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Rational L-groups of Bieberbach groups

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0. Introduction and statement of results

In this paper, we shall compute the rational L-groups of certain torsion-free groups called Bieberbach groups (cf. [7, p. 41]). A Bieberbach group Γ is the fundamental group of a compact flat Riemannian manifold M^n [14, p. 105], [2]. Therefore, we have the following exact sequence of groups

$$1 \to \mathbf{Z}^n \to \Gamma \to G \to 1 \tag{1}$$

where $\mathbf{Z}^n$ is maximal abelian in Γ and free abelian on n generators, and G is a finite group which is called the holonomy group of Γ. The exact sequence (1) comes from the regular covering

$$G \to T^n \to M^n \tag{2}$$

of M^n where T^n is the (flat) torus.

Let M^n be a manifold and $w : \Gamma = \pi_1(M^n) \to \mathbf{Z}_2 = \{\pm 1\}$ a group homomorphism. Using w, we have a local system $w\mathbf{Q}$ on M^n– each local group is isomorphic to the additive group of rationals $\mathbf{Q}$ and it is twisted to $w(g)\mathbf{Q}$ by $g \in \Gamma$.

MAIN THEOREM. *If Γ is a Bieberbach group and M^n is a compact flat Riemannian manifold with $\pi_1(M^n) = \Gamma$, then*

$$L_i^s(\Gamma, w) \otimes \mathbf{Q} \simeq \bigoplus_{j=0}^{\infty} H_{i+4j}(M^n; w\mathbf{Q})$$

$(i = 0, 1, 2, 3)$.

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When w is the 1st Stiefel–Whitney class of M^n, Poincaré duality used in conjunction with this result gives

$$L_i^s(\Gamma, w)\otimes\mathbf{Q}\simeq\bigoplus_{j=0}^{\infty} H^{n-i-4j}(M^n,\mathbf{Q}). \tag{3}$$

This theorem follows from a recent Induction Theorem of Dress [4] and an observation on the structure of a Bieberbach group Γ if the holonomy group is cyclic. It roughly goes as follows. Let

$$1\longrightarrow S\longrightarrow \Gamma\xrightarrow{p} G\longrightarrow 1 \tag{4}$$

be an extension of groups such that G is finite (and Γ is finitely presented). Let $L_i^s(\Gamma, w)$ and $L_i^h(\Gamma, w)$ $(i=0,1,2,3)$ be the surgery groups defined in [13], [12], and [10]. Let $GW(G,\mathbf{Z})$ be the equivariant Witt ring of Dress [4]. We shall make the L-groups of Γ into unital modules over $GW(G,\mathbf{Z})$. Let $\bar{\mathscr{C}}$ be the class of cyclic subgroups of G and let $\mathscr{C}$ be the class of subgroups $p^{-1}(\gamma)$ $(\gamma\in\bar{\mathscr{C}})$ of Γ. Following the argument of [4], we have isomorphisms

$$L_i^s(\Gamma, w)\otimes\mathbf{Q}\simeq L_i^h(\Gamma, w)\otimes\mathbf{Q}\simeq \lim_{\longleftarrow\gamma\in\mathscr{C}} L_i^s(\gamma, w/\gamma)\otimes\mathbf{Q}\simeq \lim_{\longleftarrow\gamma\in\mathscr{C}} L_i^h(\gamma, w/\gamma)\otimes\mathbf{Q}$$

$(i=0,1,2,3)$. We next observe that if the holonomy group G of the Bieberbach group Γ is cyclic, then Γ is a poly-$\mathbf{Z}$ group. Finally, the theorem is valid for poly-$\mathbf{Z}$ groups which follows essentially from [13] modulo technical difficulties arising from w.

By a much different approach, Bill Pardon has tentatively obtained the following result which complements ours. He proves that the surgery map

$$\theta:\left[\sum^i M^n, G/\mathrm{Top}\right]\to L_{n+i}(\pi_1 M^n)$$

is an injection onto a direct summand when M^n is a compact flat Riemannian manifold.

1. The action of $GW(G,\mathbf{Z})$ on L-groups

Consider an exact sequence $1\to S\to\Gamma\to G\to 1$ of groups with G a finite group (and Γ a finitely presented group). The purpose of this section is to define an action of Dress' equivariant Witt ring $GW(G,\mathbf{Z})$ on various L-groups for the group ring $R=\mathbf{Z}\Gamma$ and the group Γ [10, 13, 12]. Let $w:\Gamma\to\mathbf{Z}_2=\{\pm 1\}$ be a group homomorphism; use w to define an involution α of R determined by the equation: $\alpha(\gamma)=w(\gamma)\gamma^{-1}$ for $\gamma\in\Gamma$. Let M, N be right Γ-modules, then Wall [11] defines $\phi: M\times N\to R=\mathbf{Z}\Gamma$ to be an α-sesquilinear form if the following two equations hold:

$$\begin{aligned}\phi(m, n_1a_1+n_2a_2)&=\phi(m, n_1)a_1+\phi(m, n_2)a_2,\\ \phi(m_1a_1+m_2a_2, n)&=\alpha(a_1)\phi(m_1, n)+\alpha(a_2)\phi(m_2, n)\end{aligned} \tag{5}$$

for $m, m_1, m_2\in M$; $n, n_1, n_2\in N$ and $a_1, a_2\in R$. The set of all such maps is denoted by $S_\alpha(M, N)$ which is an abelian group. $S_\alpha(M)$ is an abbreviation for $S_\alpha(M, M)$.

We now give an alternate formulation of $S_\alpha(M, N)$ in terms of Eckmann's Γ-finite homomorphisms (cf. [1], pp. 358–360). We say that $\phi: M\times N\to\mathbf{Z}$ is w-sesquilinear if

(A) ϕ is $\mathbf{Z}$-bilinear;
(B) $\phi(x\gamma, y\gamma)=w(\gamma)\phi(x, y)$ for $x\in M$, $y\in N$ and $\gamma\in\Gamma$; (6)
(C) for each $x\in M$, $y\in N$ the function $\phi(x, y\gamma)$ from $\Gamma\to\mathbf{Z}$ has finite support.

Let $\bar{S}(M, N)$ denote the abelian group of all w-sesquilinear forms on $M\times N$ and $\bar{S}(M)$ denote $\bar{S}(M, M)$.

If $\phi\in\bar{S}(M, N)$, define $\hat{\phi}\in S_\alpha(M, N)$ by

$$\hat{\phi}(x, y)=\sum_{\gamma\in\Gamma}\phi(x, y\gamma^{-1})\gamma \tag{7}$$

for all $x\in M$, $y\in N$. It is easy to check that $\hat{}: \bar{S}(M, N)\to S_\alpha(M, N)$ is an isomorphism with inverse $-: S_\alpha(M, N)\to\bar{S}(M, N)$ defined by

$$\bar{\phi}(x, y)=\phi(x, y)(1) \tag{8}$$

for $x\in M$, $y\in N$.

Associated to $u=\pm1$, there is an involution of $S_\alpha(M)$ defined by

$$T_u(\phi)(x, y)=\alpha(\phi(y, x))u \tag{9}$$

for $\phi\in S_\alpha(M)$, and $x, y\in M$. Let $R_{(\alpha,u)}(M)=\operatorname{Ker}(T_u-1)$, the subset of $S_\alpha(M)$ fixed under T_u. Call the members of $R_{(\alpha,1)}(M)$ Hermitian forms on M and those of $R_{(\alpha,-1)}(M)$ skew-Hermitian. Let $Q_{(\alpha,u)}(M)=\operatorname{Cok}(T_u-1)$ be the (α, u)-quadratic forms on M and let

$$b=T_u+1: Q_{(\alpha,u)}(M)\to R_{(\alpha,u)}(M) \tag{10}$$

be the bilinearization map [11]. Correspondingly, we have $\bar{T}_u$, $\bar{Q}_u$ and $\bar{b}$ and members of $\bar{Q}_u(M)$ are called the u-quadratic forms on M. We also have the orthogonal direct sum $\theta_1 \oplus \theta_2 \in Q_{(\alpha,u)}(M_1 \oplus M_2)$ for $\theta_i \in Q_{(\alpha,u)}(M_i)$ $(i=1,2)$, and it induces the corresponding operation $\oplus$ on $\bar{Q}_u$.

Recall that Eckmann (cf. [1, p. 358]) defines $\overline{\mathrm{Hom}}\,(M, \mathbf{Z})$ to be the set of all $\mathbf{Z}$-homomorphisms $f: M \to \mathbf{Z}$ which are Γ-finite; i.e., for each $x \in M$ the function $f(x\gamma): \Gamma \to \mathbf{Z}$ has finite support; $\overline{\mathrm{Hom}}\,(M, \mathbf{Z})$ is a right Γ-module defined by $f \cdot \gamma(x) = w(\gamma) f(x\gamma^{-1})$ for $x \in M$, $\gamma \in \Gamma$. Denote it by $\bar{M}$. If M is a right Γ-module, the dual $\mathrm{Hom}_\Gamma\,(M, R)$ is a left Γ-module. We can make the dual a right Γ-module M^α by setting $f \cdot a(m) = \alpha(a) f(m)$ for $a \in R$, $m \in M$ and $f: M \to R$. We have an isomorphism $\hat{}: \bar{M} \to M^\alpha$ defined by $\hat{f}(x) = \sum_{\gamma \in \Gamma} f(x\gamma^{-1})\gamma$. Associated to $\phi \in S_\alpha(M, N)$, we have a homomorphism $A\phi: M \to N^\alpha$ defined by $A\phi(m)(n) = \phi(m, n)$ [11]. Similarly, we have $\bar{A}\phi: M \to \bar{N}$ defined by $\bar{A}\phi(x)(y) = \phi(x, y)$. Both $A\phi$, $\bar{A}\phi$ are Γ-homomorphisms and we have the following commutative diagram

$$\begin{array}{ccc} \bar{S}(M, N) & \xrightarrow{\bar{A}} & \underset{\Gamma}{\mathrm{Hom}}\,(M, \bar{N}) \\ \big\downarrow{\scriptstyle\wedge} & & \big\downarrow{\scriptstyle \mathrm{Hom}\,(\,,\hat{}\,)} \\ S_\alpha(M, N) & \xrightarrow{A} & \underset{\Gamma}{\mathrm{Hom}}\,(M, N^\alpha) \end{array} \tag{11}$$

where both vertical arrows are isomorphisms.

Throughout this paper, P denotes a finitely generated projective Γ-module. We call $\phi \in \bar{S}(P)$ non-singular if $A\phi: P \to \bar{P}$ is an isomorphism and $\theta \in \bar{Q}_u(P)$ non-singular if its bilinearization $\bar{b}_\theta$ is. An element $\phi \in \bar{S}(P)$ (respectively $\theta \in \bar{Q}_u(P)$) is non-singular if and only if $\hat{\phi} \in S_\alpha(P)$ (respectively $\hat{\theta} \in Q_{(\alpha,u)}(P)$) is. Let $\bar{\mathfrak{Q}}(\Gamma, w, u)$ be the category whose objects (P, θ) for $\theta \in \bar{Q}_u(P)$ are non-singular u-quadratic forms on P and whose morphisms $(P, \theta) \to (P', \theta')$ are isomorphisms. Let $\mathfrak{Q}(R, \alpha, u)$ be the category introduced in [10, p. 267]. The functor $\hat{}: \bar{\mathfrak{Q}}(\Gamma, w, u) \to \mathfrak{Q}(R, \alpha, u)$ is an isomorphism $-\ \hat{}$ sends the object (P, θ) to $(P, \hat{\theta})$ and leaves the morphisms alone.

Both $\bar{\mathfrak{Q}}$, $\mathfrak{Q}$ are categories with product $\oplus$ and $K_i(\bar{\mathfrak{Q}})$, $K_i(\mathfrak{Q})$ $(i=0,1)$ are defined; $\hat{}$ induces an isomorphism $\hat{}: K_i(\bar{\mathfrak{Q}}) \to K_i(\mathfrak{Q})$ $(i=0,1)$. The hyperbolic functor H, the forgetting functor F etc. of [10] induce the corresponding functors $\bar{H}$, $\bar{F}$ etc. The reason why we translate $\mathfrak{Q}$ into $\bar{\mathfrak{Q}}$ is because we can describe the action of the equivariant Witt ring $GW(G, \mathbf{Z})$ on the L-groups easier.

Let us now recall the definition of $GW(G, \mathbf{Z})$ and define the action. Let G be a finite factor group of Γ. A $\mathbf{Z}G$-lattice is a pair (M, f) where M is a finitely generated Z-free G-module and $f: M \times M \to \mathbf{Z}$ is a symmetric G-invariant non-singular form. Let $\mathfrak{D}$ be the category whose objects are $\mathbf{Z}G$-lattices and whose morphisms $(M, f) \to (M', f')$ are isomorphisms. (Dress constructs $GW(G, \mathbf{Z})$ from $\mathfrak{D}$.)

We next introduce a functor

$$G:\mathfrak{D}\times\bar{\mathfrak{Q}}(\Gamma, w, u)\to\bar{\mathfrak{Q}}(\Gamma, w, u) \tag{12}$$

defined by

$$G((M, f), (P, \theta)) = (M\otimes P, [f\otimes\phi]) \tag{13}$$

for $(M, f)\in\mathfrak{D}$, $(P, \theta)\in\bar{\mathfrak{Q}}(\Gamma, w, u)$ with $\theta=[\phi]$. Let us explain formula (13); M being a G-module is also a Γ-module and we use the diagonal action on $M\otimes P$, then $M\otimes P$ is a finitely generated projective Γ-module; $f\otimes\phi$ is defined by

$$f\otimes\phi(x\otimes y, x'\otimes y') = f(x, x')\phi(y, y') \tag{14}$$

for x, $x'\in M$ and y, $y'\in P$. One easily checks that $f\otimes\phi\in\bar{S}(M\otimes P)$ and its equivalence class in $\bar{Q}_u(M\otimes P)$ is independent of the choice of the representative ϕ of θ. Furthermore, $[f\otimes\phi]$ is non-singular. The functor G induces a bilinear map (also denoted by G)

$$G: K_0\mathfrak{D}\times K_i\bar{\mathfrak{Q}}(\Gamma, w, u)\to K_i\bar{\mathfrak{Q}}(\Gamma, w, u) \qquad (i=0, 1). \tag{15}$$

Via G, $K_i\bar{\mathfrak{Q}}(\Gamma, w, u)$ is a unital module over the ring $K_0\mathfrak{D}$.

Let (M, f) be a $\mathbf{Z}G$-lattice, then

$$G((M, f), \bar{H}(P)) \simeq \bar{H}(M\otimes P), \tag{16}$$

and this isomorphism is natural in $P(M\otimes P$ has the diagonal Γ-module structure). To construct isomorphism (16), first observe that $\bar{H}(M\otimes P)$ is isomorphic to $((M\otimes P)\oplus(M^*\otimes P), [l])$ where l is determined by the equation

$$l(m\otimes x, q\otimes p), (m'\otimes x', q'\otimes p')) = q(m')p(x') \tag{17}$$

for x, $x'\in P$; m, $m'\in M$; p, $p'\in\bar{P}$; and q, $q'\in M^*$ (where $M^*=\mathrm{Hom}\,(M, \mathbf{Z})$). Next, there is an isomorphism

$$g: M\otimes(P\oplus\bar{P})\to(M\otimes P)\oplus(M^*\otimes\bar{P}) \tag{18}$$

determined by

$$g(m\otimes(x, p)) = (m\otimes x, Af(m)\otimes p) \tag{19}$$

for $m \in M$, $x \in P$ and $p \in \bar{P}$ (where $Af: M \to M^*$ is defined by $Af(x)(y) = f(x, y)$). Also $f \otimes h$, with $h((x, p), (x', p')) = p(x')$, corresponds to l under g; this completes the construction of isomorphism (16). As a consequence of (16), image $\bar{H}$ is a $K_0\mathfrak{D}$-submodule of $K_i\bar{\mathfrak{Q}}(\Gamma, w, u)$ $(i = 0, 1)$.

Let $(P, \theta) \in \bar{\mathfrak{Q}}(\Gamma, w, u)$ and define a submodule $P' \subseteq P$ to be a subkernel if it is a subkernel of $(P, \hat{\theta}) \in \mathfrak{Q}(R, \alpha, u)$ in the sense of [10, p. 268]. Let $(M, f) \in \mathfrak{D}$ be split (cf. [4], p. 294) and N be a Lagrangian in M; i.e., N is a G-submodule such that

$$N = N^{\perp} = \{m \in M \mid f(m, n) = 0 \quad \text{for all} \quad n \in N\}. \tag{20}$$

LEMMA 1.1. *If $(P, \theta) \in \bar{\mathfrak{Q}}(\Gamma, w, u)$, then $N \otimes P$ (with the diagonal Γ-module structure) is a subkernel of $G((M, f), ((P, \theta))$.*

Proof. If S is a Γ-submodule of (P, θ), define

$$S^{\perp} = \{x \in P \mid b(\theta)(x, y) = 0 \quad \text{for all} \quad y \in S\}; \tag{21}$$

$S^{\perp}$ is clearly equal to the kernel of the composite of $\bar{i}$ with $\bar{A}b(\theta)$ where i denotes the inclusion map and $\bar{i}: \bar{P} \to \bar{S}$ is induced by i. Let j denote the inclusion map of N into M; since $\operatorname{Cok} j$ is $\mathbf{Z}$-free, $j \otimes id: N \otimes P \to M \otimes P$ is a monomorphism. Furthermore, $(N \otimes P)^{\perp}$ equals the kernel of the composite of $j^* \otimes id: M^* \otimes \bar{P} \to N^* \otimes P$ and $Af \otimes \bar{A}b(\theta)$. Since $\bar{A}b(\theta)$ is an isomorphism and P is $\mathbf{Z}$-free,

$$(N \otimes P)^{\perp} = \operatorname{Ker}(j^* \circ Af) \otimes P = N \otimes P. \tag{22}$$

Also $j^*: M^* \to N^*$ is an epimorphism, hence $M^* \otimes \bar{P} \to N^* \otimes \bar{P}$ is an epimorphism. Since $N^* \otimes \bar{P}$ is Γ-projective, $N \otimes P$ is a direct summand of $M \otimes P$. Observe that if S is a submodule of a Γ-module Q and $\varphi \in \bar{R}_u(Q)$, then

$$\{x \in Q \mid \varphi(x, y) = 0 \quad \text{for all} \quad y \in S\} = \{x \in Q \mid \hat{\varphi}(x, y) = 0 \quad \text{for all} \quad y \in S\}. \tag{23}$$

It follows from Lemma 2 of [10] that $N \otimes P$ is a subkernel of $\hat{}$ applied to $G((M, f), (P, \theta))$.

Recall that Dress [4] defines $GW(G, \mathbf{Z})$ to be the residue class ring of $K_0\mathfrak{D}$ with respect to the ideal (=additive subgroup), generated by all split lattices. Consequently, Lemma 1.1 implies that $\operatorname{Cok}(\bar{H}: K_i(\mathcal{P}(R)) \to K_i\bar{\mathfrak{Q}}(\Gamma, w, u))$ $(i = 0, 1)$ is a unital $GW(G, \mathbf{Z})$-module (where $\mathcal{P}(R)$ is the category of finitely generated projective R-modules). This is obvious when $i = 0$. When $i = 1$, let $(P, \theta) \in \bar{\mathfrak{Q}}(\Gamma, w, u)$ and $A \in \operatorname{Aut}(P, \theta)$ represent an element $x \in K_1\bar{\mathfrak{Q}}(\Gamma, w, u,)$; let $(M, f) \in \mathfrak{D}$ represent $r \in K_0\mathfrak{D}$ and assume that (M, f) is split with N a Lagrangian in M, then $rx \in K_1\bar{\mathfrak{Q}}(\Gamma, w, u)$ is represented by $id \otimes A \in \operatorname{Aut} G((M, f), (P, \theta))$.

Since $id \otimes A$ leaves the subkernel $N \otimes P$ invariant, Wall's discussion in [10, §2] shows that $rx \in \text{image } \bar{H}$ which verifies our above assertion.

Our main interest is not in $\operatorname{Cok} \bar{H}$ but in its variants the surgery groups $L_i^s(\Gamma, w)$ and $L_i^h(\Gamma, w)$ $(i = 0, 1, 2, 3)$ defined in [13], [12] and [10]. Let us recall their definitions. Let $\mathscr{X}(R)$ be the full subcategory of $\mathscr{P}(R)$ consisting of free modules and $\mathfrak{Q}^h(R, \alpha, u)$ the full subcategory of $\mathfrak{Q}(R, \alpha, u)$ with objects (P, θ) where $P \in \mathscr{X}(R)$. The hyperbolic functor restricts to $H: \mathscr{X}(R) \to \mathfrak{Q}^h(R, \alpha, u)$ and we have

$$L_{2i}^h(\Gamma, w) = \operatorname{Cok}(H: K_0\mathscr{X}(R) \to K_0\mathfrak{Q}^h(R, \alpha, (-1)^i),$$

$$L_{2i+1}^h(\Gamma, w) = \operatorname{Cok}(H: K_1\mathscr{X}(R) \to K_1\mathfrak{Q}^h(R, \alpha, (-1)^i))/\text{class of } \sigma = \begin{pmatrix} 0 & 1 \\ (-1)^i & 0 \end{pmatrix}$$

$$\text{where } \sigma \in \operatorname{Aut} H(R) \qquad (i = 0, 1). \quad (24)$$

Let $\mathscr{B}(R)$ be Wall's category [10, p. 270] of based R-modules except with determinants calculated in $Wh\Gamma$ instead of K_1R, $\mathscr{B}\mathfrak{Q}(R, \alpha, u)$ his category of based quadratic modules, and $\mathscr{L}(R, \alpha, u)$ the full subcategory of forms with zero discriminant, then there is still a hyperbolic functor $H: \mathscr{B}(R) \to \mathscr{L}(R, \alpha, u)$ and

$$L_{2i}^s(\Gamma, w) = \operatorname{Cok}(H: K_0\mathscr{B}(R) \to K_0\mathscr{L}(R, \alpha, (-1)^i)),$$

$$L_{2i+1}^s(\Gamma, w) = \operatorname{Cok}(H: K_1\mathscr{B}(R) \to$$

$$K_1\mathscr{L}(R, \alpha, (-1)^i))/\text{class of } \sigma = \begin{pmatrix} 0 & 1 \\ (-1)^i & 0 \end{pmatrix} \qquad (i = 0, 1). \quad (25)$$

There are clearly natural maps $L_i^s(\Gamma, w) \to L_i^h(\Gamma, w)$ $(i = 0, 1, 2, 3)$ and Rothenberg has shown that their kernels and cokernels have exponent 2 (cf. [13, p. 248]).

LEMMA 1.2. *The surgery groups* $L_i^s(\Gamma, w)$ *and* $L_i^h(\Gamma, w)$ $(i = 0, 1, 2, 3)$ *are unital* $GW(G, \mathbf{Z})$*-modules and the natural maps* $L_i^s(\Gamma, w) \to L_i^h(\Gamma, w)$ *are* $GW(G, \mathbf{Z})$*-homomorphisms.*

Proof. (A) Let us first show that $L_i^h(\Gamma, w)$ $(i = 0, 1, 2, 3)$ are unital $GW(G, \mathbf{Z})$-modules. Let $(F, \hat{\theta}) \in \mathfrak{Q}^h(R, \alpha, (-1)^i)$ represent an element $x \in L_{2i}^h(\Gamma, w)$ where $(F, \theta) \in \bar{\mathfrak{Q}}(\Gamma, w, (-1)^i)$ and $(M, f) \in \mathfrak{D}$ represent $r \in GW(G, \mathbf{Z})$, then $rx \in L_{2i}^h(\Gamma, w)$ is represented by $\hat{G}((M, f), (F, \theta))$ (where $\hat{G}$ is the composite of G and $\hat{}: \bar{\mathfrak{Q}}(\Gamma, w, (-1)^i) \to \mathfrak{Q}(R, \alpha, (-1)^i)$). If $A \in \operatorname{Aut}(F, \hat{\theta})$ represents $y \in L_{2i+1}^h(\Gamma, w)$ then $id \otimes A \in \operatorname{Aut} \hat{G}((M, f), (F, \theta))$ represents $ry \in L_{2i+1}^h(\Gamma, w)$. One shows that this action is well defined by using formula (16) together with Lemma 1.1 as in the proof that $\operatorname{Cok}(\bar{H})$ is a $GW(G, \mathbf{Z})$-module.

(B) Consider the category $\bar{\mathscr{L}}(\Gamma, w, u)$ whose objects are pairs (F, θ) with zero discriminant where $F \in \mathscr{B}(R)$ and $\theta \in \bar{Q}_u(F)$. It bears the same relation to $\mathscr{L}(R, \alpha, u)$ that $\bar{\mathfrak{Q}}(\Gamma, w, u)$ has to $\mathfrak{Q}(R, \alpha, u)$; in particular, $\hat{}: \bar{\mathscr{L}}(\Gamma, w, u) \mapsto \mathscr{L}(R, \alpha, u)$ is an isomorphism.

Let M, N be two finitely generated G-modules which are Z-free with bases $\{e_i\}$, $\{e_i'\}$ and $\mathscr{A}: M \to N$ a G-isomorphism. Let $P, Q \in \mathscr{B}(R)$ with bases $\{\sigma_j\}$, $\{\sigma_j'\}$ and $A: P \to Q$ a Γ-isomorphism with zero torsion in $Wh\Gamma$, then $M \otimes P$, $N \otimes Q$ with the diagonal Γ-module structures and bases $\{e_i \otimes \sigma_j\}$, $\{e_i' \otimes \sigma_j'\}$ are objects in $\mathscr{B}(R)$. Furthermore, one observes that the isomorphism $\mathscr{A} \otimes A: M \otimes P \to N \otimes Q$ has zero torsion in $Wh\Gamma$. (We leave the verification of this fact to the reader.) Because of this, we can extend the definition of the functor G to a functor (also denoted by G)

$$G: \mathfrak{D} \times \bar{\mathscr{L}}(\Gamma, w, u) \to \bar{\mathscr{L}}(\Gamma, w, u) \tag{26}$$

defined essentially by formula (13) subject to the change that now $(P, \theta) \in \bar{\mathscr{L}}(\Gamma, w, u)$ and we give $M \otimes P$ the basis $\{e_i \otimes \sigma_j\}$ where $\{e_i\}$ is an arbitrary $\mathbf{Z}$-basis for M and $\{\sigma_j\}$ is in the preferred of bases for P.

We now define an action of $GW(G, \mathbf{Z})$ on $L^s(\Gamma, w)$ in a manner completely analogous to that described in part (A). To show that this action is well defined, we observe that the isomorphism of formula (16) has zero torsion, and in this situation Lemma 1.1 can be strengthened to say that $N \otimes P$ is a based subkernel of $G((M, f), (P, \theta))$ in the sense of [10, §1]. The factoring out of the class of σ in formulas (24) and (25) is also compatible with this action.

(C) It is clear from the construction that the natural maps $L_i^s(\Gamma, w) \to L_i^h(\Gamma, w)$ are homomorphisms of $GW(G, \mathbf{Z})$-modules.

2. L-groups as Frobenius modules over $GW(G, \mathbf{Z})$

Let $i: S \to \Gamma$ be an inclusion of a subgroup of Γ of finite index. We have functors

$$\begin{aligned} i^*&: \bar{\mathfrak{Q}}^h(\Gamma, w, u) \to \bar{\mathfrak{Q}}^h(S, w, u) \\ i^*&: \bar{\mathscr{L}}(\Gamma, w, u) \to \bar{\mathscr{L}}(S, w, u) \end{aligned} \tag{27}$$

where w in $\bar{\mathfrak{Q}}^h(S, w, u)$ and $\bar{\mathscr{L}}(S, w, u)$ is w/S and $\bar{\mathfrak{Q}}^h(\Gamma, w, u)$ denotes the full subcategory of $\bar{\mathfrak{Q}}(\Gamma, w, u)$ consisting of objects (P, θ) with P free. The first i^* is the forgetful functor, and the second we describe as follows. Let $\gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma$ be a complete set of right coset representatives for S in Γ. For $(F, v) \in \mathscr{B}(\mathbf{Z}\Gamma)$ and

$\{e_1, e_2, \ldots, e_n\}$ a particular basis for F in the class v, let i^*v be the equivalence class of bases for F (considered as an S-module) containing the basis $\{e_i\gamma_j \mid 1 \le i \le n,\ 1 \le j \le m\}$, and observe that i^*v is independent of the choices of $\{e_i\}$ and $\{\gamma_j\}$. For $((F, v), \theta) \in \bar{\mathscr{L}}(\Gamma, w, u)$, set

$$i^*((F, v), \theta) = ((F, i^*v), \theta) \tag{28}$$

and leave the morphisms alone. Via the isomorphisms of categories $\hat{}$, the functors i^* induce restriction (transfer) maps

$$\begin{aligned} i^*: L_k^h(\Gamma, w) &\to L_k^h(S, w) \qquad (k = 0, 1, 2, 3)\\ i^*: L_k^s(\Gamma, w) &\to L_k^s(S, w) \qquad (k = 0, 1, 2, 3). \end{aligned} \tag{29}$$

On the other hand, we have induction homomorphisms

$$\begin{aligned} i_*: L_k^h(S, w) &\to L_k^h(\Gamma, w) \qquad (k = 0, 1, 2, 3)\\ i_*: L_k^s(S, w) &\to L_k^s(\Gamma, w) \qquad (k = 0, 1, 2, 3) \end{aligned} \tag{30}$$

for an arbitrary subgroup $S \subseteq \Gamma$ (cf. [13], pp. 49 and 65). These are induced (via $\hat{}$) by functors

$$\begin{aligned} i_*: \bar{\mathscr{Q}}^h(S, w, u) &\to \bar{\mathscr{Q}}^h(\Gamma, w, u)\\ i_*: \bar{\mathscr{L}}(S, w, u) &\to \bar{\mathscr{L}}(\Gamma, w, u). \end{aligned} \tag{31}$$

The first functor sends the object $(F, [\phi])$ to $(F \otimes_S \mathbf{Z}\Gamma, [i_*\phi])$ where $\phi \in \bar{S}(F)$ and $i_*\phi$ is determined by the formula

$$i_*\phi(x \otimes \gamma, y \otimes \beta) = \phi(x, y\varepsilon(\beta\gamma^{-1}))w(\gamma) \tag{32}$$

where $x, y \in F$; $\gamma, \beta \in \Gamma$ and $\varepsilon: \mathbf{Z}\Gamma \to \mathbf{Z}S$ is given by $\varepsilon(\gamma) = \gamma$ for $\gamma \in S$ and $\varepsilon(\gamma) = 0$ for $\gamma \notin S$. For $A \in \operatorname{Aut}(F, [\phi])$, i_* maps A to $A \otimes id \in \operatorname{Aut}(F \otimes_S \mathbf{Z}\Gamma, [i_*\phi])$. The second functor i_* is defined similarly.

We next record some properties of i_* and i^*. If $S = \Gamma$, then both i_* and i^* are the identity map; and if $T \subseteq S \subseteq \Gamma$, j denotes the inclusion map of T in S and i that of S in Γ then $i_*j_* = (ij)_*$; while if $[S:T] < \infty$, $[\Gamma:S] < \infty$ then $j^*i^* = (ij)^*$.

Now let G be a finite factor group of Γ with canonical projection $p: \Gamma \to G$ and let $\mathscr{S}$ denote the category of subgroups of G with morphisms the inclusion maps. Let $\bar{M}$ be one of the functors L_i^h or L_i^s $(i = 0, 1, 2, 3)$, and let $M(\Pi) = \bar{M}(p^{-1}(\Pi))$ for $\Pi \in \mathscr{S}$. We wish to show that $M(\ )$ is a Frobenius module over the

Frobenius functor $F(\Pi)=GW(\Pi,\mathbf{Z})$ in the sense that Lam's Axioms 1, 2, 3 [8, p. 17] hold for all subgroups Π of G. We just verified Axiom 1 and Axiom 2 is clear. Let $\Pi'\subseteq\Pi\subseteq G$ and let i denote both the inclusion map of Π' into Π and that of $p^{-1}\Pi'$ into $p^{-1}\Pi$.

LEMMA 2.1. *If* $x\in F(\Pi)$ *and* $y\in M(\Pi')$, *then* $i_*(i^*(x)y)=xi_*(y)$.

Proof. We prove the assertion first for $\bar{M}=L^h_{2k}$ $(k=0,1)$. Let $S=p^{-1}\Pi'$ and $T=p^{-1}\Pi$. Let $y\in L^h_{2k}(S,w)$ be represented by $(P,[\phi])$ and let (N,f) represent x. Then

$$((N\otimes P)\otimes_S\mathbf{Z}T,[i_*(f\otimes\phi)])\quad\text{represents}\quad i_*(i^*(x)y)$$

and

$$(N\otimes(P\otimes_S\mathbf{Z}T),[f\otimes i_*\phi])\quad\text{represents}\quad xi_*(y).$$

We have an isomorphism

$$g:(N\otimes P)\otimes_S\mathbf{Z}T\to N\otimes(P\otimes_S\mathbf{Z}T)\tag{33}$$

defined by

$$g((n\otimes x)\otimes\gamma)=n\gamma\otimes(x\otimes\gamma)\tag{34}$$

for $n\in N$, $x\in P$, $\gamma\in T$. It is a straightforward calculation to show that g maps $i_*(f\otimes\phi)$ to $f\otimes i_*\phi$.

Since g is natural in P, we have also proven the assertion for $\bar{M}=L^h_k$ $(k=1,3)$. The proof for $\bar{M}=L^s_k$ is similar.

LEMMA 2.2. *If* $x\in F(\Pi')$ *and* $y\in M(\Pi)$, *then* $i_*(xi^*y)=i_*(x)y$.

Proof. Again, consider first the case $\bar{M}=L^h_{2k}$ $(k=0,1)$ and let $S=p^{-1}\Pi'$, $T=p^{-1}\Pi$. Let (N,f) represent x and $(P,[\phi])$ represent $y\in L^h_{2k}(T,w)$, then

$$((N\otimes P)\otimes_S\mathbf{Z}T,[i_*(f\otimes\phi)])\quad\text{represents}\quad i_*(xi^*(y))$$

and

$$((N\otimes_{\Pi'}\mathbf{Z}\Pi)\otimes P,[(i_*f)\otimes\phi])\quad\text{represents}\quad i_*(x)y$$

were i_*f is determined by the formula

$$i_*f(n\otimes\gamma, n'\otimes\beta)=\begin{cases} f(n, n'\beta\gamma^{-1}) & \text{if} \quad \beta\gamma^{-1}\in\Pi' \\ 0 & \text{otherwise,}\end{cases} \tag{35}$$

for n, $n'\in M$ and β, $\gamma\in\Pi$. Define an isomorphism

$$g:(N\otimes P)\otimes_S \mathbf{Z}T\to(N\otimes_{\Pi'}\mathbf{Z}\Pi)\otimes P \tag{36}$$

by the formula

$$g((n\otimes x)\otimes\gamma)=(n\otimes\bar{\gamma})\otimes x\gamma, \tag{37}$$

for $n\in N$, $x\in P$, $\gamma\in T$, and $\bar{\gamma}=p(\gamma)\in\Pi\subseteq G$. A calculation again shows that g maps $i_*(f\otimes\phi)$ to $(i_*f)\otimes\phi$. This proves the lemma for $\bar{M}=L^h_{2k}$, and since g is natural in P also for $\bar{M}=L^h_{2k+1}$ $(k=0,1)$. The case when $\bar{M}=L^s_k$ is left to the reader.

It follows from Lemma 2.1 and Lemma 2.2 that we have the following theorem.

THEOREM 2.3. *Let G be a finite factor group of Γ with quotient map p, then $L^h_k(p^{-1}\ ,w)$ and $L^s_k(p^{-1}\ ,w)$ $(k=0,1,2,3)$ are G-Frobenius modules over $GW(\ ,\mathbf{Z})$.*

Let $\mathscr{C}$ denote the class of cyclic subgroups of G and define

$$M(G)^{\mathscr{C}}=\bigcap_{\Pi\in\mathscr{C}}\operatorname{Ker}(M(G)\xrightarrow{i^*}M(\Pi)). \tag{38}$$

COROLLARY 2.4. *Let n denote the order of G. Then $M(G)^{\mathscr{C}}$ is a torsion-group whose exponent divides $4n$, and when n is odd it divides n.*

This corollary is a consequence of [3, Theorem 1] and a standard argument about Frobenius modules [8, p. 23, lemma 2.10].

Recall that L^h_i and L^s_i $(i=0,1,2,3)$ are covariant functors on the category $\mathscr{G}$ with objects pairs (Γ, w) where Γ is a group and $w:\Gamma\to\{\pm1\}$ is a homomorphism. If $(\Gamma_j, w_j)\in\mathscr{G}$ $(j=1,2)$, then a group homomorphism $f:\Gamma_1\to\Gamma_2$ is a morphism in $\mathscr{G}$ is $w_2(f(x))=w_1(x)$ for all $x\in\Gamma_1$; f_* denotes the image of f under these functors (cf. [13], pp. 49 and 65).

Let S be a normal subgroup of Γ with finite factor group G; for each $\gamma\in\Gamma$, let

c_γ denote the outer automorphism of S given by $c_\gamma(s) = \gamma s \gamma^{-1}$ for $s \in S$. We recall Taylor's Theorem [9, Theorem 1].

LEMMA 2.5. *If $\gamma \in S$, then $(c_\gamma)_*$ is multiplication by $w(\gamma)$.*

Define an action of Γ on $M(1) = L_k^h(S, w)$ or $L_k^s(S, w)$ $(k = 0, 1, 2, 3)$ by

$$\gamma x = w(\gamma)(c_\gamma)_*(x) \tag{39}$$

for $x \in M(1)$ and $\gamma \in \Gamma$ (1 denotes the trivial group). By Lemma 2.5, this action factors through G. So, we have a left G-module structure on $M(1)$. (We note that formula (39) still defines an action of G on $L_k^h(S, w)$ and $L_k^s(S, w)$ even when G is not finite.) Recall that for any G-module M, we have $M^G = \{x \in M \mid gx = x \text{ for all } g \in G\}$ the fixed point set of G.

LEMMA 2.6. *The image of $i^*: M(G) \to M(1)$ is a subgroup of $M(1)^G$.*

This result follows from Lemma 2.5 and the following commutative diagram

$$\begin{array}{ccc} M(1) & \xrightarrow{(c_\gamma)_*} & M(1) \\ \uparrow{\scriptstyle i^*} & & \uparrow{\scriptstyle i^*} \\ M(G) & \xrightarrow{(c_\gamma)_*} & M(G) \end{array} \tag{40}$$

Let $N = \sum_{g \in G} g \in \mathbf{Z}G$ be the norm.

LEMMA 2.7. *Let i denote the inclusion of 1 into G, then $i^*(i_*(x)) = Nx$ for all $x \in M(1)$.*

Proof. We shall prove the lamma for $\bar{M} = L_{2k}^h$ $(k = 0, 1)$ and leave the other cases to the reader. Let $(P, [\phi])$ represent x, then $(P \otimes_S \mathbf{Z}\Gamma, [i_*\phi])$ represents $i^*(i_*(x))$. For each $g \in G$, pick an element $g' \in \Gamma$ with $p(g') = g$, then gx is represented by $(g'P, [w(g')\phi])$ where $g'P = P$ as abelian groups but with new S-module structure $(m \cdot s)$ defined by $m \cdot s = mg's(g')^{-1}$ for $m \in P$, $s \in S$. Hence Nx is represented by the orthogonal direct sum

$$\left(\bigoplus_{g \in G} g'P, \left[\bigoplus_{g \in G} w(g')\phi\right]\right).$$

Define an S-isomorphism

$$f: \bigoplus_{g \in G} g'P \to P \otimes_S \mathbf{Z}\Gamma$$

by $f(m) = m \otimes g'$ if $m \in g'P$; one easily sees that f carries $\oplus_{g \in G} w(g')\phi$ to $i_*\phi$.

LEMMA 2.8. *Let m be the order of G, then the homomorphism*

$$i^* \otimes id : M(G) \otimes \mathbf{Z}\left[\frac{1}{m}\right] \to M(1) \otimes \mathbf{Z}\left[\frac{1}{m}\right] \tag{41}$$

has image $(M(1) \otimes \mathbf{Z}[1/m])^G$.

Proof. By Lemma 2.6, the image of $i^* \otimes id$ is contained in $(M(1) \otimes \mathbf{Z}[1/m])^G$. Lemma 2.7 shows that $i^* \otimes id$ composed with $i_* \otimes id$ is multiplication by N, hence restricted to $(M(1) \otimes \mathbf{Z}[1/m])^G$ it is multiplication by m which is an automorphism. Therefore the image of $i^* \otimes id$ is $(M(1) \otimes \mathbf{Z}[1/m])^G$.

We also have the following lemma. It comes out of a straightforward calculation.

LEMMA 2.9. *Let the following be a commutative square of group inclusions*:

$$\begin{array}{ccc} A & \xrightarrow{k} & C \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ B & \xrightarrow{l} & D \end{array} \tag{42}$$

such that A and C are normal in B and D and l induces an isomorphism of B/A to D/C. Then, the following diagram commutes

$$\begin{array}{ccc} \bar{M}(A) & \xrightarrow{k_*} & \bar{M}(C) \\ \uparrow{\scriptstyle i^*} & & \uparrow{\scriptstyle j^*} \\ \bar{M}(B) & \xrightarrow{l_*} & \bar{M}(D) \end{array} \tag{43}$$

where $\bar{M} = L_t^h(\ ,w)$ or $L_t^s(\ ,w)$ $(t = 0, 1, 2, 3)$.

3. Poly-Z groups

In this section, we shall study poly-**Z** groups. For a poly-**Z** group Γ, it follows from [13, p. 248] and [5, Th. 29] that $L_i^h(\Gamma, w) = L_i^s(\Gamma, w)$ $(i = 0, 1, 2, 3)$ and we shall denote the common group by $L_i(\Gamma, w)$. We now state the main result of this section.

THEOREM 3.1. *Let Γ be a poly-**Z** group of finite rank and S a normal subgroup with finite factor group G of order m and let $i : S \to \Gamma$ be the inclusion map. Then,*

$$i^* \otimes id : L_k(\Gamma, w) \otimes \mathbf{Z}\left[\frac{1}{m}\right] \to \left(L_k(S, w) \otimes \mathbf{Z}\left[\frac{1}{m}\right]\right)^G \tag{44}$$

is an isomorphism for $k = 0, 1, 2, 3$.

In the proof of this theorem, we will use Wall's geometric construction of L-theory [13, §9]. Hence, we take this opportunity to rectify an apparent error in his construction. The following remark shows there is a difficulty. Let (K, x) be a pointed CW-complex and $\gamma \in \pi_1(K, x)$, then by the homotopy extension theorem, there is a map $f : (K, x) \to (K, x)$ such that $f_\# = c_\gamma$ on $\pi_1(K, x)$ and f is homotopic to *id*. By [13, p. 87], $L_n^1(f) = id$ on $L_n^1(K)$, but Taylor's Theorem, Lemma 2.5, says $(c_\gamma)_*$ is multiplication by $w(\gamma)$ on $L_n^s(\pi_1(K, x), w)$. Hence, the "isomorphism" of [13, Cor. 9.4.1] could not be functorial.

We will revise the definition of $L_n^1(K)$ by being fussier about orientations and local coefficients. Consider the category $\mathscr{C}$ with objects $\eta = (K', K)$ where η is a principal $\mathbf{Z}_2$-bundle with base space a CW-complex K and total space K'; and a morphism is a $\mathbf{Z}_2$-equivariant map $f' : (K_1', K_1) \to (K_2', K_2)$:

$$\begin{array}{ccc} K_1' & \xrightarrow{f'} & K_2' \\ \downarrow & & \downarrow \\ K_1 & \xrightarrow{f} & K_2 \end{array} \tag{45}$$

We will define functors $L_n^1(\eta)$ from $\mathscr{C}$ to the category of abelian groups. If $\eta \in \mathscr{C}$, denote the associated local **Z**-coefficients on K by $\eta\mathbf{Z}$. We construct $L_n^1(\eta)$ by the same procedure (also notation) Wall uses to define $L_n^1(K)$ (cf. [13, p. 86]) with the following modifications. We insist the orientations $[Y]$, $[N]$ be elements of the explicit groups $H_n(Y, X; \omega^*(\eta)\mathbf{Z})$ and $H_n(N, M; (\omega\phi)^*(\eta)\mathbf{Z})$ respectively. Also, the first Stiefel–Whitney class of η should be the homomorphism $w : \pi_1(K) \to \{\pm 1\}$ in Wall's definition.

Now let $f': \eta_1 \to \eta_2$ be a bundle map where $\eta_i = (K_i', K_i)$ $(i = 1, 2)$, we wish to construct an induced homomorphism $f'_*: L_n^1(\eta_1) \to L_n^1(\eta_2)$. If θ is an "object" (same notation as [13, p. 86]) representing the element $[\theta] \in L_n^1(\eta_1)$, then we set $f'_*[\theta] = [f'_*\theta]$ where $f'_*\theta$ is the new "object" defined by composing the map $\omega: Y \to K_1$ with $f: K_1 \to K_2$ and constructing new orientation classes

$$[Y]' \in H_n(Y, X; (f\omega)^*(\eta_2)\mathbf{Z}),$$
$$[N]' \in H_n(N, M; (f\omega\phi)^*(\eta_2)\mathbf{Z})$$

as follows. We have canonical local coefficient isomorphisms

$$f_1: \omega^*(\eta_1)\mathbf{Z} \to (f\omega)^*(\eta_2)\mathbf{Z},$$
$$f_2: (\omega\phi)^*(\eta_1)\mathbf{Z} \to (f\omega\phi)^*(\eta_2)\mathbf{Z}$$

both determined by f'. These induce maps on homology, and we set $[Y]'$ and $[N]'$ equal to the images of $[Y]$ and $[N]$ respectively under these maps. Clearly, $L_n^1(\eta)$ is a functor.

Let $\eta = (K', K) \in \mathscr{C}$ and $p: \bar{K} \to K$ be a finite sheeted covering space, then we define the transfer homomorphism $p^*: L_n^1(\eta) \to L_n^1(p^*\eta)$ by associating to $[\theta] \in L_n^1(\eta)$ the element in $L_n^1(p^*\eta)$ represented by the pullback "object" $p^*\theta$ defined in the obvious way.

Analogous to the result in [13, p. 87], we have the following lemma.

LEMMA 3.2. *The functor $L_n^1(\eta)$ is a homotopy functor; i.e., if f' and $g': \eta_1 \to \eta_2$ are homotopic through bundle maps, then $f'_* = g'_*$.*

Let $\eta = (K', K) \in \mathscr{C}$; a basepoint x' for η is a pair of points (x', x) such that x' lies over x. When K has a finite, connected 2-skeleton, we wish to define an isomorphism

$$\lambda(x', \ ): L_n^1(\eta) \to L_n^s(\pi_1(K, x), w) \tag{46}$$

(which depends on x'). Let θ be an "object" with respect to η (same notation as [13, p. 86]), we assume that ϕ is k-connected (where $n = 2k$ or $2k+1$); and, for convenience, that N is connected. Choose a base point y' for $(\omega\phi)^*\eta$ such that $(\omega\phi)'(y')$ is in the same component of K' as x', where $(\omega\phi)': (\omega\phi)^*\eta \to \eta$ is the canonical bundle map. We can think of y' as prescribing an orientation to N at y (cf. [13, p. 45]) and therefore determining an algebraic surgery obstruction

$\hat{\theta} \in L_n^s(\pi_1(N, y), w)$. Hence

$$(\omega\phi)_*(\hat{\theta}) \in L_n^s(\pi_1(K, \omega(\phi(y))), w);$$

let γ' be a path in K' from $(\omega\phi)'(y')$ to x' and γ its image in K from $\omega\phi(y)$ to x, then γ determines an isomorphism $\gamma_\#: \pi_1(K, \omega\phi(x)) \to \pi_1(K, x)$ (which commutes with w). We define

$$\lambda(x', [\theta]) = (\gamma_\#)_*((\omega\phi)_*(\hat{\theta})). \tag{47}$$

Let α' be another path from $(\omega\phi)'(y')$ to x', and let α be its image in K. Clearly $w(\alpha * \gamma^{-1}) = 1$. By Lemma 2.5, $((\alpha * \gamma^{-1})_\#)_*$ is multiplication by 1, the identity map; therefore, $\lambda(x', [\theta])$ is independent of the choice of the path γ'. A similar argument shows that $\lambda(x', [\theta])$ is also independent of the choice of y'. Hence, $\lambda(x', [\theta])$ is well defined.

After changing Wall's faulty "map" from $L_n^1(K) \to L_n(\pi_1(K), w)$ to

$$\lambda(x', \ ): L_n^1(\eta) \to L_n^s(\pi_1(K, x), w),$$

we can follow his original argument (cf. [13, Cor. 9.4.1]) to establish the following theorem.

THEOREM 3.3. *For $n \geq 5$, $\lambda(x', \ )$ is an isomorphism.*

If we denote the action of the non-trivial element in $\mathbf{Z}_2$ on η by $x' \to -x'$, then one can show that $\lambda(x', \ ) = -\lambda(-x', \ )$.

Let $\eta_i \in \mathscr{C}$ $(i = 1, 2)$ be such that K_i both have finite, connected 2-skeletons; let $f': \eta_1 \to \eta_2$ be a bundle map and x' a base point for η_1, then the following diagram commutes

$$\begin{tikzcd} L_n^1(\eta_1) \arrow[r, "f'_*"] \arrow[d, "\lambda(x',\,)"] & L_n^1(\eta_2) \arrow[d, "\lambda(f'(x'),\,)"] \\ L_n^s(\pi_1(K_1, x), w) \arrow[r, "(f_\#)_*"] & L_n^s(\pi_1(K_2, f(x)), w). \end{tikzcd}$$

Hence λ is functorial, which resolves the difficulty motivating this digression. The corresponding diagram relating the geometric transfer map to the algebraic restriction (transfer) map via λ also commutes.

Let $\eta = (K', K) \in \mathscr{C}$ and $q: K' \to K$ denote the projection map; assume that K has a finite, connected 2-skeleton. Let $p: \bar{K} \to K$ be a principal G-bundle where G is a finite group. We define a left G-module structure on $L_n^1(p^*\eta)$ as follows.

For each $g \in G$, $x \in \bar{K}$, $y \in K'$ such that $p(x) = q(y)$ define $g'(x, y) = (xg^{-1}, y)$, then $g' : p^*\eta \to p^*\eta$ is a bundle map; if $a \in L_n^1(p^*\eta)$, define $ga = g'_* a$. The isomorphism λ identifies this G-module structure with the algebraic G-module structure of §2 (cf. discussion following Lemma 2.5) in the appropriate way. We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Lemma 2.8 shows that $i^* \otimes id$ is an epimorphism. We shall prove it a monomorphism by an induction the rank of Γ; for rank of $\Gamma = 0$, this is trivial, hence we assume it is true for groups of rank less than n and must prove it for groups of rank n. Let $p : \Gamma \twoheadrightarrow C_\infty$ be an epimorphism such that $\Delta = \operatorname{Ker} p$ has rank $n-1$ where C_∞ denotes the infinite cyclic group; $p(S) = sC_\infty$ for some positive integer s; let $T = p^{-1}(sC_\infty)$, then $S \subseteq T \subseteq \Gamma$ and denote the inclusion map of T into Γ by i_1 and that of S into T by i_2. We now abbreviate our notation by letting $L_k(\Gamma)$ denote $L_k(\Gamma, w) \otimes \mathbf{Z}[1/m]$, i_* denote $i_* \otimes id$, i^* denote $i^* \otimes id$, etc. We wish to prove that both

$$i_1^* : L_k(\Gamma) \to L_k(T),$$

and

$$i_2^* : L_k(T) \to L_k(S) \tag{49}$$

are monic. Our proof, in both cases, is based on the fundamental result of Shaneson–Wall [13, Theorem 12.6].

I. The Homomorphism i_1^* is Monic

It follows from [13, Theorem 12.6] (cf. [5, Theorem 29]) that we have the following exact sequences

$$L_k(\Delta) \xrightarrow{t'} L_k(\Delta) \xrightarrow{j'_*} L_k(T) \xrightarrow{q'} L_{k-1}(\Delta) \tag{50}$$

$$L_k(\Delta) \xrightarrow{t} L_l(\Delta) \xrightarrow{j_*} L_k(\Gamma) \xrightarrow{q} L_{k-1}(\Gamma) \tag{51}$$

where j denotes the inclusion map of Δ into Γ, and j' the inclusion of Δ into T. We can describe t and t' in terms of a generator g for C_∞ by the equations

$$tx = (1-g)x, \qquad t'x = (1-g^s)x \tag{52}$$

where $x \in L_k(\Delta)$ and $1-g$, $1-g^s \in \mathbf{Z}C_\infty$ (cf. §2 for the definition of the action of

C_∞ on $L_k(\Delta)$). The maps q and q' are constructed geometrically. This is why we needed to correct Wall's geometric construction of his L-groups.

We obtain the following commutative diagram of exact sequences by mapping (51) to (50)

$$\begin{tikzcd} L_k(\Delta) \arrow[r, "1-g^s"] & L_k(\Delta) \arrow[r, "j'_*"] & L_k(T) \arrow[r, "q'"] & L_{k-1}(\Delta) \\ L_k(\Delta) \arrow[r, "1-g"] \arrow[u, "1"] & L_k(\Delta) \arrow[r, "j_*"] \arrow[u, "N"] & L_k(\Gamma) \arrow[r, "q"] \arrow[u, "i_1^*"] & L_{k-1}(\Delta) \arrow[u, "1"] \end{tikzcd} \tag{53}$$

where $N=1+g+g^2+\cdots+g^{s-1}$, and the maps denoted 1, $1-g$, $1-g^s$, N are multiplication by these elements. The left square of (53) clearly commutes; since j is the composite of i_1 and j', Lemma 2.7 shows the middle square commutes; and the geometric description of q (and q') can be used to prove the last square also commutes.

By chasing diagram (53), we see that j_* maps $\ker N$ onto $\ker(i_1^*)$. Denote $\ker(1-g^s)$ by M, then it is clear that $\ker N\subseteq M$ and $(1-g)M\subseteq \ker N$; consequently, $\ker(i_1^*)$ is a factor group of $\ker N/(1-g)M$. But M is a C_s-module where C_s denotes the cyclic group of order s "generated" by g. It is a standard result from group chomology [3, p. 251] that $H^1(C_s, M)=\ker N/(1-g)M$, and hence is zero since it is a $\mathbf{Z}[1/m]$-module which is annihilated by s (s divides m); consequently, i_1^* is monic.

II. The homomorphism i_2^* is Monic

Let $\Delta'=\Delta\cap S$ and $i_3:\Delta'\to\Delta$, $\hat{j}:\Delta'\to S$ denote the inclusion maps. We have the following commutative diagram of group extensions

$$\begin{tikzcd} 1 \arrow[r] & \Delta' \arrow[r, "\hat{j}"] \arrow[d, "i_3"] & S \arrow[r, "p"] \arrow[d, "i_2"] & sC_\infty \arrow[r] \arrow[d, "id"] & 1 \\ 1 \arrow[r] & \Delta' \arrow[r, "j'"] & T \arrow[r, "p"] & sC_\infty \arrow[r] & 1 \end{tikzcd} \tag{54}$$

which induces the commutative diagram of exact sequences:

$$\begin{tikzcd} L_k(\Delta') \arrow[r, "1-g^s"] & L_k(\Delta') \arrow[r, "\hat{j}_*"] & L_k(S) \arrow[r, "q''"] & L_{k-1}(\Delta') \\ L_k(\Delta) \arrow[r, "1-g^s"] \arrow[u, "i_3^*"] & L_k(\Delta) \arrow[r, "j'_*"] \arrow[u, "i_3^*"] & L_k(T) \arrow[r, "q'"] \arrow[u, "i_2^*"] & L_{k-1}(\Delta) \arrow[u, "i_3^*"] \end{tikzcd} \tag{55}$$

where q'' (and q') are constructed geometrically. One easily sees the left square of (55) commutes; the middle square commutes by Lemma 2.9, and the right square by a geometric argument. The homomorphism j' induces an isomorphism of factor groups Δ/Δ' to T/S and denote this common group by F.

The top line of (55) is an exact sequence of $\mathbf{Z}[1/m]$ (F)-modules. (Use the geometric description of the F-module structure given earlier in this section to see that q'' is an F-module map.) For these modules, the functor $M \mapsto M^F$ is exact, since $M^F = H^0(F, M)$ and $H^1(F, M) = 0$ (the order of F divides m). Applying this functor to the top line of (55), we obtain a new commutative diagram

$$\begin{tikzcd} L_k(\Delta')^F \arrow[r] & L_k(\Delta')^F \arrow[r] & L_k(S)^F \arrow[r] & L_{k-1}(\Delta')^F \arrow[r] & L_{k-1}(\Delta')^F \\ L_k(\Delta) \arrow[r] \arrow[u, "i_3^*"'] & L_k(\Delta) \arrow[r] \arrow[u, "i_3^*"'] & L_k(T) \arrow[r] \arrow[u, "i_2^*"'] & L_{k-1}(\Delta) \arrow[r] \arrow[u, "i_3^*"'] & L_{k-1}(\Delta) \arrow[u, "i_3^*"'] \end{tikzcd} \tag{56}$$

whose rows are exact. By induction and the five lemma, we have i_2^* is an isomorphism. This completes the proof of the theorem.

4. Proof of the main theorem

We need some lemmas for our proof of the main theorem.

LEMMA 4.1. *Let Γ be a torsion-free group and S a normal subgroup which is free abelian of finite rank. If the factor group Γ/S is finite cyclic then Γ is a poly-$\mathbf{Z}$ group of finite rank.*

Proof. Let us prove that there is a non-trivial homomorphism $f:\Gamma \to \mathbf{Z}$ (the additive group of integers). Then the lemma follows from this fact and an induction on the rank of S. Let γS generate Γ/S and $m = \text{order } \Gamma/S$, then γ^m is a non-trivial element in the center of Γ. Hence, we can find an element $e_1 \in S \cap \text{center } \Gamma$ which is indivisible in S and we can choose extra elements $e_2, e_3, \ldots, e_n$ such that $e_1, e_2, \ldots, e_n$ form a basis of S. Define $g: S \to \mathbf{Z}$ by

$$g(e_1) = \begin{cases} 1 & \text{if} \quad i = 1, \\ 0 & \text{if} \quad i \neq 1. \end{cases} \tag{57}$$

Averaging g over Γ/S, we have a homomorphism $g': S \to \mathbf{Z}$ given by

$$g'(x) = \sum_{\gamma S \in \Gamma/S} g(\gamma x \gamma^{-1}); \tag{58}$$

clearly g' has the following two properties:

(i) $g'(e_1) = m \neq 0$,
(ii) g' restricted to $[\Gamma, S]$ is identically zero.

Now, from the identity

$$[u, vw] = [u, v][u, w][[w, u], v] \tag{59}$$

where $[u, v] = uvu^{-1}v^{-1}$ (cf. [6, p. 49, formula (5)]) together with the fact that Γ/S is cyclic, one easily deduces that $[\Gamma, \Gamma] = [\Gamma, S]$. Define $f : \Gamma \to \mathbf{Z}$ by $f(x) = g'(x^m)$. Since $x^m y^m (xy)^{-m} \in [\Gamma, \Gamma] = [\Gamma, S]$, property (ii) implies that f is a homomorphism and (i) implies that f is non-trivial. This proves the lemma.

Let Γ be a torsion-free group, S a normal subgroup which is free abelian of rank n, and $G = \Gamma/S$ a finite group. Let $w : \Gamma \to \{\pm 1\}$ be a homomorphism; we assume that S is in its kernel. (If not, replace S with $S' = S \cap \ker w$ and G with $G' = \Gamma/S'$.) We have a right Γ-structure on $\mathbf{Q}$ defined by the equation

$$x\gamma = w(\gamma)x \quad \text{for} \quad x \in \mathbf{Q}, \qquad \gamma \in \Gamma \tag{60}$$

and denote this module by $w\mathbf{Q}$. Consider the Eilenberg–MacLane space $K(S, 1)$ which we can take to be the n-torus T^n with base point $*$. Recall the outer automorphism c_γ of S defined by $c_\gamma(s) = \gamma s \gamma^{-1}$ where $\gamma \in \Gamma$ and $s \in S$. This automorphism depends only on the coset $g = \gamma S \in G$. Consider the function $g' : (T^n, *) \to (T^n, *)$ (unique up to homotopy) such that

$$g'_{\#} : \pi_1(T^n, *) \to \pi_1(T^n, *)$$

is c_γ. Define a left G-module structure on $H_j(T^n, \mathbf{Q})$ by $gx = w(\gamma)g'_*(x)$ where $x \in H_j(T^n, \mathbf{Q})$ and $g = \gamma S$. It follows from the Lyndon–Hochschild–Serre spectral sequence that we have the following lemma.

LEMMA 4.2. *For all j, $H_j(\Gamma, w\mathbf{Q}) \simeq H_j(T^n, \mathbf{Q})^G$.*

Of course, we have $H_j(M^n, w\mathbf{Q}) = H_j(\Gamma, w\mathbf{Q})$ for the manifold M^n of the Main Theorem.

Now consider Wall's map

$$l_S : \bigoplus_{j=0}^{\infty} H_{i+4j}(T^n, \mathbf{Q}) \to L_i^s(S, w) \otimes \mathbf{Q} \tag{61}$$

$(i = 0, 1, 2, 3)$ [13, p. 266]; one easily sees that l_S is a G-module map. The next lemma follows from [13, p. 267].

LEMMA 4.3. *The map*

$$l_S : \bigoplus_{j=0}^{\infty} H_{i+4j}(T^n, \mathbf{Q})^G \to (L_i^s(S, w) \otimes \mathbf{Q})^G$$

is an isomorphism $(i = 0, 1, 2, 3)$.

Now, the Main Theorem follows directly from these three lemmas together with Lemma 2.8, Corollary 2.4, and Theorem 3.1.

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