Big\| & & \\ & & \pi'_m & \xrightarrow{i'} & k'C_m & \xrightarrow{j'} & \pi_{m-1} & \longrightarrow & 0 \end{array}$$

Then the m-skeleton $\mathcal{G}_{\bar{k}}^{(m)}$ is given by

$$0 \longrightarrow \pi'_m \xrightarrow{i'} k'C_m \xrightarrow{\partial'_m} C_{m-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow M \longrightarrow 0$$

where ∂'_m is the composite $k'C_m \xrightarrow{j'} \pi_{m-1} \hookrightarrow C_{m-1}$. This completes 6.1.

DEFINITION. An element $k \in \operatorname{Ext}^{m+1}(M, \pi'_m)$ is called *projective* if k can be realized as the k-invariant of a truncated projective resolution:

$$\mathcal{P}_k^{(m)}: 0 \to \pi'_m \to P_m \to P_{m-1} \to \cdots \to P_0 \to M \to 0$$

when compared with the base resolution $\mathcal{F}(M)$. The set of projective k-invariants of $\operatorname{Ext}^{m+1}(M, \pi'_m)$ is denoted by $\mathcal{P}(\operatorname{Ext}^{m+1}(M, \pi'_m))$.

6.2. THEOREM. *Let M be any R-module and π_m be M-realizable for $m \geq 0$. Then*

(a) $$\operatorname{Ext}_R^{m+1}(M, \pi_m) \cong \frac{\operatorname{End}(\pi_m)}{\operatorname{im}\operatorname{Hom}(C_m, \pi_m)}.$$

(b) *If* $B^m = \operatorname{im}\{\operatorname{Hom}(C_m, \pi_m) \to \operatorname{End}(\pi_m)\}$ *is a right ideal, then* $\operatorname{Ext}^{m+1}(M, \pi_m)$ *has a ring structure induced from that of* $\operatorname{End}(\pi_m)$ such that the projective *k-invariants lie between the units and right units of* $\operatorname{Ext}^{m+1}(M, \pi_m)$:

$$U(\operatorname{Ext}^{m+1}(M, \pi_m)) \subset \mathcal{P}(\operatorname{Ext}^{m+1}(M, \pi_m)) \subset RU(\operatorname{Ext}^{m+1}(M, \pi_m)).$$

(c) *If B^m is a right ideal, $\mathcal{P}(\operatorname{Ext}^{m+1}(M, \pi_m)) = U(\operatorname{Ext}^{m+1}(M, \pi_m))$, and each C_i*

($i=0, 1, \ldots, m+1$) a finitely generated free module, then the function

$$\chi_m : \mathcal{P}(\mathrm{Ext}^{m+1}(M; \pi_m)) \to \tilde{K}_0 R$$

which assigns to each $k \in \mathcal{P}$ the Euler characteristic $\chi_m(\mathcal{P}_k^{(m)}) = \sum_{i=0}^m (-1)^i [P_i] \in \tilde{K}_0 R$ of $\mathcal{P}_k^{(m)}$ is a homomorphism.

Note. (1) Theorem 6.2 is theorem 1.1 in the case $R = Z\pi$ and $M = Z$. This follows because $H^{m+1}(\pi; Z\pi) = 0$ and C_m finitely generated implies that $H^{m+1}(\pi; C_m) = 0$. Thus $H^{m+1}(\pi; \pi_m)$ is a ring (3.5) and by (5.6) right units are units because π_m is finitely generated.

(2) It follows from [11, theorem 3.1] that if $m \geq 3$, any π-module π_m realizable by a truncated *free* resolution over Z is topologically realizable as well.

(3) It follows from (4.1) that the set $\mathcal{P}_{\pi_m}$ of π_m-*projective* k-invariants is equal to the set of right units of $\mathrm{Ext}^{m+1}(M; \pi_m)$. Furthermore, (4.2) implies that any unit in $\mathrm{Ext}^{m+1}(M, \pi_m)$ must be *projective.* We do not know whether in general $\mathcal{P}$ is distinct from U or RU (see 5.4).

The following lemma is useful in the subsequent work:

6.3. LEMMA OF COCKCROFT-SWAN [3, Appendix]. Let $\xi_i^{(m)}: 0 \to \pi_m \to E_m^i \to P_{m-1}^i \to \cdots \to P_0^i \to M \to 0$ ($i = 1, 2$) be resolutions of M with each P_j^i ($j = 0, 1, \ldots, m-1$) projective. Let $f: \xi_1^{(m)} \to \xi_2^{(m)}$ be a chain map covering the identity on M and inducing an isomorphism on π_m. Then

$$E_m^1 \oplus P_{m-1}^2 \oplus P_{m-2}^1 \oplus \cdots \cong E_m^2 \oplus P_{m-1}^1 \oplus P_{m-2}^2 \oplus \cdots$$

Note the similarity between this and Schanuel's lemma [11].

6.4. COROLLARY. *Let $\xi_1^{(m)}$ be projective (i.e., E_m^1 is projective) and suppose $k(\xi_1^{(m)}) = k(\xi_2^{(m)})$ when compared to $\mathcal{F}$. Then*

$$E_m^1 \oplus P_{m-1}^2 \oplus P_{m-2}^1 \oplus \cdots \cong E_m^2 \oplus P_{m-1}^1 \oplus P_{m-2}^2 \oplus \cdots$$

and hence $\xi_2^{(m)}$ is projective also.

Proof. By standard obstruction arguments, there exists a chain map $\xi_1^{(m)} \to \xi_2^{(m)}$ inducing the identity on M *and* π_m. Then apply (6.3).

Proof of 6.2. We will only show that if $\mathcal{P} = U$, then $\chi: \mathcal{P} \to \tilde{K}_0 R$ is a homomorphism. Let k, $k' \in \mathrm{End}(\pi_m)$ represent projective k-invariants in $\mathrm{Ext}^{m+1}(M; \pi_m)$. We will show that

$$(k' \circ k) C_m \oplus C_m \oplus C_{m+1} \cong k C_m \oplus k' C_m \oplus C_{m+1}.$$

Let $\partial k' \in \operatorname{End}(\pi_{m+1})$ be any map determined by extending k':

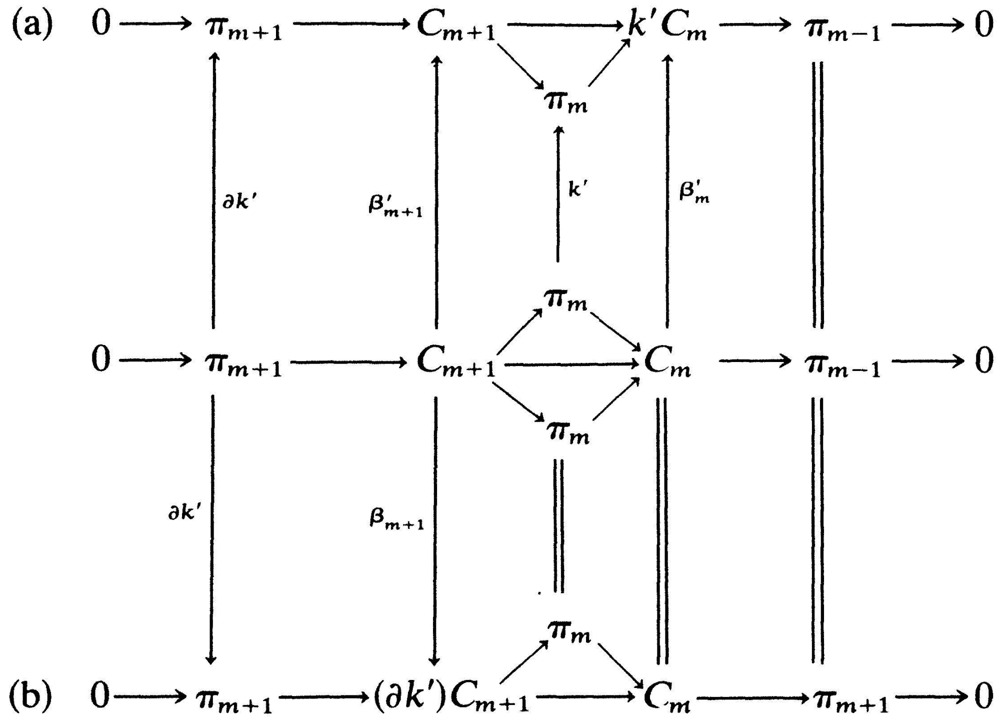
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_{m+1} & \longrightarrow & C_{m+1} & \longrightarrow & \pi_m & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial k'} & & \downarrow{\scriptstyle \beta'_{m+1}} & & \downarrow{\scriptstyle k'} & & \\ 0 & \longrightarrow & \pi_{m+1} & \longrightarrow & C_{m+1} & \longrightarrow & \pi_m & \longrightarrow & 0 \end{array}$$

The correspondence $\{k'\} \to \{\partial k'\}$ gives the boundary homomorphism

$$\partial : \operatorname{Ext}^{m+1}(M; \pi_m) \to \operatorname{Ext}^{m+2}(M; \pi_{m+1}).$$

6.5. LEMMA. *Let $k' \in \operatorname{End}(\pi_m)$ be projective. Then $(\partial k')C_{m+1} \oplus k'C_m \cong C_m \oplus C_{m+1}$. Hence $(\partial k')C_{m+1}$ is projective and $[(\partial k')C_{m+1}] + [k'C_m] = 0$ in $\tilde{K}_0R$.*

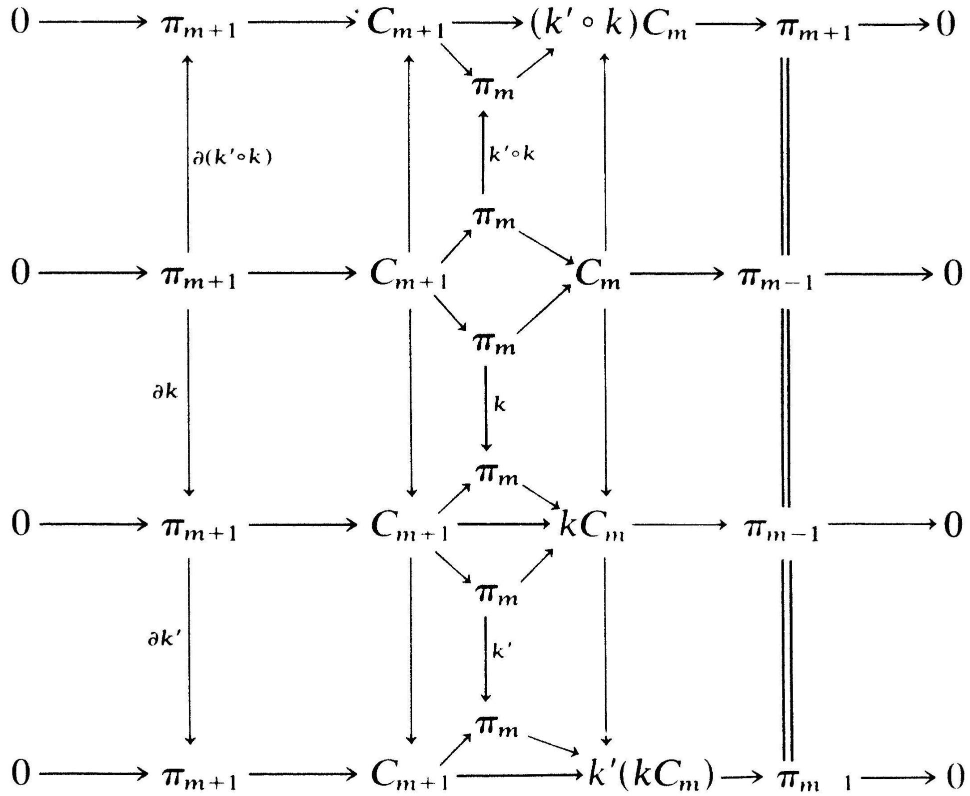
Proof. Consider the resolutions



These resolutions (a) and (b) necessarily have the same k-invariant, (a) is projective; hence (b) is also projective by lemma 6.4. $(\partial k')C_{m+1} \oplus k'C_m \cong C_{m+1} \oplus C_m$ follows from (6.4).

6.6. LEMMA. *k is projective and $k' \circ k - 1 \in B^m$ implies $C_{m+1} \oplus kC_m \cong (k' \circ k)C_m \oplus (\partial k')C_{m+1}$.*

Proof. Realize the k-invariant $\{\partial(k' \circ k)\} = \{\partial k' \circ \partial k\} \in \operatorname{Ext}^{m+2}(M; \pi_{m+1})$ in three ways:

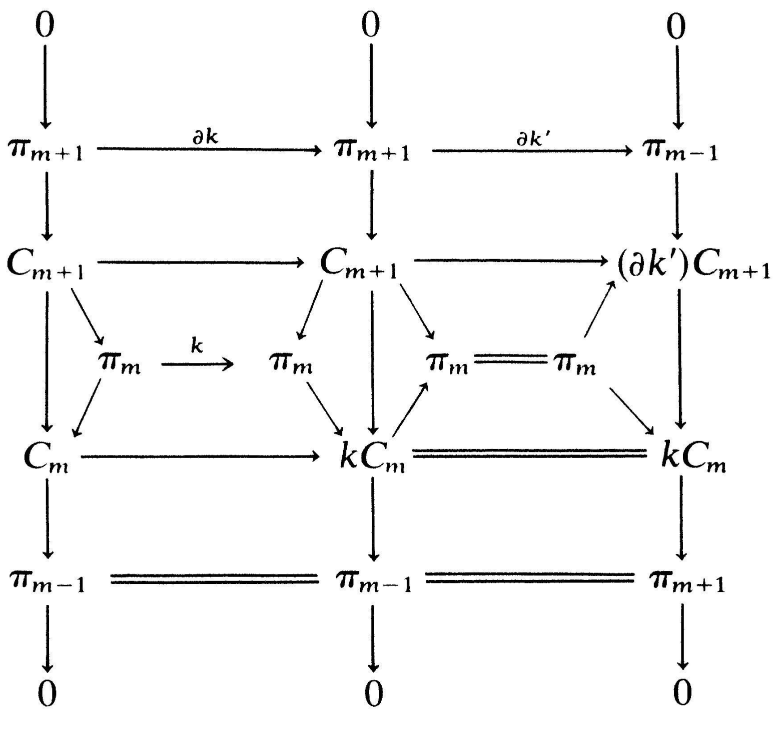


It follows that

$$(k' \circ k)C_m \cong k'(kC_m)$$

via a map inducing identity on π_{m-1} and π_m because the k-invariants are the same. Thus $\{\partial(k' \circ k)\} = \{\partial k' \circ \partial k\}$. Note that $k' \circ k$ is projective because it is a unit.

Furthermore, the following also has k-invariant $\partial k' \circ \partial k$:



Thus, by another application of lemma 6.4, we have $C_{m+1} \oplus kC_m \cong (k' \circ k)C_m \oplus (\partial k')C_{m+1}$. (6.5) and (6.6) taken together prove (c).

CONJECTURE (see [11, lemma 6.1 (c)]).

$(k' \circ k)C_m \oplus C_m \cong kC_m \oplus k'C_m$.

Let $\partial : \mathrm{Ext}^{m+1}(M, \pi_m) \to \mathrm{Ext}^{m+2}(M, \pi_{m+1})$ be the boundary operator in the coefficient exact sequence associated with the functor $\mathrm{Ext}^i(M, -)$ and the exact sequence

$$0 \to \pi_{m+1} \to C_{m+1} \to \pi_m \to 0.$$

The previous proof shows that ∂ is a *ring homomorphism*, provided the domain and range are rings.

Furthermore, we see that because C_i is finitely generated and free for $i = 0, \ldots, m+1$, then $\mathrm{im}\, \chi_m \subset \mathrm{im}\, \chi_{m+1}$. This follows from the commutative diagram:

$$\begin{array}{ccc} \mathcal{P}(\mathrm{Ext}^{m+2}(M, \pi_{m+1})) & \searrow^{\chi_{m+1}} & \\ \uparrow{\scriptstyle \partial} & & \tilde{K}_0 R \\ \mathcal{P}(\mathrm{Ext}^{m+1}(M, \pi_m)) & \nearrow_{\chi_m} & \end{array}$$

The conditions of section 3 have obvious analogs in this setting:

6.7. $(C(m))$. *The composition in* $\mathrm{End}(\pi_m)$ *is commutative modulo* B^m.

6.8. (RE_m). *Each map* $k \in \mathrm{End}(\pi_m)$ *extends to a map in*

$$\mathrm{End}(C_m) \Leftrightarrow \partial : \mathrm{Ext}^m(M, \pi_{m-1}) \to \mathrm{Ext}^{m+1}(M, \pi_m) \text{ is surjective}$$
$$\Leftrightarrow \mathrm{Ext}^{m+1}(M; C_m) \to \mathrm{Ext}^{m+1}(M; \pi_{m-1}) \text{ is monic.}$$

6.9. (E_m). *Each map* $f \in \mathrm{Hom}(\pi_m, C_m)$ *extends to a map in*

$$\mathrm{End}(C_m) \Leftrightarrow \mathrm{Ext}^1(\pi_{m-1}, C_m) = \mathrm{Ext}^{m+1}(M; C_m) = 0$$

Again: $(E_m) \Rightarrow (RE_m) \Rightarrow B^m$ is a right ideal $\Leftarrow (C(m))$

At the present writing, I know of no examples where $C(m)$ is not satisfied.

We can "dualize" RE_m as follows:

6.10. (RE^m). *Any map* $k \in \mathrm{End}(\pi_m)$ *which coextends to* C_{m+1} *extends to* C_m. Thus, in the following diagram,

$$\begin{array}{ccccc} & & \pi_m & \xhookrightarrow{i} & C_m \\ & \swarrow^{\exists \alpha} & \downarrow{\scriptstyle k} & \swarrow_{\exists \beta} & \\ C_{m+1} & \xrightarrow{j} & \pi_m & & \end{array}$$

the existence of α such that $j \circ \alpha = k$ implies the existence of a β such that $\beta \circ i = k$. The converse is always true because C_m is projective.

6.11. PROPOSITION. *Any map $k \in \operatorname{End}(\pi_m)$ which coextends to C_{m+1} extends to C_m iff $\partial : \operatorname{Ext}^{m+1}(M, \pi_m) \to \operatorname{Ext}^{m+2}(M, \pi_{m+1})$ is a monomorphism iff $i_* : \operatorname{Ext}^{m+1}(M, \pi_{m+1}) \to \operatorname{Ext}^{m+1}(M, C_{m+1})$ is surjective.*

6.12. PROPOSITION. *If each $k \in \operatorname{End}(\pi_m)$ which coextends to C_{m+1} also extends to C_m, then $\operatorname{Ext}^{m+1}(M; \pi_m)$ is a ring.*

Proof. Let k, $\bar{k} \in \operatorname{End}(\pi_m)$, let k extend to C_m. We must show that $k \circ \bar{k}$ extends to C_m. But k extends to C_m implies that k coextends to C_{m+1} by (6.10). Thus $k \circ k'$ coextends to C_{m+1}. But condition RE^m implies that $k \circ k'$ extends to C_m. This proves (6.12).

Note that $(RE_m) \Leftarrow (E_m) \Rightarrow (RE^m)$.

Notice that it follows from (6.6) that if $\{k\} \in \operatorname{Ext}^m(M, \pi_{m-1})$ is projective and $\{k' \circ k\} = 1$, then $\{\partial k'\} \in \operatorname{Ext}^{m+1}(M; \pi_m)$ is projective. Also, (6.5) implies that $\partial\{k\}$ is projective if $\{k\}$ is.

6.13. COROLLARY. *If $\partial : \operatorname{Ext}^{m+1}(M; \pi_m) \to \operatorname{Ext}^{m+2}(M; \pi_{m+1})$ is a monomorphism (condition RE^m), then each projective k-invariant is a unit.*

Proof. Let $\{k\} \in \operatorname{Ext}^{m+1}(M; \pi_m)$ be projective. By (5.3), there is a $k' \in \operatorname{End}(\pi_m)$ such that $k' \circ k' - 1 \in B^m$. Thus $\partial k' \circ \partial k - 1 \in B^{m+1}$. By (6.6), $\{\partial k'\}$ is projective. By (5.3) again, $\{\partial k \circ \partial k'\} = 1 = \{\partial k' \circ \partial k\}$. Since ∂ is a monomorphism, $\operatorname{im} \partial$ a ring, and $\partial\{k \circ k'\} = \{\partial k \circ \partial k'\}$, then $\{k \circ k'\} = 1 = \{k' \circ k\}$. This completes (6.13).

The proof of the following corollary is similar to 6.13.

6.14. COROLLARY. *If $\partial|_{\mathcal{P}} : \mathcal{P}(\operatorname{Ext}^m(M, \pi_{m-1}) \to \mathcal{P}(\operatorname{Ext}^{m+1}(M, \pi_m))$ is surjective, then each projective k-invariant in $\operatorname{Ext}^{m+1}(M, \pi_m)$ is a unit.*

Questions. (a) If $M = Z$, is B^m always a right ideal? For example, if $A(\pi)$ is the augmentation ideal in $Z\pi$, is $H^1(\pi; A(\pi))$ a ring?

(b) If B^m is a right ideal, is $\mathcal{P}(\operatorname{Ext}^{m+1}(M; \pi_m))$ a *semigroup* under composition?

Appendix: Groups Having $H^i(\pi; Z\pi) = 0$

We will give some results that show that the hypothesis of theorem 1.1 is often satisfied.

(a) If π is a finite group, then $H^i(\pi; Z\pi) = 0$ $(i > 0)$. This follows because any projective π-module is weakly injective.

(b) If π is a (Poincare) duality group with cohomological dimension m, then $H^i(\pi; Z\pi) = 0$ $(i \neq m)$ [1].

(c) If F is a free abelian group of countable rank, then $H^i(F; ZF) = 0$ for all $i \geq 0$.

(d) [1, Proposition 3.1] *If S is a subgroup of G with finite index* (not necessarily normal), *then $H^i(S; ZS) \cong H^i(G; ZG)$ as right S-modules.* Thus if $S < G$ such that $[G : S] < \infty$, then $H^k(S; ZS) = 0 \Leftrightarrow H^k(G; ZG) = 0$.

For example, if $0 \to C \to G \to T \to 0$ is an exact sequence of groups where C is a group of cohomological dimension n and T is finite, then $H^i(G; ZG) = 0$ for $i > n$. Thus, any finitely generated abelian group A of rank n has $H^i(A; ZA) = 0$ for $i \neq n$.

(e) The following theorem is an easy consequence of the spectral sequence of a group extension: *Let $1 \to N \to \pi \to G \to 1$ be an exact sequence of groups. Let N be finite. Then $H^i(\pi; Z\pi) \cong H^i(G; ZG)$ for all $i > 0$.*

For example, if π is an extension of a finite group by a duality group of cohomological dimension n, then $H^i(\pi; Z\pi) = 0$ for $i \neq n$. Also any one relator group G [8] is such that $H^i(G; ZG) = 0$ for $i \geq 3$.

(f) We say that a group π has *property* $\mathcal{P}^n$ if $H^i(\pi; Z\pi) = 0$, $0 < i < n$. The functor $H^*(\pi, -)$ is *strongly additive* if it commutes with arbitrary direct sums. For example, if π admits a projective resolution of finite type

$$\cdots \to P_m \to P_{m-1} \to \cdots \to P_0 \to Z \to 0$$

of the trivial π-module Z (i.e., each P_i is a finitely generated projective π-module), then $H^*(\pi; -)$ is strongly additive. The following is then true: *Let $1 \to A \to \pi \to B \to 1$ be an exact sequence of groups such that $H^*(A; -)$ is strongly additive. Then A has $\mathcal{P}^i$ and B has $\mathcal{P}^j$ implies that π has $\mathcal{P}^k$, where $k = \min(i, j)$.*

(g) Let $n(G)$ denote the smallest integer $\leq \infty$ such that $H^i(G; ZG) = 0$ for all $i > n(G)$. Let $\mathcal{L}$ be the class of all groups G such that $n(G)$ is finite. It follows easily from (d) and (e) that $\mathcal{L}$ contains all polycyclic (= soluble with maximum condition on subgroups) groups. More generally, if $\mathcal{A}$ is a class of groups, we say that a group G is poly ($\mathcal{A}$) if there exists a *finite* sequence of subgroups

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n = 1$$

such that $G_{i+1} \triangleleft G_i$ and G_i/G_{i+1} is a member of $\mathcal{A}$. Let *fcd* denote the class of groups of finite cohomological dimension. By the use of (d) and (e) one may show the following:

THEOREM. *If G is poly (finitely generated abelian) or poly (finite or fcd) then G is a member of $\mathscr{L}$.*

Furthermore, it follows from [13, page 138] that $\mathscr{L}$ is closed under finite sums. It is closed under infinite sums provided that each of the summands G_i has $n(G_i) < k$, k being independent of i. $\mathscr{L}$ is closed under amalgamated sums by [2]. If $G = \bigcup_{i \in Z} G_i$ is a countable union of subgroups G_i such that $n(G_i) \leq M < \infty$ for all $i \in \infty$, then $n(G) \leq M+1$ (R. Bieri). Thus any countable torsion group G has $n(G) \leq 1$, because G is the countable union of finite subgroups. There are simple examples to show that $\mathscr{L}$ is not closed under arbitrary direct limits.

BIBLIOGRAPHY

[1] Bieri, R., and Eckmann, B., *Groups with homological duality generalizing Poincare duality*, Inv. Math., *20* (1973), 103–124.

[2] ——, *Amalgamated free products of groups and homological duality*, Comment Math. Helv., *49* (1974), 460–478.

[3] Cockcroft, W., and Swan, R., *On the homotopy types of certain two-dimensional complexes*, Proc. London, Math. Soc., *3* (11) 1961, 194–202.

[4] Dyer, M., *Homotopy classification of (π, m)-complexes* (to appear in Jour. Pure and Applied Algebra).

[5] Eilenberg, S., and MacLane, S., *Homology of spaces with operators* II, Trans. Amer. Math. Soc., *65* (1949) 49–99.

[6] ——, *Cohomology theory in abstract groups* III, Ann. Math. Soc., *50* (1949), 736–761.

[7] Jacobson, N., *Some Remarks on one sided inverses*, Proc. Amer. Math. Soc., *1* (1950), 352–355.

[8] Lyndon, R. C., *Cohomology of groups with a single defining relator*, Ann. Math. *52* (1950), 650–666.

[9] MacLane, S., *Homology*, Springer-Verlag, New York, 1963.

[10] Milnor, J., *An introduction to algebraic K-theory*, Annals of Mathematics Studies No. 72, Princeton University Press, Princeton 1971.

[11] Swan, R., *Periodic resolutions for finite groups*, Ann. Math., *72* (1960), 267–291.

[12] Wall, C. T. C., *Finiteness conditions for CW complexes* I, *Ann. Math.*, (2) *81* (1965), 59–69.

[13] Gruenberg, K. W., *Cohomological topics in group theory*, Vol. 143, Lecture notes in mathematics, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

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