Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 51 (1976)

Artikel: Polynominal Growth in Holonomy Groups of Foliations

Autor: Plante, J.F. / Thurston, W.P.

DOI: https://doi.org/10.5169/seals-39459

Comment. Math. Helvetici 39(51) 567–584 Birkhäuser Verlag, Basel

Polynomial Growth in Holonomy Groups of Foliations

J. F. PLANTE and W. P. THURSTON

Introduction

A notion of growth in finitely generated groups has been introduced by Milnor [8]. One type of growth considered there is called polynomial growth and it has been conjectured that a finitely generated group has polynomial growth if, and only if, it has a nilpotent subgroup of finite index. That groups which have a nilpotent subgroup of finite index have polynomial growth has been shown by Wolf [22] (see also Bass [1]). So far, the converse has been proved for solvable groups (Milnor-Wolf [9, 22]) and for linear groups (Tits [21]). In the present note we show that the conjecture is true for finitely generated groups of differentiable germs. There are also related results about groups of homeomorphisms which may be of independent interest. Our interest in groups of germs was motivated by the study of holonomy groups of foliations and we give some applications in that direction. It turns out, for example, that holonomy groups of codimension one, transversely oriented foliations of class C^2 which have polynomial growth, must actually be abelian. These results are applied in the last section to generalize a result of Haefliger concerning analytic foliations of codimension one.

1. Polynomial Growth

This section reviews the notion of polynomial growth in discrete groups introduced by Milnor [8]. For further background on this subject the reader is also referred to [1], [9], and [22]. Since it will be necessary to allow for the possibility of groups which are not finitely generated (in passing to subgroups), our definition will be somewhat more general than that in [8].

Assume that G is a (discrete) group with metric $d: G\times G \to [0,\infty)$ which is invariant under the action of G by left translation. The growth function of the pair (G, d) is defined as follows: If $t>0$, $\gamma(t)$ is defined to be the cardinality of the set $\{g\in G \mid d(e, g)\leq t\}$, where e denotes the identity element of G. If $\gamma(t)$ is finite for every t we call $\gamma: \mathbf{R}^+\to \mathbf{R}^+$ the growth function of (G, d).

DEFINITION. (G, d) *has polynomial growth of degree* k if there is a polynomial $p(x)$ of degree k such that $\gamma(t) \leq p(t)$ for every $t > 0$. For convenience we extend the notion of polynomial to permit terms of the form ax^r, where $r \geq 0$ need not be an integer.

Note that a group having polynomial growth must be countable. It is also clear from this definition that every subgroup of a group with polynomial growth must have polynomial growth of the same (and possibly, lower) degree.

Suppose G is generated by a set S. Let $n_1 : S \to \mathbf{R}^+$ be any function such that for each $r \in \mathbf{R}^+$, $n_1^{-1}(0, r]$ is finite. Now for $g \in G$ we define a function $n(g)$ as the minimum of $\sum_{i=1}^{l} n_1(s_i)$ where $s_1, s_2, \ldots, s_l$ is a sequence of generators such that $\prod_{i=1}^{l} s_i^{\pm 1} = g$. It is clear that $n(gh) \leq n(g) + n(h)$; hence the formula

$$d(g, h) = n(g^{-1}h)$$

defines a metric on G. With such a metric, $\gamma(t)$ is finite for each t.

If $S = G$, every left-invariant metric is obtained by this construction. If S is finite, then G has polynomial growth for some metric iff it has polynomial growth for the metric defined above with $n_1 = 1$. If S is infinite, on the other hand, it is obvious we can prescribe n_1 so that $\gamma(t)$ exceeds any given function, for all $t > 1$. The interest, however, is in finding n_1 such that the growth rate of γ is low. In this regard we have

1.1. PROPOSITION. *For every* $\varepsilon > 0$, *a countable group* G *has a metric with polynomial growth of degree* $d + \varepsilon$ *if every finitely generated subgroup has a metric with polynomial growth of degree* d.

Proof. Let $S = \{s_1, s_2, \ldots\}$ be a generating set; let G_i be the subgroup generated by $s_1, s_2, \ldots, s_i$. Inductively, we will construct a function n_1 so that the growth $\gamma_i(t)$ of G_i satisfies

$$\gamma_i(t) \leq 1 + (1 - 2^i) t^{d+\varepsilon}.$$

This is clear when $i = 0$ and G_i is the trivial group. Suppose n_1 has been so chosen for $i = k$. Let n_1' be defined on $s_1, \ldots, s_{k+1}$ by

$$n_1'(s_i) = n_1(s_i) \quad \text{if} \quad i \leq k$$
$$n_1'(s_{k+1}) = 1.$$

By hypothesis, the growth function γ_{k+1}' of this group is dominated by some polynomial P of degree d. Then, there is some constant $C \geqslant 0$ such that

$$\gamma'_{k+1}(t) \leqslant P(t) \leqslant 2^{-(k+1)} \cdot (t+C)^{d+\varepsilon}.$$

Let $n_1(s_{k+1}) = C+1$, and γ_{k+1} be the associated growth function. For $t < C+1$, $\gamma_{k+1}(t) = \gamma_k(t)$. For $t \geqslant C+1$, the set of elements of G_{i+1} which have at least one s_{i+1} in their minimal representation has cardinality $\leq \gamma'_{k+1}(t-C) \leq 2^{-(k+1)} t^{d+\varepsilon}$, while those without s_{k+1} in the representation have cardinality $\leq \gamma_k(t) \leq 1+(1-2^{-k})t^{d+\varepsilon}$. Combining, we have $\gamma_{k+1}(t) \leq 1+(1-2^{-(k+1)})t^{d+\varepsilon}$.

Note that this gives a necessary and sufficient condition for G to have a metric of polynomial growth in terms of its finitely generated subgroups.

Example. Every finitely generated subgroup of the rationals, $\mathbf{Q}$ is infinite cyclic, hence has polynomial growth of degree one. Hence $\mathbf{Q}$ has metrics of polynomial growth; in particular the function $n_1(1/n) = n!$ defines a metric with quadratic growth. On the other hand, it is easy to verify that $\mathbf{Q}$ can have no metric with linear growth. (The same is true for any non-finitely generated subgroup of $\mathbf{Q}$.)

Wolf [22] has shown that a finitely generated group which has a nilpotent subgroup of finite index has polynomial growth. On the other hand, it is reasonable to conjecture that a group having polynomial growth must have a nilpotent subgroup of finite index. This conjecture has been proved for finitely generated solvable groups by Wolf-Milnor [22, 9]. It has also been proved by Tits [21] for finitely generated subgroups of $GL(n, \mathbf{R})$. In Section 3 we give some extensions of this last result.

2. Polynomial Growth Versus Polycyclic

A group G is said to be *polycyclic* if there is a finite descending chain of subgroups

$$G = G_1 \supset \cdots \supset G_k = \{e\}$$

such that each subgroup is normal in the preceding one and the corresponding quotient groups are all cyclic. We refer the reader to [22] for a discussion of other conditions on G which are equivalent to polycyclic. The following result is proved in [22].

2.1. THEOREM. *A polycyclic group G has polynomial growth if and only if it has a nilpotent subgroup of finite index.*

The purpose of this section is to show that in certain cases a group having polynomial growth must be polycyclic. We assume that our group G has a fixed left invariant metric.

2.2. THEOREM. *Let G be a group having polynomial growth and suppose that* $\mathrm{Hom}(H;\mathbf{R})\neq 0$ *for every non-trivial finitely generated subgroup H of G. Then* $\mathrm{Hom}(G;\mathbf{R})\neq 0$.

The proof of (2.2) will follow soon. It should be noted that **R** may be replaced by **Q** in the statement of (2.2).

If G is countable, it can be written as an increasing union

$$G=\bigcup_{i=1}^{\infty} G_i, \qquad G_i\subset G_{i+1}$$

of finitely generated subgroups. The vector space $\mathrm{Hom}(G;\mathbf{R})$ is just $\overleftarrow{\lim}\,\{\mathrm{Hom}(G_i;\mathbf{R})\}$ where the homomorphism

$$\mathrm{Hom}(G_i;\mathbf{R})\to\mathrm{Hom}(G_j;\mathbf{R}), \qquad i>j$$

is defined by restriction.

We need a lemma which relates polynomial growth to a chain condition.

2.3. LEMMA. *Suppose that G has polynomial growth of degree k and that*

$$H_0\subset H_1\subset\cdots\subset H_n\subset G$$

is a finite sequence of subgroups such that for each i ($1\leq i\leq n$) *there is a non-zero homomorphism* $f_i: H_i\to\mathbf{R}$ *such that* $H_{i-1}\subset\ker f_i$. *Then* $n\leq k$.

Proof. Let $a_i\in H_i-\ker f_i$. The words of the form $a_1^{p_1}\cdots a_n^{p_n}$ represent distinct elements of G. This means that the growth function of G dominates a polynomial of degree n with the coefficient of x^n being positive. Hence, we must have $n\leq k$.

Proof of (2.2). Let G be an increasing union of finitely generated groups G_i and let V_i denote the finite dimensional real vector space $\mathrm{Hom}(G_i;\mathbf{R})$. When $i\leq j$ let V_{ij} be the subspace of V_i consisting of homomorphisms which extend to G_j. If

$j<k$ then $V_{ik}\subset V_{ij}$. Define

$$W_i=\bigcap_{j\geq i} V_{ij}.$$

Note that $W_i=0$ if and only if $V_{ij}=0$ for j sufficiently large. We claim that for i sufficiently large $W_i\neq 0$. Suppose this is not the case, i.e., there exist arbitrarily large i such that $W_i=0$. $W_i=0$ means that $V_{ij}=0$ for j sufficiently large. In particular, we may choose $j>1$ such that $W_j=0$. Thus, we have a non-zero homomorphism $f: G_j \to \mathbf{R}$ such that $G_i\subset \ker f$. By repeating this process we would obtain an infinite increasing chain contradicting (2.3). This proves the claim.

For $k>i$ the restriction $V_{kj}\to V_{ij}$ is surjective for every $j\geq k$. Hence, the restriction $W_k \to W_i$ is also surjective. Since the W_i are non-zero for i sufficiently large, this shows that the inverse limit is non-zero and completes the proof of (2.2).

2.4. COROLLARY. *If G has a polynomial growth* (*with respect to some invariant metric*) *and every finitely generated subgroup of G admits a non-zero real-valued homomorphism then G is solvable. If G is also finitely generated then G is polycyclic.*

Proof. Using (2.2) construct inductively a chain of subgroups

$$G=G_0\supset G_1\supset\cdots$$

and corresponding non-zero homomorphisms $f_i\in \operatorname{Hom}(G_i;\mathbf{R})$ such that $G_{i+1}=\ker f_i$. By (2.3) the chain must reach the trivial subgroup after a finite number of steps which implies that G is solvable. If G is also finitely generated then it must be polycyclic by a result of Milnor [9].

Note. The example ($G=\mathbf{Q}$) following Proposition 1.1 shows that G need not be finitely generated—in which case it cannot be polycyclic.

3. Groups of Diffeomorphisms and their Germs

Denote by $\mathscr{G}^r(\mathbf{R}^n, 0)$ the groups of germs of C^r ($r>1$) diffeomorphisms of $\mathbf{R}^n$ which fix the origin.

3.1. THEOREM. *If $G\subset\mathscr{G}^r(\mathbf{R}^n,0)$, $r\geq 1$, is a finitely generated group having* *polynomial growth then G has a nilpotent subgroup of finite index. If we also assume $n=1$, then G is polycyclic.*

Proof. Let

$$D:\mathcal{G}^r(\mathbf{R}^n, 0)\to GL(n, \mathbf{R})$$

be the homomorphism which takes a germ to its derivative at the origin. Thus, $G_0 = G\cap \ker D$ is a normal subgroup of G and the quotient G/G_0 is isomorphic to a subgroup of $GL(n, \mathbf{R})$. By the generalized Reeb Stability Theorem [19] every finitely generated subgroup of G_0 has a non-zero real-valued homomorphism. From (2.4) it follows that G_0 is solvable. By Tits [21] the quotient group G/G_0 must have a nilpotent subgroup of finite index since it has polynomial growth and is isomorphic to a subgroup of $GL(n, \mathbf{R})$. This means that G has a solvable subgroup of finite index and this subgroup is finitely generated (cf. page 90 of [6]). Since this solvable subgroup has polynomial growth it has a nilpotent subgroup of finite index. Since this nilpotent group has finite index in G the first statement of (3.1) is proved. When $n=1$, G/G_0 is abelian and, hence, G is solvable and, therefore, polycyclic. This completes the proof of (3.1).

Remark. (3.1) may be thought of as an extension of Tits' result that a finitely generated group of linear isomorphisms of $\mathbf{R}^n$ having polynomial growth must have a nilpotent subgroup of finite index. Any linear group has a corresponding group of germs which uniquely determines the linear group.

The following is a related result about groups of diffeomorphisms with compact support.

3.2. PROPOSITION. *Let G be a group of C^r ($r\geq 1$) diffeomorphisms with compact support of an open manifold M. If G has polynomial growth then it is solvable. If G is also finitely generated then it is polycyclic and has a nilpotent subgroup of finite index.*

Proof. We show that every finitely generated subgroup of G has a non-zero real-valued homomorphism. If a subgroup H of G is generated by $g_1, \ldots, g_n$ let

$$U_i = \{x\in M \mid g_i(x)\neq x\}.$$

Let x_0 be a boundary point of the set $U_1\cup\cdots\cup U_n$. Since the derivative of g_i at x_0 is the identity ($i=1,\ldots,n$) the generalized Reeb Stability Theorem yields a non-zero element of $\operatorname{Hom}(H;\mathbf{R})$. (3.2) now follows from (2.4).

4. Groups of Homeomorphisms of One Dimensional Manifolds

Denote by Homeo ($\mathbf{R}$) and Homeo (S^1) the groups of homeomorphisms of $\mathbf{R}$ and S^1, respectively.

4.1. PROPOSITION. *If $G \subset$ Homeo ($\mathbf{R}$) is a group having polynomial growth then G is solvable. If G is also finitely generated then G is polycyclic and has a nilpotent subgroup of finite index.*

Proof. Let $G_0 \subset G$ be the subgroup (of index at most 2) consisting of orientation preserving homeomorphisms. It suffices to show that G_0 is polycyclic. Let H be a non-trivial finitely generated subgroup of G_0. Choose a point $x \in \mathbf{R}$ which is moved by some element of H and let $a, b \in \mathbf{R}$ be, respectively, the inf and sup of the set $\{h(x) \mid h \in H\}$. Since H has polynomial growth and the interval (a, b) is homeomorphic to $\mathbf{R}$, the fact that Hom $(H; \mathbf{R}) \neq 0$ follows from (5.5) of [15]. Now (2.4) implies that G_0 and, hence, G are solvable and polycyclic if finitely generated.

4.2. COROLLARY. *If $G \subset$ Homeo (S^1) is a group having polynomial growth then G is solvable. If G is also finitely generated then G is polycyclic and has a nilpotent subgroup of finite index.*

Proof. Let $G_0 \subset G$ be the subgroup consisting of orientation preserving homeomorphisms. Since G_0 has polynomial growth it preserves a Borel measure μ on S^1 such that $\mu(S^1) = 1$. In [14] it is shown that the rotation number of $g \in G_0$ is $\mu([x, g(x))$ where $[x, y)$ denotes the half open interval going from x to y in the positive direction (according to the orientation of S^1). Furthermore, it is also shown in [14] that the rotation number map

$$\rho : G_0 \rightarrow [0, 1) \bmod 1$$

is a homomorphism. The image $\rho(G_0)$ is abelian and every element of the subgroup $\ker \rho$ fixes every point x in the support of μ. By cutting the circle at such an x the action of $\ker \rho$ on S^1 may be thought of as an action on an interval. Thus, by (4.1), $\ker \rho$ and, hence, G are solvable. If G is finitely generated it is polycyclic.

It turns out that somewhat more can be said if the homeomorphisms in question are of class C^2. This will be based on a result of N. Kopell which is proved in [5] and may be stated as follows.

4.3. THEOREM. *Suppose* $f, g:[a, b)\to[a, \infty)$ *are* C^2 *diffeomorphisms (not necessarily onto but* $f(a)=a=g(a)$*) such that the following conditions are satisfied:*

i) $f(x)<x$ *for all* $x\in(a, b)$
ii) $g(x)=x$ *for some* $x\in(a, b)$
iii) $f(g(x))=g(f(x))$ *for all* $x\in[a, b)$.

Then $g(x)=x$ *for all* $x\in[a, b)$.

We will also need the following observation of Moussu [10]. We say that a group G acts *without fixed points* on (a, b) if $g(x)=x$ for some $g\in G$ and $x\in(a, b)$ implies that $g(x)=x$ for *all* $x\in(a, b)$. We say that G acts *essentially* on (a, b) if $g(x)=x$ for all $x\in(a, b)$ implies that g is the identity element of G.

4.4. LEMMA. *If a group G acts essentially and without fixed points on an open interval* (a, b) *then G is (torsion free) abelian.*

Proof. Let $x\in(a, b)$ and define a partial ordering on G as follows:

$$f<g \quad \text{if} \quad f(x)<g(x).$$

Since G acts on (a, b) essentially and without fixed points this definition is independent of $x\in(a, b)$ and gives an Archimedean total ordering of G. By a theorem of Hölder (cf. [2] page 226) G is isomorphic to a subgroup of $\mathbf{R}$ and is therefore abelian.

4.5. THEOREM. *If G is a nilpotent group of diffeomorphisms of* $[0, \infty)$ *of class* C^2 *then G is (torsion free) abelian.*

Proof. If the restriction of G to $(0, \infty)$ acts without fixed points then we are done by (4.4). Otherwise, let f be a non-trivial element of $Z(G)$ (center of G) and let $a<b\le\infty$ be such that $f(a)=a$, $f(b)=b$ (if $b<\infty$) and f has no fixed points between a and b. By taking an inverse, if necessary, assume that $f(x)<x$ for x in (a, b). If $g\in G$ we claim that either g is the identity on $[a, b)$ or $g(x)\ne x$ for every x in (a, b). Suppose $g(x)=x$ for some x in (a, b). Since f and g commute, $f^n(x)$ is fixed by g for every integer n and, hence, so is $\lim_{n\to\infty} f^n(x)=a$. It now follows from (4.3) that g is the identity on $[a, b)$. A similar argument (reversing the roles of f and g) shows that, in either of the above cases, $g(a)=a$ and $g(b)=b$ (if $b<\infty$). Let K_1 denote the closure of the set

$$\{x\in[0, \infty) \mid f(x)\ne x \quad \text{for some} \quad f\in Z(G)\}.$$

(4.4) and the claim proved above imply that G commutes on K_1. $Z(G)$ acts trivially on the complement of K_1 so we can think of $G/Z(G)$ as acting on this complement. If $G/Z(G)$ acts trivially we are done. Otherwise, since $G/Z(G)$ is nilpotent, we repeat the above argument to find a closed set $K_2 \supset K_1$ $(K_2 \neq K_1)$ such that G commutes on K_2 and $Z(G/Z(G))$ acts trivially on the complement of K_2. Since G is nilpotent this process can be repeated only a finite number of times and we eventually obtain a closed set K_q such that G commutes on K_q and acts trivially on its complement. Thus, G is abelian and the proof of (4.5) is complete.

4.6. COROLLARY. *If G is a group of C^2 diffeomorphisms of $[0, \infty)$ which has polynomial growth then G is (torsion free) abelian.*

Proof. Passing to a subgroup we may assume that G is finitely generated. Since $\mathbf{R}$ is homeomorphic to $(0, \infty)$, (4.1) implies that G has a nilpotent subgroup G_0 of finite index. By (4.5) G_0 is abelian. From the proof of (4.5) we see that there is a closed set C such that every element of G_0 fixes every point in C and that on each maximal open interval in the complement of C, G_0 acts without fixed points. Since each element of G has a power in G_0, it is clear that every element of G fixes every point of C. Let (a, b) be a maximal open interval in the complement of C and restrict the action of G to the interval (a, b). Suppose $g \in G$ and $x \in (a, b)$ are such that $g(x) = x$ and let k be an integer such that $g^k \in G_0$. Since $g^k(x) = x$ and G_0 acts without fixed points g^k and, hence, g must be the identity on (a, b). Hence, G itself acts without fixed points on (a, b). Doing this for every maximal open interval in the complement of C we conclude that G is abelian.

4.7. COROLLARY. *If G is a finitely generated group of C^2 diffeomorphisms of S^1 which has polynomial growth then G has an abelian subgroup of finite index.*

Proof. Without loss of generality, we may suppose that g contains only orientation preserving diffeomorphisms. As in the proof of (4.2) we consider the rotation number homomorphism ρ. The image of ρ must either be finite or dense. If the image of ρ is dense then some element of G has irrational rotation number and by Denjoy's theorem is conjugate to an irrational rotation. This implies that the support of the G-invariant Borel measure must be all of S^1. In this case, [14] ((4.2) or proof of (2.3)) implies that G is conjugate to a group of rotations and is abelian. On the other hand, if the image of ρ is finite then $\ker \rho$ has finite index in G and every element of $\ker \rho$ fixes some $x \in S^1$. Thus, $\ker \rho$ can be identified with a group of C^2 orientation preserving diffeomorphisms of $[0, \infty)$. Now $\ker \rho$ is abelian by (4.6) and the proof of (4.7) is complete.

Remarks. The differentiability assumption in (4.6) is crucial. A C^0 counterexample (which can be made C^1 using an unpublished construction of D. Pixton) may be constructed as follows. Let $\psi:[0,1]\to[0,1]$ be a C^∞ diffeomorphism such that $\psi(0)=0$, $\psi(1)=1$, ψ has infinite order contact with the identity at 0 and 1, and $\psi(x)>x$ for $x\in(0,1)$. Now define a C^∞ orientation preserving diffeomorphism $f:\mathbf{R}\to\mathbf{R}$ by

$$f(x)=\begin{cases}\psi(x-[x])+[x] & \text{if } [x]\equiv 0 \mod 4\\ x & \text{if } [x]\equiv 1 \text{ or } 3 \mod 4\\ \psi^{-1}(x-[x])+[x] & \text{if } [x]\equiv 2 \mod 4\end{cases}$$

Define $g:\mathbf{R}\to\mathbf{R}$ by $g(x)=x+2$. The group G generated by f and g has polynomial growth but is not abelian (it is isomorphic to the fundamental group of the Klein bottle). If we identify $\mathbf{R}$ with $(0,\infty)$ we can realize G as a group of homeomorphisms of $[0,\infty)$.

The arguments in the proof of (4.6) show that a group of orientation preserving C^2 diffeomorphisms of a closed interval which has polynomial growth must be abelian. The above example shows that this assertion is false for open intervals.

5. Groups of Germs in Dimension One

In this section we establish results analogous to those of the previous section for groups of germs. Let $\mathscr{G}_+^r$ denote the group of germs of C^r $(r\geq 0)$ diffeomorphisms of $[0,\infty)$ which fix 0. Let $\mathscr{L}_+^r$ denote the pseudogroup consisting of local C^r diffeomorphisms which are defined in a neighborhood of 0. There is a natural map $\mathscr{L}_+^r\to\mathscr{G}_+^r$ which takes a local diffeomorphism to its germ at 0. If $\Gamma\subset\mathscr{L}_+^r$ is a subpseudogroup which goes to a group $G\subset\mathscr{G}_+^r$ via the above map then Γ is called a *realization* of the group of germs G. It is clear that any group of germs has many realizations but we will be interested in having certain properties of the group G carry over to the pseudogroup Γ. If the group G has polynomial growth, for example, we would like to be able to make a similar statement about Γ. If G is finitely presented then this can be done. Suppose that G is generated by $g_1,\dots,g_n$ and that G is determined by finitely many relations $R_1,\dots,R_m$ involving the g_i's. For each g_i we choose a representative $f_i\in\mathscr{L}_+^r$. For each relation R_j we restrict the domains of the f_i's so that the relation R_j holds (for those $x\in[0,\infty)$ for which it makes sense) with g_i's replaced by f_i's. Restrict $f_1,\dots,f_n$ so that they satisfy each of the relations R_j, $j=1,\dots,m$. These restricted f_i's generate a pseudogroup Γ and if G has polynomial growth then it is clear that each orbit of Γ has polynomial growth (in the obvious sense [15]).

5.1. PROPOSITION. *If $G \subset \mathscr{G}^2_+$ is a finitely generated nilpotent group then G is free abelian.*

Proof. In this situation it is clear that G has no torsion and, hence, by results of Mal'cev [7], G is the fundamental group of a compact nilmanifold. Since such a manifold can be triangulated, G must be finitely presented. From the discussion above there is a finitely generated realization $\Gamma \subset \mathscr{L}^2_+$ of G such that each orbit of Γ has polynomial growth. By (5.1) and (5.4) of [15] there are two possibilities:

i) There is an $x > 0$ which is fixed by every element of Γ, or
ii) There exists $f \in \Gamma$ such that the restriction of f to some interval of the form $[0, \varepsilon)$ has no fixed points other than 0. (We will say in this case that the germ of f at 0 is fixed point free.)

In case i), Γ restricts to an action of G on the interval $[0, x)$ and, hence, G is abelian by (4.5). In case ii) we claim that every element of G is fixed point free. If $f_0 \in \Gamma$ corresponds to a non-trivial element in the center of G then by (4.3) the germ of f_0 at 0 is fixed point free. The same argument now shows that every element of G other than the identity is fixed point free. Finally, the argument used in the proof of (4.4) shows that G is abelian and (5.1) is proved.

5.2. THEOREM. *If a subgroup $G \subset \mathscr{G}^2_+$ has polynomial growth then G is a torsion free abelian group.*

Proof. By passing to a subgroup we may assume that G is finitely generated. (3.1) implies that G has a nilpotent subgroup G_0 of finite index which by (5.1) must be abelian. Furthermore, there is a realization of G_0 which satisfies i) or ii) above. Since every element of G has a power in G_0 we conclude that either G is realized by an action on an interval of the form $[0, x)$, $x > 0$ or G is a fixed point free group of germs. In either case G is abelian and the proof of (5.2) is complete.

The following is immediate from (5.2).

5.3. COROLLARY. *If $G \subset \mathscr{G}^r(\mathbf{R}, 0)$, $r \geq 2$, is a group of orientation preserving germs which has polynomial growth then G is a torsion free abelian group.*

We conclude this section with some observations about analytic germs which will be used in Section 7. Let $\mathscr{G}^\omega_+(\mathbf{R}, 0)$ denote the germs of orientation preserving analytic diffeomorphisms of $\mathbf{R}$ which fix zero. Such a germ is determined by its

Taylor series. Suppose f is a local diffeomorphism represented by a power series of the form

$$f(x)=x+a_kx^k+\text{higher order terms.}$$

If $a_k\neq 0$ then we say that the germ of f at zero has *order* k. Define a map $\delta_k:\mathcal{G}_+^\omega(\mathbf{R},0)\to\mathbf{R}$ by

$$\delta_k(f)=\begin{cases}\log(1+a_1) & \text{if} \quad k=1\\ a_k & \text{if} \quad k>1.\end{cases}$$

5.4. LEMMA. *If $G\subset\mathcal{G}_+^\omega(\mathbf{R},0)$ is a group of germs such that every element of G other than the identity has order $\geq k$ then the restriction of δ_k to G is a homomorphism. If every element other than the identity has order k then this homomorphism is injective.*

The proof of (5.4) is straight forward and may be obtained either by substitution of power series or by use of Leibnitz' rule. The second statement of (5.4) follows from the first.

5.5. LEMMA. *If two germs in $\mathcal{G}_+^\omega(\mathbf{R},0)$ commute and neither is the identity then both germs have the same order.*

Proof. Suppose the commuting germs are represented by

$$f(x)=x+a_kx^k+\text{higher order terms}$$
$$g(x)=x+b_jx^j+\text{higher order terms}$$

where $a_k\neq 0$, $b_j\neq 0$. Substituting to get power series for fg and gf and equating coefficients of x^{k+j-1} we conclude that $k=j$, i.e., the germs of f and g have the same order.

6. Holonomy Groups of Foliations

Let M be a smooth manifold and $\mathcal{F}$ a C^r ($r\geq 1$) foliation of M of codimension k. We begin by recalling briefly the notion of holonomy groups. If L is a leaf of $\mathcal{F}$ we choose an embedding of $\mathbf{R}^k$ in M which is transverse to $\mathcal{F}$ and such that the origin in $\mathbf{R}^k$ is sent to a point of L which we take as the basepoint. A based loop in L determines, by sliding along leaves near L, a local diffeomorphism of $\mathbf{R}^k$ at the origin. If γ denotes a loop we denote by $h(\gamma)$ the germ of the corresponding local diffeomorphism. It turns out that $h(\gamma)$ depends only on the homotopy class $[\gamma]\in\pi_1(L)$ and that the map $[\gamma]\to h(\gamma)$ determines a homomorphism (or anti-homomorphism depending on conventions) from $\pi_1(L)$ to $\mathscr{G}^r(\mathbf{R}^k, 0)$. The image of h is called the holonomy group of the leaf L and will be denoted by $H(L)$. $H(L)$ depends on the original embedding of $\mathbf{R}^k$ in M but its isomorphism class does not. The next two results are immediate from (3.1) and (5.3).

6.1. PROPOSITION. *If L is a leaf of a C^r foliation $(r\geq 1)$ of codimension k such that $H(L)$ is finitely generated and has polynomial growth then $H(L)$ has a nilpotent subgroup of finite index. If, further, $k=1$ then $H(L)$ is polycyclic.*

6.2. PROPOSITION. *If L is a leaf of a transversely oriented codimension one C^r foliation $(r\geq 2)$ and $H(L)$ has polynomial growth then $H(L)$ is a torsion free abelian group.*

Remark. The hypothesis regarding $H(L)$ in (6.1) or (6.2) holds, for example, if $\pi_1(L)$ is a finitely generated group having polynomial growth.

The following result gives a case in which the structure of holonomy groups is related to the fundamental group of the manifold M. We say that a codimension one foliation $\mathscr{F}$ of M has a *null transversal* if there is a loop in M which is everywhere transverse to $\mathscr{F}$ and which is freely homotopic to zero.

6.3. PROPOSITION. *Let $\mathscr{F}$ be a transversely oriented codimension one foliation of class C^2 of a manifold M. If $\pi_1(M)$ has polynomial growth and $\mathscr{F}$ has no null transversals then the holonomy group of every leaf is abelian.*

Proof. Let L be a leaf of $\mathscr{F}$ and assume that $\pi_1(L)$ is finitely generated. By (6.2) it is sufficient to show that $H(L)$ has polynomial growth. Let $x_0\in L$ be the basepoint, $i:L\to M$ the inclusion map, and $i_\#$ the induced homomorphism between fundamental groups. $i_\#$ induces a homomorphism

$$H(L)\cong\frac{\pi_1(L)}{\ker h}\to\frac{i_\#(\pi_1(L))}{i_\#(\ker h)}.$$

We claim that this homomorphism is injective. If not then there is a based loop γ such that $h[\gamma]\neq 0$ but $i_\#[\gamma]=0$. Since $h[\gamma]\neq 0$, γ is freely homotopic to a loop of the form $\alpha*\beta$ where α is a path in a single leaf of $\mathscr{F}$ and β is a path transverse to $\mathscr{F}$. On the other hand, by a standard argument, $\alpha*\beta$ is freely homotopic to a closed curve transverse to $\mathscr{F}$. If $i_\#[\gamma]=0$ then we would have a null transversal and contradict the hypothesis of (6.3). Thus, $H(L)$ has polynomial growth and must be abelian. If $\pi_1(L)$ is not finitely generated then we replace $\pi_1(L)$ by an arbitrary finitely generated subgroup in the above argument. The corresponding subgroups of $H(L)$ are all abelian and, hence, so is $H(L)$. This proves (6.3).

Remark. The hypothesis in (6.3) concerning null transversals arises naturally. For example, it is satisfied if $\mathcal{F}$ comes from a locally free Lie group action [12] or if $\mathcal{F}$ is real analytic [3, 4].

7. Analytic Foliations of Codimension One

Let M be a compact C^∞ manifold. It is known [20] that if M has Euler characteristic zero then M admits a C^∞ foliation of codimension one. On the other hand, Haefliger [3, 4] proved that if M has a real analytic codimension one foliation then the fundamental group of M is infinite. In particular, if $\pi_1(M)$ is finite M does not have such a foliation. Other examples of manifolds not admitting analytic codimension one foliations have been given by Novikov [11], Thurston [18], and Goodman. In each case the examples are 3-dimensional and are obtained as a by-product of a compact leaf theorem. Analytic foliations are interesting because they tend to have nice geometric properties, which are also common to foliations that arise in nature. Almost nothing is known about analytic foliations except in codimension one. In this section we give an extension of Haefliger's result about codimension one analytic foliations on n-manifolds.

7.1. THEOREM. *Let M be a compact manifold such that $\pi_1(M)$ has polynomial growth. If M admits a transversely oriented real analytic foliation of codimension one then $H^1(M, \mathbf{R}) \neq 0$.*

7.2. COROLLARY. *If M is compact, $\pi_1(M)$ has polynomial growth, and $H^1(M, \mathbf{R}) = 0$ then M does not admit a transversely oriented analytic foliation of codimension one. If $H^1(M; Z_2)$ is also zero, M does not admit any analytic codimension one foliation.*

The proof of (7.1) will require the following preliminary result.

7.3. LEMMA. *Let M be a compact manifold with boundary (possibly empty) and let $L_1, \ldots, L_p$ be disjoint compact connected submanifolds of codimension one, each of which separates a connected component of M. Let $V_1, \ldots, V_q$ be connected* *manifolds with boundary such that*

$$M-\bigcup_{i=1}^{p} L_i=\bigcup_{i=1}^{q} \text{interior } V_i.$$

Let $\Phi_i \in H^1(V_i;\mathbf{R})$ *and if* $L_k \subset V_i$ *denote by* Φ_{ik} *the restriction of* Φ_i *to* L_k. *Assume that* $\Phi_i \neq 0$ $(i=1,\ldots,q)$ *and that* Φ_{ik} *and* Φ_{jk} *are linearly dependent if* $L_k = V_i \cap V_j$. *Then there is a non-zero class* $\Phi \in H^1(M;\mathbf{R})$ *whose restriction to each* V_i *is a multiple* (*not always zero*) *of* Φ_i.

Proof. If $q=1$ then $p=0$ and the assertion is obvious. Assume that (7.3) is known for $q-1$. We suppose that $V_1,\ldots,V_q$ are ordered so that $V_1\cap V_q \neq \emptyset$. From the induction hypothesis we have a non-zero class $\Phi_0 \in H^1(\bigcup_{i=1}^{q-1} V_i;\mathbf{R})$. Since the restriction of Φ_0 to V_1 is a multiple of Φ_1 there exist a and b, not both zero, such that $(a\Phi_0, b\Phi_q)$ has image zero in the Mayer-Vietoris sequence

$$H^1(M;\mathbf{R}) \to H^1\left(\bigcup_{i=1}^{q-1} V_i;\mathbf{R}\right)\oplus H^1(V_q;\mathbf{R}) \to H^1(V_1\cap V_q;\mathbf{R}).$$

Choosing $\Phi \in H^1(M;\mathbf{R})$ to have image $(a\Phi_0, b\Phi_q)$ completes the proof of (7.3).

Proof of (7.1). Suppose that M has an analytic transversely oriented foliation $\mathcal{F}$ of codimension one. By [3, 4], $\mathcal{F}$ has no null transversals. Since $\pi_1(M)$ has polynomial growth, standard arguments [15] imply that every leaf of $\mathcal{F}$ has polynomial growth. If $\mathcal{F}$ has no compact leaves then (7.1) follows from (6.4) of [15]. On the other hand, in [3, 4] it is shown that either every leaf of $\mathcal{F}$ is compact, in which case M fibers over S^1 and $H^1(M;\mathbf{R})\neq 0$, or $\mathcal{F}$ has finitely many compact leaves $L_1,\ldots,L_p$. If some leaf L_i does not separate M, then it is dual to a non-zero element of $H^1(M;\mathbf{R})$, since it has a trivial normal bundle. Thus, we may assume that $\mathcal{F}$ has a finite (positive) number of compact leaves, each of which separates M. If we remove the compact leaves we are left with finitely many connected components. Let L be one of the compact leaves and let V_+ and V_- be the connected components on either side of L (with + and − determined by the orientation transverse to $\mathcal{F}$). The foliations determined by restricting $\mathcal{F}$ to V_+ and V_- admit non-zero invariant measures in the sense of [15] since every leaf of $\mathcal{F}$ has polynomial growth. Pick such measures for V_+ and V_- and let $\Phi_+ \in H^1(V_+;\mathbf{R})$ and $\Phi_- \in H^1(V_-;\mathbf{R})$ be the corresponding cohomology classes. Since the inclusions $V_+ \subset \bar{V}_+$, $V_- \subset \bar{V}_-$ are homotopy equivalences we may think of Φ_+ and Φ_- as cohomology classes for $\bar{V}_+$ and $\bar{V}_-$, respectively. Let $i_+ : L \to V_+$, $i_- : L \to V_-$ be inclusion maps. We claim that the classes $i_+^*\Phi_+$ and $i_-^*\Phi_-$ are linearly dependent in $H^1(L;\mathbf{R})$. This follows immediately if either of them is zero so we assume that they are both non-zero. (This amounts to assuming that the support sets for both invariant measures are asymptotic to L.) Take coordinates for a neighborhood of L in $\bar{V}_+$ which is diffeomorphic to $L\times[0,1)$ and such that the foliation induced from $\mathcal{F}$ is transverse to the $[0,1)$ factor. The $\mathcal{F}$-invariant measure on V^+ induces a measure on $(0,1)$ which is invariant under the action of the holonomy group $H(L)$. (Note that we do not need to distinguish the holonomy group on each side of L since $\mathcal{F}$ is analytic.) The cohomology class $i_+^*\Phi_+$ is represented (up to sign) by the composition

$$\pi_1(L)\xrightarrow{h} H(L)\xrightarrow{\tau}\mathbf{R}$$

where τ is the "translation number" homomorphism defined in (5.3) of [15]. Since $H(L)$ is abelian by (6.3), it follows from (4.3) and the definition of τ that τ is injective. By (5.4) and (5.5) the map $\delta_k: H(L)\to\mathbf{R}$ is injective for some $k\geq 1$. (Here we are thinking of $H(L)$ as a group of germs.) The group $H(L)$ has an ordering determined by germ $(f)<$ germ (g) if $f(x)<g(x)$ for all sufficiently small $x>0$. Note that the homomorphisms δ_k and $-\tau$ are both order preserving. This implies that one is a constant multiple of the other. Hence, $i_+^*\Phi_+$ is represented by a multiple of the composition

$$\pi_1(L)\xrightarrow{h} H(L)\xrightarrow{\delta_k}\mathbf{R}.$$

The same argument shows that this is also the case for $i_-^*\Phi_-$, thus proving that $i_+^*\Phi_+$ and $i_-^*\Phi_-$ are linearly dependent. (7.1) now follows from (7.3).

Remarks and Examples. 1) For the case $\dim M=3$, (7.1) has been proved by S. Goodman.

2) In the proof of (7.1) we have used only the properties
   a) that when the holonomy around a compact leaf is non-trivial, it is non-trivial in some r-jet, and
   b) that there are no null transversals.

3) If Γ is a uniform discrete subgroup of $SL(2,\mathbf{R})$ such that $H^1(SL(2,\mathbf{R})/\Gamma;\mathbf{R})=0$ (as in [19], for example) then $SL(2,\mathbf{R})/\Gamma$ has an oriented analytic foliation of codimension one so the hypothesis that $\pi_1(M)$ has polynomial growth cannot be dropped.

4) Let N be a bundle over the Klein bottle with fiber $[0,1]$ which is twisted to make N orientable. If two copies of N are appropriately attached along their

boundary tori ([15]) the resulting compact 3-manifold M satisfies the hypotheses of (7.2). N has an obvious foliation of codimension one by compact leaves (one Klein bottle, the rest tori) which is not transversely orientable. Attaching as above we conclude that the transverse orientability assumption cannot be dropped. (The resulting foliation of M actually has a "bundle-like metric" and the statements of (14.1) of [17] and (1.5) of [13] should be modified to require transverse orientability.) Note that $\pi_1(M)$ is infinite; in fact, the universal covering space of M is $\mathbf{R}^3$ and every element of $\pi_1(M)$ has infinite order.

5) Examples in higher dimensions of manifolds not admitting C^ω transversely oriented foliations may be obtained by noting that the hypotheses of Corollary 7.2 depend only on $\pi_1(M)$.

REFERENCES

[1] H. Bass, *The degree of polynomial growth of finitely generated nilpotent groups*, Proc. London Math. Soc., *25* (1972), 603–614.

[2] G. Birkhoff, *Lattice Theory*, A.M.S. Colloquium Publication Vol. 25 (1948).

[3] A. Haefliger, *Structure feuilletées et cohomologie à valeur dans un faisceau de groupoides*, Commentarii Math. Helv., *32* (1958), 248–329.

[4] ——, *Variétés feuilletées*, Ann. Scuola Norm. Sup. Pisa, *16* (1962), 367–397.

[5] N. Kopell, *Commuting diffeomorphisms*, Proc. Symposia in Pure Math. Vol. *14*, A.M.S. (1969), 165–184.

[6] W. Magnus, A. Karass, and D. Solitar, *Combinatorial Group Theory*, Interscience, New York (1966).

[7] A. Mal'cev, *On a class of homogeneous spaces* (Russian), Izv. Akad. Nauk. (Ser. Mat.) *13* (1949), 9–32.

[8] J. Milnor, *A note on curvature and the fundamental group*, J. of Differential Geometry, *2* (1968), 1–7.

[9] ——, *Growth of finitely generated solvable groups*, J. of Differential Geometry *2* (1968), 447–449.

[10] R. Moussu, *Sur les feuilletages de codimension un*, Thèse, Orsay (1971).

[11] S. Novikov, *Topology of foliations* (Russian), Trudy Mosk. Mat. Obsch., *14* (1965), 248–277 (A.M.S. translation 1967).

[12] J. Plante, *Asympototic properties of foliations*, Commentarii Math. Helv. *47* (1972), 449–456.

[13] ——, *On the existence of exceptional minimal sets in foliations of codimension one*, J. of Differential Equations, *15* (1974), 178–194.

[14] ——, *Measure preserving pseudogroups and a theorem of Sacksteder*, Annales de l'Institut Fourier *25* (1975), 237–249.

[15] ——, *Foliations with measure preserving holonomy*, Annals of Math., *102* (1975), 327–361.

[16] G. Reeb, *Sur certaines propriétés topologiques des variétés feuilletées*, Hermann, Paris (1952).

[17] R. Sacksteder, *Foliations and pseudogroups*, American J. of Math., *87* (1965), 79–102.

[18] W. Thurston, *Foliations of 3-manifolds which are circle bundles*, Thesis, U. of California, Berkeley (1972).

[19] ——, *A generalization of the Reeb Stability Theorem*, Topology, *13* (1974), 347–352.

[20] ——, *Existence of codimension one foliations*, Annals of Math. (to appear).

[21] J. Tits, *Free subgroups in linear groups*, J. of Algebra, *20* (1972), 250–270.

[22] J. Wolf, *Growth of finitely generated solvable groups and curvature of Riemannian manifolds*, J. of Differential Geometry, *2* (1968), 421–446.

Department of Mathematics
University of North Carolina
Chapel Hill, NC 27514

Department of Mathematics
Princeton University
Princeton, NJ 08540

Received April 1976.