Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 51 (1976)

Artikel: Projective k-invariants

Autor: Dyer, Michael N.

DOI: https://doi.org/10.5169/seals-39442

Comment. Math. Helvetici *51* (1976) 259–277 Birkhäuser Verlag, Basel

Projective k-invariants

MICHEAL N. DYER

1. Introduction

Let π be a group. A (π, m)-*complex* X is a finite connected m-dimensional CW complex having fundamental group π and trivial homotopy modules $\pi_i(X) = 0$ in dimensions $i = 2, \ldots, m-1$. A π-module π_m is said to be *topologically realizable* if $\pi_m \approx \pi_m(X)$ for some (π, m)-complex X. The classification problem for (π, m)-complexes is the problem of describing the set HT (π, m) of homotopy types of (π, m)-complexes.

For π a finite group of order n, $H^{m+1}(\pi; \pi_m) \cong Z_n$ as a ring. An important aspect in this classification is the boundary operator $\partial : Z_n^* = \text{Units}\ (H^{m+1}(\pi; \pi_m)) \to \tilde{K}_0 Z\pi$, the (reduced) projective class group of the integral group ring $Z\pi$, associated with the Milnor Mayer-Vietoris sequence in algebraic K-theory [10].

This arises as follows. The cellular chain complex $C_*(\tilde{X})$ of the universal cover $\tilde{X}$ is a truncated resolution of the trivial π-module Z:

$$0 \longrightarrow \pi_m \longrightarrow C_m(\tilde{X}) \xrightarrow{\partial_m} \cdots \xrightarrow{\partial_1} C_0(\tilde{X}) \xrightarrow{\epsilon} Z \longrightarrow 0.$$

The algebraic m-type $T(X)$ of X is the triple $(\pi, \pi_m(X), k(X))$ where $k(X) \in H^{m+1}(\pi, \pi_m)$ is the k-invariant which arises by comparing the truncated resolution above with a standard resolution (see section 6; also [5], [6]). One can show that $k(X) \in \text{Units}\ (H^{m+1}(\pi; \pi_m))$; furthermore any $k \in Z_n^*$ can be the k-invariant of a finitely generated truncated projective resolution

$$(*) \quad \mathcal{P}_k : 0 \to \pi_m \to P_m \to P_{m-1} \to \cdots \to P_0 \to Z \to 0.$$

Also the assignment $(\pi, \pi_m, k) \to$ Euler characteristic $\chi(\mathcal{P}_k) = \sum_{i=0}^{m} (-1)^i [P_i]$ ($[P]$ is the class of the projective P in $\tilde{K}_0 Z\pi$) is the negative of the Milnor boundary ∂. Then (π, π_m, k) ($k \in Z_n^*$, $m \geq 3$) is the m-type of a (π, m)-complex iff $k \in \ker \partial$ [4].

The purpose of this paper is to generalize the above to groups other than finite groups.

1.1. THEOREM. *Let π be a group and m be an integer $m \geq 0$ such that $H^{m+1}(\pi; Z\pi) = 0$. Let π_m be any finitely generated topologically realizable π-module. Then*

(a) *$H^{m+1}(\pi; \pi_m)$ has the structure of a ring with identity such that the units $U(H^{m+1}(\pi, \pi_m))$ are the projective k-invariants,* i.e., *those* k*-invariants realizable by a resolution of the form* (*).

(b) *The function $\chi_m : U(H^{m+1}(\pi; \pi_m)) \to \tilde{K}_0 Z\pi$ which assigns to each $k \in U$ the Euler characteristic of a truncated resolution $\mathcal{P}_k$ realizing the m-type (π, π_m, k) is a homomorphism.*

We say that an m-type (π, π_m, k) comes from a (π, m)-complex if there exists a (π, m)-complex X such that $T(X) \cong (\pi, \pi_m, k)$ in the appropriate sense (see [4], [6] for a definition).

1.2. COROLLARY. *If $m \geq 3$ and $H^{m+1}(\pi; Z\pi) = 0$, then* ker *χ_m is the set of k-invariants which come from (π, m)-complexes.*

The corollary follows from a theorem of J. Milnor [11, theorem 3.1] concerning the realizability of a resolution by a (π, m)-complex.

DEFINITION. The subgroup im $\chi_m \subset \tilde{K}_0 Z\pi$ is called the *Swan subgroup of $\tilde{K}_0 Z\pi$ in dimension m.*

If π is a finite group of order n, let $N = \sum_{x \in \pi} x \in Z\pi$ be the norm element. The left ideal (p, N) of $Z\pi$ is projective provided p is prime to n. For π finite, im $\chi_m = \text{im}\, \partial = \{[(p, N)] \in \tilde{K}_0 Z\pi \mid 1 \leq p < n,\ (p, n) = 1\}$. If π is a (Poincaré) duality group of cohomological dimension m, then im $\chi_{m-i} = 0$ $(2 \leq i \leq m)$.

The Swan subgroup im χ_m is important because the *Wall obstruction* of any CW complex having fundamental group π and realizable π_m, which is dominated by a (π', m)-complex lies in im χ_m [12].

The organization of the paper is as follows. Let R be a ring. Section 2 gives certain constructions associated with the exact sequence of R-modules $0 \to K \to P \to C \to 0$. We say that P is K-projective if ∂: End $(K) \to$ Ext (C, K) is *surjective.* Section 3 gives conditions under which Ext (C, K) inherits a ring structure from End (K), provided P is K-projective. Section 4 shows that elements in End (K) which determine K-projective extensions are right units in Ext (C, K). Section 5 studies conditions under which each K-projective element in End (K) is a unit in Ext (C, K). Theorem 1 is proved in section 6. In an appendix we study conditions under which $H^i(\pi; Z\pi) = 0$.

The author would like to acknowlege the helpful comments of A. J. Sieradski and C. W. Curtis. Also, thanks are due to Robert Bieri for suggestions and corrections in the appendix.

Finally, the author wishes to extend his gratitude to Albrecht Dold and the Mathematical Institute of the University of Heidelberg for their support during part of this research.

2. Extensions as Pushouts and Pull-backs.

Let R be a ring. All modules are left R-modules. Let C be a given R-module and $\xi: 0 \longrightarrow K \xrightarrow{i} P \xrightarrow{j} C \longrightarrow 0$ be an exact sequence of R-modules.

It is shown in [9, page 66] that given any module homomorphism $k: K \to K'$ there exists a module kP and a homomorphism $k\beta: P \to kP$ such that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] \arrow[d, "k"'] & P \arrow[r, "j"] \arrow[d, "k\beta"'] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K' \arrow[r, "i_k"'] & kP \arrow[r, "j_k"'] & C \arrow[r] & 0 \end{tikzcd} \tag{2.1}$$

Here the bottom row is exact also. kP is defined as the pushout of i and k.

Furthermore, given any module homomorphism $s: C \to C$, there exists a module Ps and a homomorphism $\beta s: Ps \to P$ such that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i^s"] \arrow[d, equal] & Ps \arrow[r, "j^s"] \arrow[d, "\beta s"] & C \arrow[r] \arrow[d, "s"] & 0 \\ 0 \arrow[r] & K \arrow[r, "i"] & P \arrow[r, "j"] & C \arrow[r] & 0 \end{tikzcd} \tag{2.2}$$

Ps is defined to be the pullback of j and s.

3. $\mathrm{Ext}_R(C, K)$ as a Ring.

Let R be a ring and

$$\xi : 0 \longrightarrow K \xrightarrow{i} P \xrightarrow{j} C \longrightarrow 0$$

be an exact sequence of (left) R-modules.

DEFINITION We say that P is *K-projective* if

$i^* : \mathrm{Ext}^1_R(P, K) \to \mathrm{Ext}^1_R(K, K)$

is a monomorphism.

Of course, it follows from the long exact sequence for $\mathrm{Ext}^i_R(-, K)$ [9, page 74] associated with ξ that P is K-projective iff the boundary operator $\partial : \mathrm{End}_R(K) \to \mathrm{Ext}^1_R(C, K)$ is surjective. Here $\partial(k)$ equals the equivalence class of the extension kP for any $k \in \mathrm{End}(K)$. If $\mathrm{Ext}_R(P, K) = 0$, then P is K-projective; in particular, any projective R-module is K-projective.

3.1. THEOREM. *If* $0 \longrightarrow K \xrightarrow{i} P \xrightarrow{j} C \longrightarrow 0$ *is an exact sequence of R-modules with P K-projective, then the boundary operator ∂ induces an isomorphism*

$$\bar{\partial} : \frac{\mathrm{End}_R(K)}{i^*(\mathrm{Hom}_R(P, K))} \to \mathrm{Ext}^1_R(C, K).$$

For each $k \in \mathrm{End}(K)$, let $\{k\}$ denote the element $\partial(k)$ in $\mathrm{Ext}^1_R(C, K)$.

End(K) has a ring structure under composition. The question is: when is $B = i^* \mathrm{Hom}(P, K)$ a two-sided ideal? If we denote the composition $K \xrightarrow{\alpha} K \xrightarrow{\beta} K$ by $\beta\alpha$, then

$$B = \{\alpha : K \to K \mid \alpha \text{ extends to a map } \alpha' : P \to K\}$$

is always a left ideal. For, if $\alpha \in B$, $\beta \in \mathrm{End}(K)$ and $\alpha' \in \mathrm{Hom}(P, K)$ extends α, then $\beta\alpha'$ extends $\beta\alpha$. Thus B is a right ideal and $B \neq \mathrm{End}(K)$ implies that $\mathrm{Ext}(C, K)$ is a ring with identity.

We will now delineate a sequence of sufficient conditions that imply that B is a right ideal.

3.2. (*C*). *The composition in* End(K) *is commutative modulo B.*

3.3. (*RE*). *Each homomorphism in* $\mathrm{End}(K)$ *extends to a homomorphism in* $\mathrm{End}(P)$.

3.4. (*E*). *Each homomorphism in* $\mathrm{Hom}(K, P)$ *extends to a homomorphism in* $\mathrm{End}(P)$.

Note that the following implications hold:

$(E) \Rightarrow (RE) \Rightarrow B$ is a right ideal $\Leftarrow (C)$.

3.5. If $\mathrm{Ext}(C, P) = 0$, *then* (E) *is true*. This follows because $\mathrm{Ext}(C, P) = 0$ implies $i^*: \mathrm{End}(P) \to \mathrm{Hom}(K, P)$ is surjective. If $\mathrm{Ext}(P, P) = 0$, then (E) *is equivalent to* $\mathrm{Ext}(C, P) = 0$. In particular, this is true if P is projective.

3.6. Also, one can easily see that (RE) iff *the boundary homomorphism* $\partial: \mathrm{End}(C) \to \mathrm{Ext}(C, K)$ *is surjective* iff $j_*: \mathrm{Ext}(C, P) \to \mathrm{Ext}(C, C)$ *is a monomorphism.*

Note that $\mathrm{Ext}(C, K)$ is cyclic automatically implies (C).

We may call P *C-injective* if $j_*: \mathrm{Ext}(C, P) \to \mathrm{Ext}(C, C)$ is a monomorphism. Thus $\mathrm{Ext}(C, K)$ has a ring structure as above if P is C-injective *and* K-projective.

More generally, we may proceed as follows: *let P be K-projective.*

DEFINITION. Let $\mathrm{Ext}(C, K)_K$ denote the subset of $\mathrm{Ext}(C, K)$ such that $\{k\} \in \mathrm{Ext}(C, K)_K$ iff $Bk \subset B$.

It is clear that

(a) $\mathrm{Ext}(C, K)_K$ is a subgroup of $\mathrm{Ext}(C, K)$.
(b) $\mathrm{Ext}(C, K)_K$ is a ring with identity under composition.
(c) The image of the center of $\mathrm{End}(K)$ is contained in $\mathrm{Ext}(C, K)_K$.

$\mathrm{Ext}(C, K)_K$ is called the *maximal K-ring of* $\mathrm{Ext}(C, K)$.

Let $\partial_C: \mathrm{End}(C) \to \mathrm{Ext}(C, K)$ be the boundary operator in the exact sequence for $\mathrm{Ext}^i(C, -)$ associated with the extension $\xi: 0 \to K \to P \to C \to 0$. $\partial_C(r)$ is the equivalence class of the extension Pr (see 2.2).

3.7. PROPOSITION.

(a) $\mathrm{End}(C)$ *always induces a ring structure on the subgroup* $\mathrm{im}\, \partial_C = {}_C\mathrm{Ext}(C, K)$.

(b) ${}_C\mathrm{Ext}(C, K)$ *is a subring of* $\mathrm{Ext}(C, K)_K$

(c) *If ∂C is surjective, then* ${}_C\mathrm{Ext}(C, K) \cong \mathrm{Ext}(C, K)_K$ *as rings.*

Proof.

(a) P is K-projective implies that $\mathrm{im}\{j_*:\mathrm{Hom}(C, P)\to \mathrm{End}(C)\}$ is a two-sided ideal. This follows because each homomorphism in $\mathrm{End}(C)$ extends to a homomorphism in $\mathrm{End}(P)$. Consider $l\in \mathrm{End}(C)$ and the extension Pl. Then P is K-projective implies that there exists a $k\in \mathrm{End}(K)$ such kP and Pl are equivalent extensions. Thus there is an isomorphism $\alpha: kP\to Pl$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[dd, "k"'] & P \arrow[r] \arrow[d, "l\beta"] & C \arrow[r] \arrow[dd, equal] & 0 \\
 & & kP \arrow[dr] \arrow[dd] & & \\
0 \arrow[r] & K \arrow[ur] \arrow[dr] \arrow[dd, equal] & & C \arrow[r] \arrow[dd, "l"] & 0 \\
 & & Pl \arrow[ur] \arrow[d, "\beta l"] & & \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

(b) Any $\{k\}\in \mathrm{Ext}(C, K)$ ($k\in \mathrm{End}(K)$) which is in the image of ∂_C clearly satisfies $Bk\subset B$. Let $\partial_C(l)=\{k\}$. Then we may choose an extension as in (a) so that the following commutes

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "k"'] & P \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "l"] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Now $\alpha\in B$ iff α *extends the zero map* $0: C\to C$, i.e., the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\alpha"'] & P \arrow[r] \arrow[d, "\beta_\alpha"] & C \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

But $\alpha\in B$ and $\{k\}\in \mathrm{im}\,\partial_C$ implies that $\alpha\circ k$ extends $0\circ l=0$. Thus (b) is proved.

(c) follows easily from (a) and (b). We only note that the ring isomorphism is given by the correspondence $\partial_C(l)\mapsto\{k\}$ where $k\in \mathrm{End}(K)$ extends $l\in \mathrm{End}(C)$. This completes 3.7.

Note that ∂_C is surjective iff condition (RE).

We now give a simple example to show that B is not always a right ideal. Let $R=Z$ and let the basic extension be given by

$$\begin{tikzcd}
0 \arrow[r] & Z\oplus Z \arrow[r, "i"] \arrow[d, equal] & Z\oplus Z \arrow[r, "j"] \arrow[d, equal] & Z_3\oplus Z_2 \arrow[r] \arrow[d, equal] & 0 \\
 & K & P & C &
\end{tikzcd}$$

where i has matrix $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ with respect to the natural bases. Then $B \subset \operatorname{End}(Z \oplus Z)$ is the set of all 2×2 matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ over Z with the first column divisible by 3, the second by 2. $\operatorname{Ext}(C, K) \cong Z_3^2 \oplus Z_2^2$. Representatives of the cosets modulo B are given by

$$\mathcal{R} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| \begin{matrix} 0 \leq a_{i1} \leq 2 \\ 0 \leq a_{i2} \leq 1 \end{matrix}, \quad i = 1, 2 \right\}$$

It is easy to check that only the diagonal matrices in $\mathcal{R}$ have the property that $B \circ k \subset B$. Hence $\operatorname{Ext}(C, K)_K \cong Z_3 \oplus Z_2 \hookrightarrow \operatorname{Ext}(C, K)$ by embedding in the first and fourth coordinates.

4. *K*-Projective *k*-Invariants

Throughout this section we assume that $i^* : \operatorname{End}(K) \to \operatorname{Ext}(C, K)$ is surjective; i.e., that P is K-projective.

DEFINITION. The class $\{k\} \in \operatorname{Ext}(C, K)$ determined by $k \in \operatorname{End}(K)$ is called the *k-invariant of the extension kP*. A k-invariant $\{k\}$ is called *K-projective* if kP is a K-projective R-module. An element $k \in \operatorname{End}(K)$ is also called K-projective if $\{k\}$ is K-projective. Let $\mathcal{P}_K(\operatorname{Ext}(C, K))$ denote the set of K-projective k-invariants in $\operatorname{Ext}(C, K)$, $\mathcal{P}_K(\operatorname{End}(K))$ the set of K-projective elements $\operatorname{End}(K)$.

DEFINITION. Let E be a ring with identity. An element $\alpha \in E$ is a *right unit* if there exists $\beta \in E$ such that $\beta\alpha = 1$. The set of (right) units of E is denoted by $(R)U(E)$.

For each $\alpha \in E$, let α^* denote the abelian group homomorphism $E \to E$ given by right multiplication by α. α is a right unit iff α^* is surjective.

4.1. THEOREM. *Let* $\operatorname{Ext}(C, K)$ *inherit a ring structure from* $\operatorname{End}(K)$. $\{k\}$ *is a K-projective k-invariant iff* $\{k\}$ *is a right unit.*

Proof. Suppose that k is K-projective. Then $\partial_k : \operatorname{End}(K) \to \operatorname{Ext}(C, K)$ ($\partial_k(\alpha) = (\alpha \circ k)P$, $\alpha \in \operatorname{End}(K)$) is surjective. Thus there is a $k' \in \operatorname{End}(K)$ such that $(k' \circ k)P$ is equivalent to P as extensions. Hence $k' \circ k - 1 \in B$, and k is a right unit.

If $k' \circ k - 1 \in B$, we will show that kP is K-projective. P and $(k' \circ k)P$ are equivalent extensions, so there is a commutative diagram

$$\begin{tikzcd}
 & & P \arrow[dr, "j"] & & \\
0 \arrow[r] & K \arrow[ur, "i"] \arrow[dr] & & C \arrow[r] & 0 \\
 & & (k' \circ k)P \arrow[uu, "\cong"] \arrow[ur] & & \\
0 \arrow[r] & K \arrow[uu, "k'"] \arrow[r, "i_k"] & kP \arrow[u] \arrow[r, "j_k"] & C \arrow[uu, equal] \arrow[r] & 0
\end{tikzcd}$$

Call the resulting map $\beta : kP \to P$. Apply Ext$(-, \mathrm{K})$ to this diagram to obtain the commutative diagram:

$$\begin{tikzcd}
\operatorname{Ext}(C, K) \arrow[r, "j^*"] \arrow[d, equal] & \operatorname{Ext}(P, K) \arrow[r, "i^*"] \arrow[d, "\beta^*"] & \operatorname{Ext}(K, K) \arrow[d, "k'^*"] \\
\operatorname{Ext}(C, K) \arrow[r, "j_k^*"] & \operatorname{Ext}(kP, K) \arrow[r, "i_k{}^*"] & \operatorname{Ext}(K, K)
\end{tikzcd}$$

Thus $j_k^* = \beta^* j^* = 0$ because $j^* = 0$. Thus i_k^* is a monomorphism. This completes 4.1.

4.2. THEOREM. *If* $\{k \circ k'\} = \{k \circ k'\} = \{1\}$ *in* $\operatorname{Ext}(C, K)$, *then* $\operatorname{Ext}(kP, M) = 0$ *iff* $\operatorname{Ext}(P, M) = 0$, *where M is an R-module.*

If we were to define the "degree of projectivity" of k by the class of modules $\mathcal{M}_k$ such that $M \in \mathcal{M}_k$ iff $\operatorname{Ext}(kP, M) = 0$, then the above says that $\{k\}$ is a unit implies that $\mathcal{M}_k = \mathcal{M}_1$; i.e., kP is "just as projective" as P is.

Proof. Because $k' \circ k - 1 \in B$, the argument of (4.1) implies the existence of the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i_k"] & kP \arrow[r, "j_k"] & C \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[u, "k"] \arrow[r, "i"] & P \arrow[u, "\beta"] \arrow[r, "j"] & C \arrow[u, equal] \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[u, "k'"] \arrow[r, "i_k"] & kP \arrow[u, "\beta'"] \arrow[r, "j_k"] & C \arrow[u, equal] \arrow[r] & 0
\end{tikzcd}$$

Now $k \circ k' = 1 + \alpha' \circ i$, where $\alpha' \in \operatorname{Hom}(P, K)$. Let M be any R-module such that $\operatorname{Ext}(P, M) = 0$. Apply the functor $\operatorname{Ext}(-, M)$ to the above diagram.

$$\begin{tikzcd}
\mathrm{Ext}(C, M) \arrow[r, "j_k^*"] \arrow[dd, equal] & \mathrm{Ext}(kP, M) \arrow[r, "i_k^*"] \arrow[d, "\beta^*"] & \mathrm{Ext}(K, M) \arrow[dd, "(k \circ k')^*"] \\
& \mathrm{Ext}(P, M) \arrow[d, "\beta'^*"] & \\
\mathrm{Ext}(C, M) \arrow[r, "j_k^*"'] & \mathrm{Ext}(kP, M) \arrow[r, "i_k^*"'] & \mathrm{Ext}(K, M)
\end{tikzcd}$$

The rows are exact at $\mathrm{Ext}(kP, M)$. $(k \circ k')^* = (1 + \alpha' \circ i)^* = 1 + (\alpha' \circ i)^* = 1$, since $(\alpha' \circ i)^* = 0$. Thus $\beta'^* \circ \beta^*$ is an isomorphism. Then $\mathrm{Ext}(P, M) = 0$ implies $\mathrm{Ext}(kP, M) = 0$. A similar argument shows the converse. This completes (4.2).

Since the set of right units is a semigroup under composition, the following is clear.

4.3. COROLLARY. *Let* $\mathrm{Ext}(C, K)$ *have a ring structure as above. Then the set* $\mathcal{P}_k(\mathrm{Ext}(C, K))$ *of* K*-projective* k*-invariants is a semigroup with identity under composition.* $\mathcal{P}_k$ *is a group iff each* K*-projective* k*-invariant is a unit.*

5. k-Invariants as Units.

In this section we will study conditions under which right units are units in the ring $\mathrm{Ext}(C, K)$. We continue our assumption that P is K-projective. We also assume in this section that *B is a right ideal.*

DEFINITION. For each $k \in \mathrm{End}(K)$, let $B_k = \mathrm{im}\{\mathrm{Hom}(kP, K) \to \mathrm{End}(K)\} = \ker\{\partial_k : \mathrm{End}(K) \to \mathrm{Ext}(C, K)\}$, where $\partial_k(\alpha) = (\alpha \circ k)P$ $(\alpha \in \mathrm{End}(K))$.

5.1. LEMMA. $B = \mathrm{im}\{\mathrm{Hom}(P, K) \to \mathrm{End}(K)\}$ *is a right ideal iff* $B \subset B_k$ *for all* $k \in \mathrm{End}(K)$.

Proof. Let $\alpha \in B$. For any $k \in \mathrm{End}(K)$, $\alpha \circ k \in B$ since B is a right ideal. Thus $(\alpha \circ k)P \cong \alpha(kP)$ is trivial implies that $\alpha \in B_k$. Conversely, if $B \subset B_k$ for all $k \in \mathrm{End}(K)$, then let $\alpha \in B$, and consider $\alpha \circ k$ $(k \in \mathrm{End}(K))$. $\alpha \in B_k$ implies $\alpha(kP) \cong (\alpha \circ k)P \cong K \times C$ which in turn implies that $\alpha \circ k \in B$.

We say that $\{k\} \in \mathrm{Ext}(C, K)$ is a *right zero divisor* if there exists a $\{k'\} \neq 0$ such that $\{k' \circ k\} = 0$.

5.2. PROPOSITION. *$\{k\} \in \mathrm{Ext}(C, K)$ is not a right zero divisor iff $B = B_k$. If k is K-projective, then $\{k\}$ is a unit iff $B = B_k$.*

Proof. For each $k \in \mathrm{End}(K)$, let $k^*: \mathrm{Ext}(C, K) \to \mathrm{Ext}(C, K)$ be the function defined by right multiplication by $\{k\}$. It is a homomorphism of the underlying abelian group structure. Thus $\{k\}$ *is not a right zero divisor* iff k^* *is a monomorphism.* But k^* is a monomorphism iff $B = B_k$ follows from the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, hook] & \mathrm{End}(K) \arrow[r, "\partial"] \arrow[d, equal] & \mathrm{Ext}(C, K) \arrow[r] \arrow[d, "k^*"] & 0 \\ 0 \arrow[r] & B_k \arrow[r] & \mathrm{End}(K) \arrow[r, "\partial_k"] & \mathrm{Ext}(C, K) \arrow[r] & \mathrm{Ext}(kP, K) \arrow[r] & \cdots \end{tikzcd}$$

Here $\partial(\alpha) = \alpha P$, $\partial_k(\alpha) = \alpha(kP) = (\alpha \circ k)P$ and the horizontal sequences are exact. Furthermore, k^* is an isomorphism implies that ∂_k is surjective and hence $B = B_k$. $B = B_k$ together with ∂_k surjective implies k^* is an isomorphism.

5.3. LEMMA. *Let $k \in \mathrm{End}(K)$ and suppose there exists $k' \in \mathrm{End}(K)$ such that $k' \circ k - 1 \in B$. Then $B = B_{k'}$.*

Proof. Consider the homomorphisms k^*, k'^* as in the proof of (5.2). The composite $k^* \circ k'^* = (k' \circ k)^* = 1$. Thus k'^* is a monomorphism and, by (5.2), $B = B_{k'}$.

We will now give several conditions under which K-projective k-invariants are units. Clearly, if $\mathrm{Ext}(C, K)$ is *commutative* or has *no zero divisors*, then every right unit is a unit. Furthermore a theorem of N. Jacobson [7] shows that any ring having right units which are not units must be very large. The following is just a restatement of theorem 1 of [7].

5.4. THEOREM. *If $E = \mathrm{Ext}(C, K)$ has either the ascending or descending chain condition for principal right ideals generated by idempotent elements, then right units are units.*

Thus it follows that if E is finitely generated as a left (or right) E module, then right units are units in E. For example, if R is *commutative* and K is a *finitely generated* R-module, then $\mathrm{Ext}(C, K)$ is a finitely generated R-module and hence, by (5.4), right units are units.

Now let P be a *projective* R-module and consider any exact sequence

$$0 \longrightarrow K_1 \xrightarrow{\iota_1} P_1 \xrightarrow{j_1} K \longrightarrow 0$$

of R-modules where P_1 is projective. The boundary operator

$$\partial : \mathrm{Ext}^1(C, K) \to \mathrm{Ext}^2(C, K_1) = \mathrm{Ext}^1(K, K_1)$$

is given by $\partial(\{k\}) =$ class of the extension $P_1 k$ (see 2.2).

5.5. THEOREM. *If* $\partial : \mathrm{Ext}^1(C, K) \to \mathrm{Ext}^2(C, K_1)$ *is a monomorphism, then projective k-invariants are units in* $\mathrm{Ext}(C, K)$.

5.6. COROLLARY. *If* $\mathrm{Ext}(C, R) = 0$ *and K is finitely generated as an R-module, then projective k-invariants are units in* $\mathrm{Ext}(C, K)$.

The proof of (5.5) is postponed to (6.13). The corollary follows because K is finitely generated implies P_1 may be chosen to be finitely generated. $\mathrm{Ext}(C, R) = 0$ then yields $\mathrm{Ext}(C, P_1) = 0$ and this implies that ∂ is a monomorphism.

6. The k-Invariant of a Truncated Resolution.

Let M be an R-module. Choose a *projective* resolution

$$\mathcal{F}(M): \cdots \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} C_{m-2} \longrightarrow \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} M \longrightarrow 0$$

of M, where each C_i is projective R-module. $\mathcal{F}(M)$ is called the *base resolution;* each $\pi_m = \ker \partial_m (m \geq 0)$ is called an *M-realizable* R-module. If $M = Z$, the trivial R-module, then π_m is *realizable* means it is *Z-realizable.* We say that a resolution $\mathcal{F}$ is of *finite type* if each C_i is a finitely generated R-module.

Let

$$\mathcal{G}(M): \cdots \longrightarrow G_m \xrightarrow{g_m} G_{m-1} \xrightarrow{g_{m-1}} \cdots \xrightarrow{g_1} G_0 \xrightarrow{g_0} M \longrightarrow 0$$

be a (not necessarily projective) resolution of M. Let π'_m denote $\ker g_m$. The *k-invariant of $\mathcal{G}$ in dimension m* relative to $\mathcal{F}$ is the element $\{k\} \in \mathrm{Ext}_R^{m+1}(M, \pi'_m)$ determined by a chain map $f : \mathcal{F}(M) \to \mathcal{G}(M)$ covering the identity on M. Thus f is a sequence of maps making the following diagram commute:

$$\begin{tikzcd}
 & C_{m+1} \arrow[r, "\partial_{m+1}"] \arrow[d, dashed, "k"] & C_m \arrow[r, "\partial_m"] \arrow[d, "f_m"] & C_{m-1} \arrow[r] \arrow[d, "f_{m-1}"] & \cdots \arrow[r] & C_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi'_m \arrow[r] & G_m \arrow[r, "g_m"] & G_{m-1} \arrow[r] & \cdots \arrow[r] & G_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

The map $k = f_m \circ \partial_{m+1} : C_{m+1} \to \pi'_m$ determines an element $\{k\} \in \mathrm{Ext}_R^{m+1}(M, \pi'_m)$. This is well-defined by a standard argument [5].

6.1. LEMMA. *For each $m \geq 0$ and each element $\bar{k} \in \mathrm{Ext}_R^{m+1}(M, \pi'_m)$ $\exists$ a resolution $\mathcal{G}_{\bar{k}}$ of M realizing $\bar{k}$. If C_i $(i = 0, 1, \ldots m)$ and π'_m are finitely generated, then $\mathcal{G}_{\bar{k}}^{(m)}$ may be chosen to be of finite type.*

Proof. Consider $k : C_{m+1} \to \pi'_m$ realizing $\bar{k}$; $k \cdot \partial_{m+2} = 0$ implies that k defines a map $k' : \pi_m \to \pi'_m$. Use the construction of section 2 to build

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_m & \longrightarrow & C_m & \longrightarrow & \pi_{m-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle k'} & & \downarrow & & \| & & \\
 & & \pi'_m & \xrightarrow{i'} & k'C_m & \xrightarrow{j'} & \pi_{m-1} & \longrightarrow & 0
\end{array}$$

Then the m-skeleton $\mathcal{G}_{\bar{k}}^{(m)}$ is given by

$$0 \longrightarrow \pi'_m \xrightarrow{i'} k'C_m \xrightarrow{\partial'_m} C_{m-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow M \longrightarrow 0$$

where ∂'_m is the composite $k'C_m \xrightarrow{j'} \pi_{m-1} \hookrightarrow C_{m-1}$. This completes 6.1.

DEFINITION. An element $k \in \mathrm{Ext}^{m+1}(M, \pi'_m)$ is called *projective* if k can be realized as the k-invariant of a truncated projective resolution:

$$\mathcal{P}_k^{(m)} : 0 \to \pi'_m \to P_m \to P_{m-1} \to \cdots \to P_0 \to M \to 0$$

when compared with the base resolution $\mathcal{F}(M)$. The set of projective k-invariants of $\mathrm{Ext}^{m+1}(M, \pi'_m)$ is denoted by $\mathcal{P}(\mathrm{Ext}^{m+1}(M, \pi'_m))$.

6.2. THEOREM. *Let M be any R-module and π_m be M-realizable for $m \geq 0$. Then*

(a) $$\mathrm{Ext}_R^{m+1}(M, \pi_m) \cong \frac{\mathrm{End}(\pi_m)}{\mathrm{im}\,\mathrm{Hom}(C_m, \pi_m)}.$$

(b) *If* $B^m = \mathrm{im}\{\mathrm{Hom}(C_m, \pi_m) \to \mathrm{End}(\pi_m)\}$ *is a right ideal, then* $\mathrm{Ext}^{m+1}(M, \pi_m)$ *has a ring structure induced from that of* $\mathrm{End}(\pi_m)$ such that the projective *k-invariants lie between the units and right units of* $\mathrm{Ext}^{m+1}(M, \pi_m)$:

$$U(\mathrm{Ext}^{m+1}(M, \pi_m)) \subset \mathcal{P}(\mathrm{Ext}^{m+1}(M, \pi_m)) \subset RU(\mathrm{Ext}^{m+1}(M, \pi_m)).$$

(c) *If B^m is a right ideal, $\mathcal{P}(\mathrm{Ext}^{m+1}(M, \pi_m)) = U(\mathrm{Ext}^{m+1}(M, \pi_m))$, and each C_i*

$(i=0, 1, \ldots, m+1)$ *a finitely generated free module, then the function*

$$\chi_m : \mathcal{P}(\operatorname{Ext}^{m+1}(M; \pi_m)) \to \tilde{K}_0 R$$

which assigns to each $k \in \mathcal{P}$ *the Euler characteristic* $\chi_m(\mathcal{P}_k^{(m)}) = \sum_{i=0}^{m} (-1)^i [P_i] \in \tilde{K}_0 R$ *of* $\mathcal{P}_k^{(m)}$ *is a homomorphism.*

Note. (1) Theorem 6.2 is theorem 1.1 in the case $R = Z\pi$ and $M = Z$. This follows because $H^{m+1}(\pi; Z\pi) = 0$ and C_m finitely generated implies that $H^{m+1}(\pi; C_m) = 0$. Thus $H^{m+1}(\pi; \pi_m)$ is a ring (3.5) and by (5.6) right units are units because π_m is finitely generated.

(2) It follows from [11, theorem 3.1] that if $m \geq 3$, any π-module π_m realizable by a truncated *free* resolution over Z is topologically realizable as well.

(3) It follows from (4.1) that the set $\mathcal{P}_{\pi_m}$ of π_m-*projective* k-invariants is equal to the set of right units of $\operatorname{Ext}^{m+1}(M; \pi_m)$. Furthermore, (4.2) implies that any unit in $\operatorname{Ext}^{m+1}(M, \pi_m)$ must be *projective*. We do not know whether in general $\mathcal{P}$ is distinct from U or RU (see 5.4).

The following lemma is useful in the subsequent work:

6.3. LEMMA OF COCKCROFT-SWAN [3, Appendix]. Let $\xi_i^{(m)}: 0 \to \pi_m \to E_m^i \to P_{m-1}^i \to \cdots \to P_0^i \to M \to 0$ $(i = 1, 2)$ be resolutions of M with each P_j^i $(j = 0, 1, \ldots, m-1)$ projective. Let $f: \xi_1^{(m)} \to \xi_2^{(m)}$ be a chain map covering the identity on M and inducing an isomorphism on π_m. Then

$$E_m^1 \oplus P_{m-1}^2 \oplus P_{m-2}^1 \oplus \cdots \cong E_m^2 \oplus P_{m-1}^1 \oplus P_{m-2}^2 \oplus \cdots$$

Note the similarity between this and Schanuel's lemma [11].

6.4. COROLLARY. *Let* $\xi_1^{(m)}$ *be projective (i.e.,* E_m^1 *is projective) and suppose* $k(\xi_1^{(m)}) = k(\xi_2^{(m)})$ *when compared to* $\mathcal{F}$. *Then*

$$E_m^1 \oplus P_{m-1}^2 \oplus P_{m-2}^1 \oplus \cdots \cong E_m^2 \oplus P_{m-1}^1 \oplus P_{m-2}^2 \oplus \cdots$$

and hence $\xi_2^{(m)}$ *is projective also.*

Proof. By standard obstruction arguments, there exists a chain map $\xi_1^{(m)} \to \xi_2^{(m)}$ inducing the identity on M *and* π_m. Then apply (6.3).

Proof of 6.2. We will only show that if $\mathcal{P} = U$, then $\chi: \mathcal{P} \to \tilde{K}_0 R$ is a homomorphism. Let k, $k' \in \operatorname{End}(\pi_m)$ represent projective k-invariants in $\operatorname{Ext}^{m+1}(M; \pi_m)$. We will show that

$$(k' \circ k) C_m \oplus C_m \oplus C_{m+1} \cong k C_m \oplus k' C_m \oplus C_{m+1}.$$

Let $\partial k' \in \operatorname{End}(\pi_{m+1})$ be any map determined by extending k':

$$\begin{tikzcd}
0 \arrow[r, dash] & \pi_{m+1} \arrow[r] \arrow[d, dashed, "\partial k'"'] & C_{m+1} \arrow[r] \arrow[d, dashed, "\beta'_{m+1}"] & \pi_m \arrow[r] \arrow[d, "k'"] & 0 \\
0 \arrow[r] & \pi_{m+1} \arrow[r] & C_{m+1} \arrow[r] & \pi_m \arrow[r] & 0
\end{tikzcd}$$

The correspondence $\{k'\} \to \{\partial k'\}$ gives the boundary homomorphism

$$\partial : \operatorname{Ext}^{m+1}(M; \pi_m) \to \operatorname{Ext}^{m+2}(M; \pi_{m+1}).$$

6.5. LEMMA. *Let $k' \in \operatorname{End}(\pi_m)$ be projective. Then $(\partial k')C_{m+1} \oplus k'C_m \cong C_m \oplus C_{m+1}$. Hence $(\partial k')C_{m+1}$ is projective and $[(\partial k')C_{m+1}] + [k'C_m] = 0$ in $\tilde{K}_0 R$.*

Proof. Consider the resolutions

$$\begin{tikzcd}
\text{(a)}\ 0 \arrow[r] & \pi_{m+1} \arrow[r] & C_{m+1} \arrow[rr] \arrow[dr] & & k'C_m \arrow[r] & \pi_{m-1} \arrow[r] & 0 \\
& & & \pi_m \arrow[ur] & & & \\
& & & \pi_m \arrow[u, "k'"] \arrow[dr] & & & \\
0 \arrow[r] & \pi_{m+1} \arrow[r] \arrow[uuu, "\partial k'"] \arrow[ddd, "\partial k'"'] & C_{m+1} \arrow[rr] \arrow[ur] \arrow[dr] \arrow[uuu, "\beta'_{m+1}"'] \arrow[ddd, "\beta_{m+1}"'] & & C_m \arrow[r] \arrow[uuu, "\beta'_m"'] \arrow[ddd, equal] & \pi_{m-1} \arrow[r] \arrow[uuu, equal] \arrow[ddd, equal] & 0 \\
& & & \pi_m \arrow[ur] \arrow[d, equal] & & & \\
& & & \pi_m \arrow[dr] & & & \\
\text{(b)}\ 0 \arrow[r] & \pi_{m+1} \arrow[r] & (\partial k')C_{m+1} \arrow[rr] \arrow[ur] & & C_m \arrow[r] & \pi_{m+1} \arrow[r] & 0
\end{tikzcd}$$

These resolutions (a) and (b) necessarily have the same k-invariant, (a) is projective; hence (b) is also projective by lemma 6.4. $(\partial k')C_{m+1} \oplus k'C_m \cong C_{m+1} \oplus C_m$ follows from (6.4).

6.6. LEMMA. *k is projective and $k' \circ k - 1 \in B^m$ implies $C_{m+1} \oplus kC_m \cong (k' \circ k)C_m \oplus (\partial k')C_{m+1}$.*

Proof. Realize the k-invariant $\{\partial(k' \circ k)\} = \{\partial k' \circ \partial k\} \in \operatorname{Ext}^{m+2}(M; \pi_{m+1})$ in three ways:

$$\begin{tikzcd}
0 \arrow[r] & \pi_{m+1} \arrow[r] & C_{m+1} \arrow[rr] \arrow[dr] & & (k'\circ k)C_m \arrow[r] & \pi_{m+1} \arrow[r] & 0 \\
& & & \pi_m \arrow[ur] & & & \\
& & & \pi_m \arrow[u, "k'\circ k"] \arrow[dr] & & & \\
0 \arrow[r] & \pi_{m+1} \arrow[r] \arrow[uuu, "\partial(k'\circ k)"] \arrow[ddd, "\partial k"'] & C_{m+1} \arrow[ur] \arrow[dr] \arrow[uuu] \arrow[ddd] & & C_m \arrow[r] \arrow[uuu] \arrow[ddd] & \pi_{m-1} \arrow[r] \arrow[uuu, equal] \arrow[ddd, equal] & 0 \\
& & & \pi_m \arrow[ur] \arrow[d, "k"] & & & \\
& & & \pi_m \arrow[dr] & & & \\
0 \arrow[r] & \pi_{m+1} \arrow[r] \arrow[ddd, "\partial k'"'] & C_{m+1} \arrow[rr] \arrow[ur] \arrow[dr] \arrow[ddd] & & kC_m \arrow[r] \arrow[ddd] & \pi_{m-1} \arrow[r] \arrow[ddd, equal] & 0 \\
& & & \pi_m \arrow[ur] \arrow[d, "k'"] & & & \\
& & & \pi_m \arrow[dr] & & & \\
0 \arrow[r] & \pi_{m+1} \arrow[r] & C_{m+1} \arrow[rr] \arrow[ur] & & k'(kC_m) \arrow[r] & \pi_{m-1} \arrow[r] & 0
\end{tikzcd}$$

It follows that

$$(k'\circ k)C_m \cong k'(kC_m)$$

via a map inducing identity on π_{m-1} and π_m because the k-invariants are the same. Thus $\{\partial(k'\circ k)\} = \{\partial k'\circ \partial k\}$. Note that $k'\circ k$ is projective because it is a unit.

Furthermore, the following also has k-invariant $\partial k'\circ\partial k$:

$$\begin{tikzcd}
0 \arrow[d] & & & 0 \arrow[d] & & & 0 \arrow[d] \\
\pi_{m+1} \arrow[rrr, "\partial k"] \arrow[d] & & & \pi_{m+1} \arrow[rrr, "\partial k'"] \arrow[d] & & & \pi_{m-1} \arrow[d] \\
C_{m+1} \arrow[rrr] \arrow[dr] \arrow[dd] & & & C_{m+1} \arrow[rrr] \arrow[dl] \arrow[dr] \arrow[dd] & & & (\partial k')C_{m+1} \arrow[dd] \\
& \pi_m \arrow[r, "k"] \arrow[dl] & \pi_m \arrow[dr] & & \pi_m \arrow[r, equal] & \pi_m \arrow[ur] \arrow[dr] & \\
C_m \arrow[rrr] \arrow[d] & & & kC_m \arrow[rrr, equal] \arrow[ur] \arrow[d] & & & kC_m \arrow[d] \\
\pi_{m-1} \arrow[rrr, equal] \arrow[d] & & & \pi_{m-1} \arrow[rrr, equal] \arrow[d] & & & \pi_{m+1} \arrow[d] \\
0 & & & 0 & & & 0
\end{tikzcd}$$

Thus, by another application of lemma 6.4, we have $C_{m+1} \oplus kC_m \cong (k' \circ k)C_m \oplus (\partial k')C_{m+1}$. (6.5) and (6.6) taken together prove (c).

CONJECTURE (see [11, lemma 6.1 (c)]).

$(k' \circ k)C_m \oplus C_m \cong kC_m \oplus k'C_m$.

Let $\partial : \mathrm{Ext}^{m+1}(M, \pi_m) \to \mathrm{Ext}^{m+2}(M, \pi_{m+1})$ be the boundary operator in the coefficient exact sequence associated with the functor $\mathrm{Ext}^i(M, -)$ and the exact sequence

$$0 \to \pi_{m+1} \to C_{m+1} \to \pi_m \to 0.$$

The previous proof shows that ∂ is a *ring homomorphism*, provided the domain and range are rings.

Furthermore, we see that because C_i is finitely generated and free for $i = 0, \ldots, m+1$, then $\mathrm{im}\ \chi_m \subset \mathrm{im}\ \chi_{m+1}$. This follows from the commutative diagram:

$$\begin{tikzcd}
\mathcal{P}(\mathrm{Ext}^{m+2}(M, \pi_{m+1})) \arrow[dr, "\chi_{m+1}"] & \\
& \tilde{K}_0 R \\
\mathcal{P}(\mathrm{Ext}^{m+1}(M, \pi_m)) \arrow[uu, "\partial"] \arrow[ur, "\chi_m"'] &
\end{tikzcd}$$

The conditions of section 3 have obvious analogs in this setting:

6.7. $(C(m))$. *The composition in* $\mathrm{End}(\pi_m)$ *is commutative modulo* B^m.

6.8. (RE_m). *Each map* $k \in \mathrm{End}(\pi_m)$ *extends to a map in*

$$\mathrm{End}(C_m) \Leftrightarrow \partial : \mathrm{Ext}^m(M, \pi_{m-1}) \to \mathrm{Ext}^{m+1}(M, \pi_m) \text{ is surjective}$$
$$\Leftrightarrow \mathrm{Ext}^{m+1}(M; C_m) \to \mathrm{Ext}^{m+1}(M; \pi_{m-1}) \text{ is monic.}$$

6.9. (E_m). *Each map* $f \in \mathrm{Hom}(\pi_m, C_m)$ *extends to a map in*

$$\mathrm{End}(C_m) \Leftrightarrow \mathrm{Ext}^1(\pi_{m-1}, C_m) = \mathrm{Ext}^{m+1}(M; C_m) = 0$$

Again: $(E_m) \Rightarrow (RE_m) \Rightarrow B^m$ is a right ideal $\Leftarrow (C(m))$

At the present writing, I know of no examples where $C(m)$ is not satisfied.

We can "dualize" RE_m as follows:

6.10. (RE^m). *Any map* $k \in \mathrm{End}(\pi_m)$ *which coextends to* C_{m+1} *extends to* C_m. Thus, in the following diagram,

$$\begin{tikzcd}
& \pi_m \arrow[r, hook, "i"] \arrow[dl, dashed, "\exists\alpha"'] \arrow[d, "k"] & C_m \arrow[dl, dashed, "\exists\beta"] \\
C_{m+1} \arrow[r, "j"] & \pi_m &
\end{tikzcd}$$

the existence of α such that $j \circ \alpha = k$ implies the existence of a β such that $\beta \circ i = k$. The converse is always true because C_m is projective.

6.11. PROPOSITION. *Any map* $k \in \text{End}(\pi_m)$ *which coextends to* C_{m+1} *extends to* C_m *iff* $\partial : \text{Ext}^{m+1}(M, \pi_m) \to \text{Ext}^{m+2}(M, \pi_{m+1})$ *is a monomorphism iff* $i_*: \text{Ext}^{m+1}(M, \pi_{m+1}) \to \text{Ext}^{m+1}(M, C_{m+1})$ *is surjective.*

6.12. PROPOSITION. *If each* $k \in \text{End}(\pi_m)$ *which coextends to* C_{m+1} *also extends to* C_m, *then* $\text{Ext}^{m+1}(M; \pi_m)$ *is a ring.*

Proof. Let k, $\bar{k} \in \text{End}(\pi_m)$, let k extend to C_m. We must show that $k \circ \bar{k}$ extends to C_m. But k extends to C_m implies that k coextends to C_{m+1} by (6.10). Thus $k \circ k'$ coextends to C_{m+1}. But condition RE^m implies that $k \circ k'$ extends to C_m. This proves (6.12).

Note that $(RE_m) \Leftarrow (E_m) \Rightarrow (RE^m)$.

Notice that it follows from (6.6) that if $\{k\} \in \text{Ext}^m(M, \pi_{m-1})$ is projective and $\{k' \circ k\} = 1$, then $\{\partial k'\} \in \text{Ext}^{m+1}(M; \pi_m)$ is projective. Also, (6.5) implies that $\partial\{k\}$ is projective if $\{k\}$ is.

6.13. COROLLARY. *If* $\partial : \text{Ext}^{m+1}(M; \pi_m) \to \text{Ext}^{m+2}(M; \pi_{m+1})$ *is a monomorphism (condition* RE^m*), then each projective k-invariant is a unit.*

Proof. Let $\{k\} \in \text{Ext}^{m+1}(M; \pi_m)$ be projective. By (5.3), there is a $k' \in \text{End}(\pi_m)$ such that $k' \circ k' - 1 \in B^m$. Thus $\partial k' \circ \partial k - 1 \in B^{m+1}$. By (6.6), $\{\partial k'\}$ is projective. By (5.3) again, $\{\partial k \circ \partial k'\} = 1 = \{\partial k' \circ \partial k\}$. Since ∂ is a monomorphism, $\text{im}\,\partial$ a ring, and $\partial\{k \circ k'\} = \{\partial k \circ \partial k'\}$, then $\{k \circ k'\} = 1 = \{k' \circ k\}$. This completes (6.13).

The proof of the following corollary is similar to 6.13.

6.14. COROLLARY. *If* $\partial|_{\mathcal{P}} : \mathcal{P}(\text{Ext}^m(M, \pi_{m-1}) \to \mathcal{P}(\text{Ext}^{m+1}(M, \pi_m))$ *is surjective, then each projective k-invariant in* $\text{Ext}^{m+1}(M, \pi_m)$ *is a unit.*

Questions. (a) If $M = Z$, is B^m always a right ideal? For example, if $A(\pi)$ is the augmentation ideal in $Z\pi$, is $H^1(\pi; A(\pi))$ a ring?

(b) If B^m is a right ideal, is $\mathcal{P}(\text{Ext}^{m+1}(M; \pi_m))$ a *semigroup* under composition?

Appendix: Groups Having $H^i(\pi; Z\pi) = 0$

We will give some results that show that the hypothesis of theorem 1.1 is often satisfied.

(a) If π is a finite group, then $H^i(\pi; Z\pi) = 0$ $(i > 0)$. This follows because any projective π-module is weakly injective.

(b) If π is a (Poincare) duality group with cohomological dimension m, then $H^i(\pi; Z\pi) = 0$ $(i \neq m)$ [1].

(c) If F is a free abelian group of countable rank, then $H^i(F; ZF) = 0$ for all $i \geq 0$.

(d) [1, Proposition 3.1] *If S is a subgroup of G with finite index* (not necessarily normal), *then $H^i(S; ZS) \cong H^i(G; ZG)$ as right S-modules.* Thus if $S < G$ such that $[G:S] < \infty$, then $H^k(S; ZS) = 0 \Leftrightarrow H^k(G; ZG) = 0$.

For example, if $0 \to C \to G \to T \to 0$ is an exact sequence of groups where C is a group of cohomological dimension n and T is finite, then $H^i(G; ZG) = 0$ for $i > n$. Thus, any finitely generated abelian group A of rank n has $H^i(A; ZA) = 0$ for $i \neq n$.

(e) The following theorem is an easy consequence of the spectral sequence of a group extension: *Let $1 \to N \to \pi \to G \to 1$ be an exact sequence of groups. Let N be finite. Then $H^i(\pi; Z\pi) \cong H^i(G; ZG)$ for all $i > 0$.*

For example, if π is an extension of a finite group by a duality group of cohomological dimension n, then $H^i(\pi; Z\pi) = 0$ for $i \neq n$. Also any one relator group G [8] is such that $H^i(G; ZG) = 0$ for $i \geq 3$.

(f) We say that a group π has *property* $\mathcal{P}^n$ if $H^i(\pi; Z\pi) = 0$, $0 < i < n$. The functor $H^*(\pi, -)$ is *strongly additive* if it commutes with arbitrary direct sums. For example, if π admits a projective resolution of finite type

$$\cdots \to P_m \to P_{m-1} \to \cdots \to P_0 \to Z \to 0$$

of the trivial π-module Z (i.e., each P_i is a finitely generated projective π-module), then $H^*(\pi; -)$ is strongly additive. The following is then true: *Let $1 \to A \to \pi \to B \to 1$ be an exact sequence of groups such that $H^*(A; -)$ is strongly additive. Then A has $\mathcal{P}^i$ and B has $\mathcal{P}^j$ implies that π has $\mathcal{P}^k$, where $k = \min(i, j)$.*

(g) Let $n(G)$ denote the smallest integer $\leq \infty$ such that $H^i(G; ZG) = 0$ for all $i > n(G)$. Let $\mathcal{L}$ be the class of all groups G such that $n(G)$ is finite. It follows easily from (d) and (e) that $\mathcal{L}$ contains all polycyclic (= soluble with maximum condition on subgroups) groups. More generally, if $\mathcal{A}$ is a class of groups, we say that a group G is poly $(\mathcal{A})$ if there exists a *finite* sequence of subgroups

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n = 1$$

such that $G_{i+1} \lhd G_i$ and G_i/G_{i+1} is a member of $\mathcal{A}$. Let *fcd* denote the class of groups of finite cohomological dimension. By the use of (d) and (e) one may show the following:

THEOREM. *If G is poly (finitely generated abelian) or poly (finite or fcd) then G is a member of $\mathscr{L}$.*

Furthermore, it follows from [13, page 138] that $\mathscr{L}$ is closed under finite sums. It is closed under infinite sums provided that each of the summands G_i has $n(G_i)<k$, k being independent of i. $\mathscr{L}$ is closed under amalgamated sums by [2]. If $G=\bigcup_{i\in Z} G_i$ is a countable union of subgroups G_i such that $n(G_i)\leq M<\infty$ for all $i\in\infty$, then $n(G)\leq M+1$ (R. Bieri). Thus any countable torsion group G has $n(G)\leq 1$, because G is the countable union of finite subgroups. There are simple examples to show that $\mathscr{L}$ is not closed under arbitrary direct limits.

BIBLIOGRAPHY

[1] BIERI, R., and ECKMANN, B., *Groups with homological duality generalizing Poincare duality*, Inv. Math., *20* (1973), 103–124.
[2] ——, *Amalgamated free products of groups and homological duality*, Comment Math. Helv., *49* (1974), 460–478.
[3] COCKCROFT, W., and SWAN, R., *On the homotopy types of certain two-dimensional complexes*, Proc. London, Math. Soc., *3* (11) 1961, 194–202.
[4] DYER, M., *Homotopy classification of* (π, m)*-complexes* (to appear in Jour. Pure and Applied Algebra).
[5] EILENBERG, S., and MACLANE, S., *Homology of spaces with operators* II, Trans. Amer. Math. Soc., *65* (1949) 49–99.
[6] ——, *Cohomology theory in abstract groups* III, Ann. Math. Soc., *50* (1949), 736–761.
[7] JACOBSON, N., *Some Remarks on one sided inverses*, Proc. Amer. Math. Soc., *1* (1950), 352–355.
[8] LYNDON, R. C., *Cohomology of groups with a single defining relator*, Ann. Math. *52* (1950), 650–666.
[9] MACLANE, S., *Homology*, Springer-Verlag, New York, 1963.
[10] MILNOR, J., *An introduction to algebraic K-theory*, Annals of Mathematics Studies No. 72, Princeton University Press, Princeton 1971.
[11] SWAN, R., *Periodic resolutions for finite groups*, Ann. Math., *72* (1960), 267–291.
[12] WALL, C. T. C., *Finiteness conditions for CW complexes* I, *Ann. Math.*, (2) *81* (1965), 59–69.
[13] GRUENBERG, K. W., *Cohomological topics in group theory*, Vol. 143, Lecture notes in mathematics, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

Mathematical Institute
University of Heidelberg, and
University of Oregon,
Eugene, Oregon

Received May 6, 1975