A Poncelet theorem in space.

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A Poncelet theorem in space

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One of the most important and also most beautiful theorems in classical projective geometry is that of Poncelet concerning closed polygons which are inscribed in one conic and circumscribed about another.[(1)] His proof was synthetic and somewhat elaborate in what was to become the predominant style in projective geometry of the last century. Slightly thereafter, Jacobi gave another argument based on the addition theorem[(2)] for elliptic functions. In fact, as will be seen below, the Poncelet theorem and addition theorem are essentially equivalent, so that at least in principle Poncelet gave a synthetic derivation of the group law on an elliptic curve.

Because of the appeal of the Poncelet theorem it seems reasonable to look for higher-dimensional analogues. Another motivation comes from trying to understand "addition theorems" pertaining to higher codimension.[(3)] Although this has not yet turned out to be the case in the Poncelet-type of problem, what did turn up is a class of closed polyhedra in three space which are both *inscribed in* and *circumscribed about* a pair of quadric surfaces in general position. These figures are thus even more symmetric than those arising from the classical Poncelet construction, and the object of this paper is to show how again they may be constructed analytically from elementary properties of the group law on an elliptic curve.

In Section 1 we shall discuss the classical Poncelet theorem in the plane in such a way that similar considerations will apply to the analogous problem in space. In Section 2 we prove a Poncelet-type theorem for the polyhedra in space,

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[1] J. V. Poncelet, *Traité des propriétés projectives des figures*, Mett-Paris 1822.

[2] C. G. J. Jacobi, *Über die Anwendung der elliptischen Transcendenten auf ein bekanntes Problem der Elementargeometrie*, Gesammelte Werke, Vol. I (1881), pp. 278–293.

[3] P. Griffiths, *Variations on a theorem of Abel*, Inventiones Math. Vol. 35 (1976), ppg. 321–390. This paper also contains a variant of Jacobi's discussion of the classical Poncelet problem.

and discuss some matters concerning actually constructing the figure which are obvious in the plane case. Because of the elementary character of the question we have attempted to keep the discussion reasonably self-contained in the hope that the proofs can be understood by nonspecialists in algebraic geometry.

It is a great pleasure to thank Robert Steinberg for helping us past a preliminary false version of our theorem and providing an important observation which appears in Section 2 below.

1. The classical Poncelet theorem

This deals with the following question: Given two smooth conics C, C' in the plane $\mathbb{R}^2$ with C lying inside C', when does there exist a closed polygon[4] inscribed in C' and circumscribed about C? Rephrased in terms of a construction the problem goes as follows: We first note, as is clear from Fig. 1 below, that through every point of C' there will pass two tangent lines to C and each of these lines will meet C' in two points. Beginning with a point $P_0 \in C'$ and tangent line L_0 to C passing through P_0, we let L_1 be the other tangent line through P_0 meeting C' in P_1, L_2 the other tangent line through P_1 meeting C' in P_2, and so forth. Let $\Pi_{(P_0, L_0)}$ denote the configuration of lines $\{L_i\}_{i=0,1,\ldots}$ obtained in this way. It is clear that any closed polygon inscribed in C' and circumscribed about C which contains P_0 as vertex must be the configuration $\Pi_{(P_0, L_0)}$. Consequently, our question is equivalent to asking when $\Pi_{(P_0, L_0)}$ is finite—i.e., when does $L_n = L_0$ for some n? Although it seems difficult to write down an explicit general answer to this problem,† Poncelet did succeed in proving the beautiful and remarkable

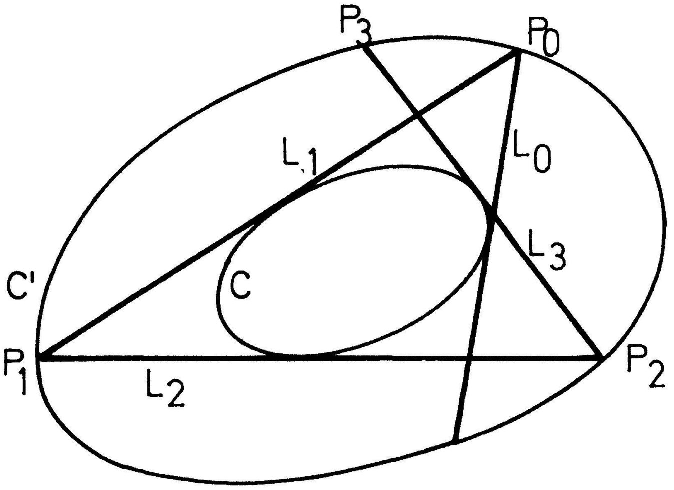

Figure 1

[4] We shall allow the polygon to have self-intersections.

† c.f. footnote [12] at the end of this paper.

THEOREM. *For any pair of ellipses as above, there exists a closed polygon inscribed in C' and circumscribed about C if, and only if, there exist infinitely many such polygons, one with a vertex at any given point $P \in C'$.*

In other words,

The condition that the configuration $\Pi_{(P_0, L_0)}$ be finite is independent of the starting point (P_0, L_0).

As mentioned before, the original proof by Poncelet was synthetic, and we shall follow the lead of Jacobi and give an analytic discussion of the question based on elliptic functions. Consequently, we consider the problem in the projective plane $\mathbb{P}^2$ over the complex numbers. Let $x = [x_0, x_1, x_2]$ be homogeneous coordinates and consider a plane conic C given by a quadratic equation

$$(Qx, x) = \sum_{i,j} q_{ij} x_i x_j = 0,$$

where the coefficient matrix $Q = (q_{ij})$ is symmetric. The condition that the algebraic curve C be smooth is that the matrix Q be nonsingular. In this case C is a complex submanifold of $\mathbb{P}^2$, and therefore defines a Riemann surface.

Since C has degree 2, every line L in $\mathbb{P}^2$ meets this conic in two points counting multiplicities. Taking a fixed point $P_0 \in C$ and line L_0 not containing P_0, the stereographic projection mapping $P \to Q$ establishes a biholomorphic mapping $C \to \mathbb{P}^1$ between C and the line L_0 (c.f. Fig. 2).

At a point $P \in C$ with coordinates x the projective tangent line $T_P(C)$ to C at

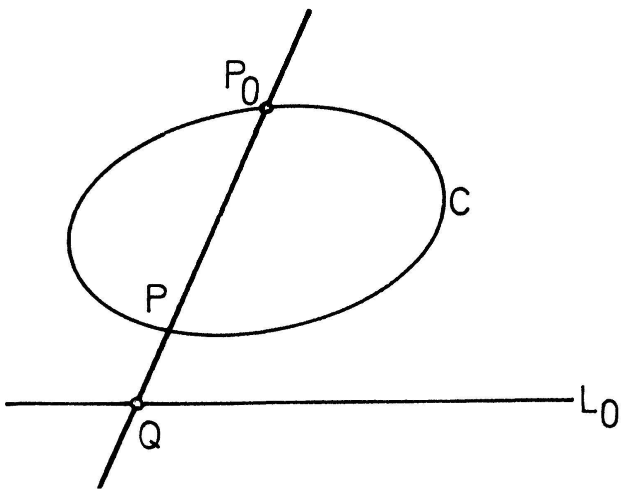

Figure 2

P is given by the equation

$$0=\tfrac{1}{2}\sum \frac{\partial}{\partial x_j}(Qx, x)y_j=\sum_j\left(\sum_i x_i q_{ji}\right)y_j=(Qx, y).$$

Consequently, in terms of the dual coordinates $\xi=[\xi_0, \xi_1, \xi_2]$ on the dual projective space $\mathbb{P}^{2*}$ of lines in $\mathbb{P}^2$, the *Gauss mapping* $P\to T_P(C)$ is given by $\xi=Qx$. It follows that the *dual curve* $C^*\subset\mathbb{P}^{2*}$ of tangent lines to C is again a smooth conic with equation $(Q^{-1}\xi, \xi)=0$, where Q^{-1} is the inverse matrix to Q. Dualizing Fig. 2, we find that through each point $P\in\mathbb{P}^2$ there pass two tangent lines to C, again

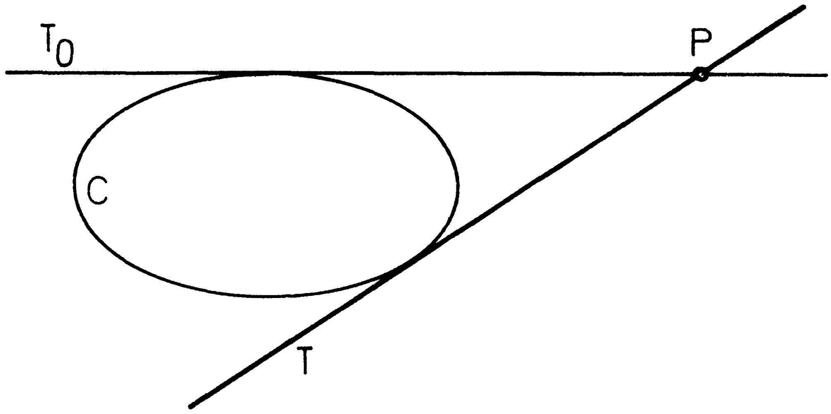

Figure 3

counting multiplicities, and C^* may be rationally parametrized by the mapping $P\to T$, where P varies on a fixed tangent line T_0 and T is the other tangent to C passing through P as depicted in Fig. 3.

Now let C' be another conic in the plane $\mathbb{P}^2$ and assume that C and C' are nowhere tangent, so that they meet transversely at four points. Then the same will be true of the dual curves C^* and C'^* in $\mathbb{P}^{2*}$, and in the product $C'\times C^*\simeq\mathbb{P}^1\times P^1$ we consider the *incidence correspondence* $E=\{(P, L): P\in L\}$ of points $P\in C'$ and tangent lines L to C such that $P\in L$. It is clear that E is the basic variety underlying the Poncelet construction. E is an algebraic curve in $\mathbb{P}^1\times\mathbb{P}^1$; the transversality of C and C' assures that for every point $(P, T)\in E$ at least one of the coordinate axes $\{P\}\times C^*$ and $C'\times\{T\}$ through (P, T) meets E in two distinct points, from which it follows that E is nonsingular. The *adjunction formula* for curves on an algebraic surface then gives that the *genus* of E is one.

These facts will also come out directly during the proof of the Poncelet theorem. Namely, the construction in Fig. 1 suggests considering the pair of *involutions*

$$i: E\to E \quad\text{and}\quad i': E\to E$$

defined respectively by

$$i(P, L) = (P, L') \quad \text{and} \quad i'(P, L) = (P', L)$$

where P' is the residual intersection of L with C' and L' is the other tangent to C through P as depicted in Fig. 4. In terms of the notation in Fig. 1,

$$i(P_i, L_i) = (P_i, L_{i+1}), \qquad i'(P_i, L_{i+1}) = (P_{i+1}, L_{i+1}),$$

so that the composition $j = i' \circ i$ satisfies $j(P_i, L_i) = (P_{i+1}, L_{i+1})$. Consequently, $j^n(P_0, L_0) = (P_n, L_n)$, from which we conclude that:

The configuration $\Pi_{(P_0, L_0)}$ will be finite exactly when $j^n(P_0, L_0) = (P_0, L_0)$ for some integer n.

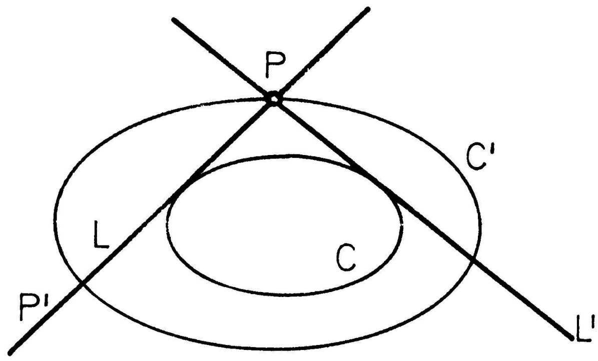

Figure 4

Evidently now our problem is equivalent to looking for fixed points of the automorphism j^n on the curve E, to which we now turn. Taking the quotient of E by the first involution i is the same as considering the map $E \to C^*$ given by $(P, L) \to L$. This realizes E as a 2-sheeted covering of the Riemann sphere $\mathbb{P}^1$. The branch points correspond to the fixed points of i, and these occur for the lines L which are tangent to C' as well as to C, i.e., to the four distinct points in $C'^* \cap C^*$. It is now clear that E is a nonsingular Riemann surface with Euler characteristic $\chi(E) = 2\chi(\mathbb{P}^1) - 4 = 0$, and, consequently, E has genus 1.

The basic deep fact underlying Poncelet is that there exists on E a nonvanishing holomorphic differential ω such that the map

$$u(p) = \int_{p_0}^{p} \omega$$

gives an analytic isomorphism of Riemann surfaces $E \to \mathbb{C}/\Lambda$ *where* Λ *is the period lattice of* ω.

Now the involutions i and i' on E are induced by automorphisms $\tilde{\imath}$ and $\tilde{\imath}'$ on the universal covering $\mathbb{C}$ of E; we can write

$$\tilde{\imath}(u) = \alpha u + \tau, \qquad \tilde{\imath}'(u) = \alpha' u + \tau'.$$

Then, since $i^2 = 0$, we have $\tilde{\imath}^2(u) = \alpha^2 u + (\alpha + 1)\tau \equiv u + \lambda$ for some $\lambda \in \Lambda$. Thus $\alpha = \pm 1$, and in fact $\alpha = -1$ since otherwise i would have no fixed points in contradiction to Fig. 5. It follows that $\tilde{\jmath}^n(u) = u + n(\tau' - \tau)$, and hence $j^n(P, L) = (P, L)$ if, and only if, $n(\tau' - \tau) \in \Lambda$. This condition is clearly independent of $(P, L) \in E$ and so the Poncelet theorem is proved.

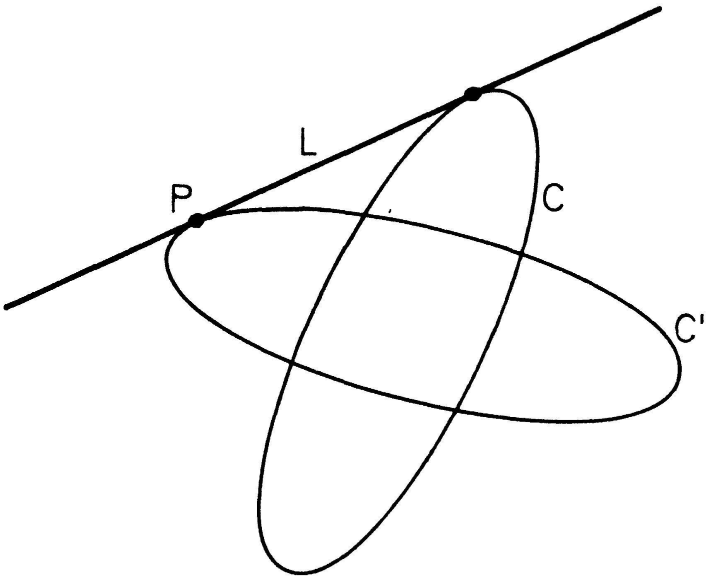

Figure 5

We want to discuss this condition in more detail. Given our plane conics C and C', by linear algebra we may change coordinates so that

$$C = \left\{ \sum_{i=0}^{2} x_i^2 = 0 \right\}, \qquad C' = \left\{ \sum_{i=0}^{2} \beta_i x_i^2 = 0 \right\},$$

where $\beta = [\beta_0, \beta_1, \beta_2]$ is a point in $\mathbb{P}^2_{(\beta)}$. The resulting map $\mathbb{P}^2_{(\beta)} \to$ {pairs of conics} is generally finite, and those pairs of conics for which $n(\tau' - \tau) \in \Lambda$ determine an algebraic curve $V_n \subset \mathbb{P}^2_{(\beta)}$. In fact, it is clear that V_n is locally determined by one analytic condition on β, and also that V_n is globally of an algebraic character.

Consequently:

The Poncelet condition holds for pairs of conics corresponding to a countable set of curves in the parameter space $\mathbb{P}^2_{(\beta)}$.

2. Poncelet theorem for quadrics in space

Following the Poncelet theorem in the plane it seems natural to look for polyhedra in $\mathbb{R}^3$ which are inscribed in one surface and circumscribed about another. At first glance this doesn't seem to work, since there are in general ∞^1 tangent planes to an algebraic surface passing through a point in space. However, upon closer inspection it is possible to generate polyhedra from a pair of quadric surfaces, and the question of whether or not we obtain a finite figure turns out to again repose on elementary properties of elliptic curves. We now describe how this goes.

Let $x=[x_0, x_1, x_2, x_3]$ be homogeneous coordinates in complex projective space $\mathbb{P}^3$. A quadric surface S is defined by

$$(Qx, x)=\sum_{i,j} q_{ij}x_ix_j=0,$$

where $Q=(q_{ij})$ is a symmetric matrix. S is nonsingular exactly when $\det Q\neq 0$, and in this case S is a complex submanifold in $\mathbb{P}^3$. As before, for $P\in S$ with coordinates x the projective tangent plane $T_P(S)$ to S at P is given by the equation $(Qx, y)=0$. Denoting by $\xi=[\xi_0, \xi_1, \xi_2, \xi_3]$ the dual coordinates in the dual projective space $\mathbb{P}^{3*}$ of planes $\{\sum_{i=0}^3 \xi_i x_i=0\}$ in $\mathbb{P}^3$, the *Gauss mapping* of S to its dual surface S^* of tangent planes to S is given by $\xi=Qx$. Consequently, S^* is a nonsingular quadric surface with equation $(Q^{-1}\xi, \xi)=0$.

If S and S' are two quadric surfaces meeting transversely, then the same is true of the dual surfaces S^* and S'^*.[5] In this case the intersections

$$C=S\cap S', \qquad E=S^*\cap S'^*$$

are nonsingular algebraic curves. Geometrically, E is the set of *bitangent planes* to the pair of surfaces S and S'.

Especially noteworthy about a smooth quadric surface S are the lines contained in it. These may be described in the following three steps:

(i) Since S has degree 2, any line meeting S in three or more points must lie

[5] This may be verified analytically using the equations of S and S' given below.

entirely in the surface. Therefore, if Q is any point in the intersection $S \cap T_P(S)$ of S with its tangent plane at P, the line PQ will lie in this intersection. It follows that:

The intersection $S \cap T_P(S)$ consists of two straight lines; any line lying in S and passing through P must be one of these.[6]

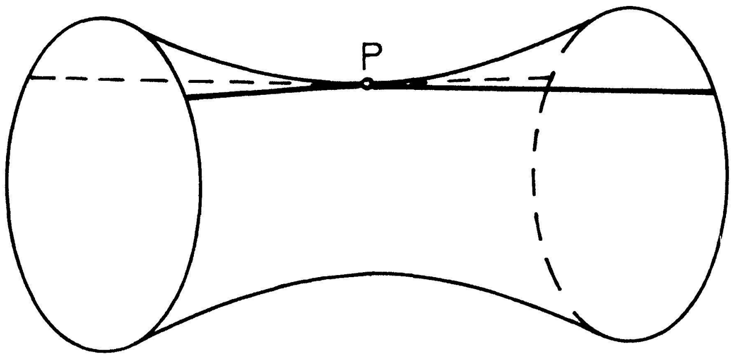

Figure 6

(ii) To describe all lines lying in S, we pick one such L_0, and call any line L an A-line if it is either equal to or disjoint from L_0 and a *B-line* if L meets L_0 in one point. In this case L and L_0 span a 2-plane $L \wedge L_0$.

If two lines $L, L' \neq L_0$ meet in a point, then the plane they span must meet L_0 in a point, and so one of the two is an A-line and the other a B-line.[7] Conversely, if L is an A-line and L' a B-line, then the plane $L_0 \wedge L'$ will meet L in a point which must therefore be a point on $L \cap L'$. In summary,

The lines on S fall into two disjoint families, the A-lines and B-lines. Each A-line L meets each B-line L' once at a point $P = L \cap L'$ whose tangent plane $T_P(S)$ is $L \wedge L'$. Two distinct A-lines or B-lines fail to meet.

Since any B-line meets L_0 once, it follows that the A-lines and B-lines are each parametrized by $\mathbb{P}^1$. In this way S is a *doubly ruled surface,* i.e., it is the surface traced out by ∞^1 lines in two distinct ways. This also shows that, as a complex manifold, $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

(iii) For any line $L \subset \mathbb{P}^3$ the set of all planes containing L is a line L^* in the dual projective space $\mathbb{P}^{3*}$. The line L lies on S if, and only if, L^* lies on S^*. If this

[6] We note that $S \cap T_P(S)$ is a plane conic containing a line, and hence must be two lines through P as in Fig. 6. These lines are distinct since $\det Q \neq 0$.

[7] Since $S \cap (L \wedge L')$ contains and therefore is equal to the plane conic $L + L'$, it follows that $L \wedge L'$ is the tangent plane to S at $L \cap L'$. In particular, L_0 does not lie in $L \wedge L'$ nor pass through $L \cap L'$.

is not the case, then L^* will meet S^* in two points. Thus:

Through a line $L \subset \mathbb{P}^3$ not lying on S there pass exactly two tangent planes to S. Any plane T containing a line $L \subset S$ is tangent to S somewhere along L.

Now we can describe the analogue of the Poncelet construction relative to a pair of smooth quadric surfaces S and S' intersecting transversely in $\mathbb{P}^3$.[8] Letting $E = S^* \cap S'^*$ be the curve of bitangent planes, each plane $T \in E$ will meet $S \cup S'$ in a figure of the sort consisting of the pair of lines L_A, L_B on S, together with a pair of lines L'_A, L'_B on S'. This is by (i) above. The four lines are distinct since S and S' meet transversely, and $P = L_A \cap L_B$, $P' = L'_A \cap L'_B$ are the points where T is tangent to S, S', respectively.

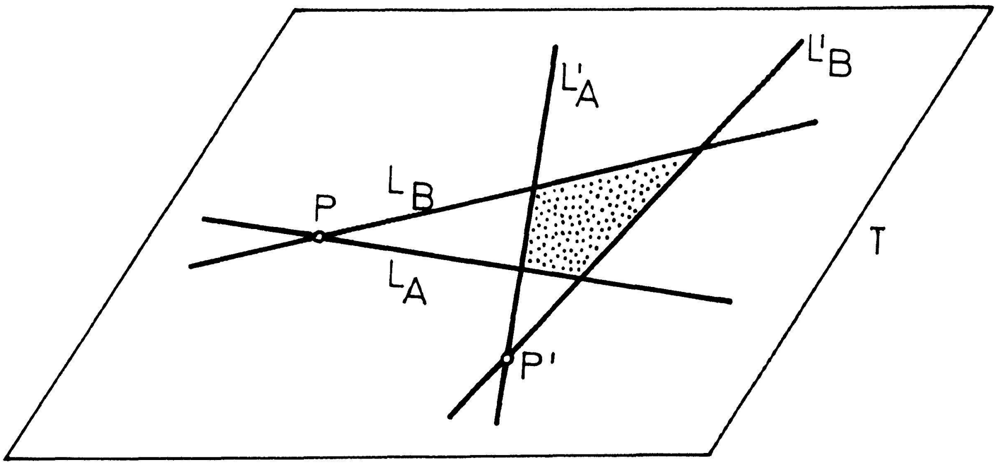

Figure 7

According to (iii) there will be one plane $\tilde{T}$ other than T passing through L_A and tangent to S', and by (iii) again this plane will be tangent to S somewhere along L_A. The picture of T, together with $\tilde{T}$, is something like that shown in Fig. 8. There we are letting

$$\begin{cases} T \cap S = L_A + L_B, & T \cap S' = L'_A + L'_B \\ \tilde{T} \cap S = \tilde{L}_A + \tilde{L}_B, & \tilde{T} \cap S' = \tilde{L}'_A + \tilde{L}'_B \\ \tilde{L}_A = L_A \end{cases}$$

It is important to note that by (ii) the line L'_B meets $L_A = \tilde{L}_A$ in the same point as $\tilde{L}'_A$, etc., so that when flattened out the shaded regions form a configuration as shown in Fig. 9. We set $\tilde{T} = i_A(T)$, and in this way define four involutions

[8] We will comment on the construction in $\mathbb{R}^3$ at the end of the paper.

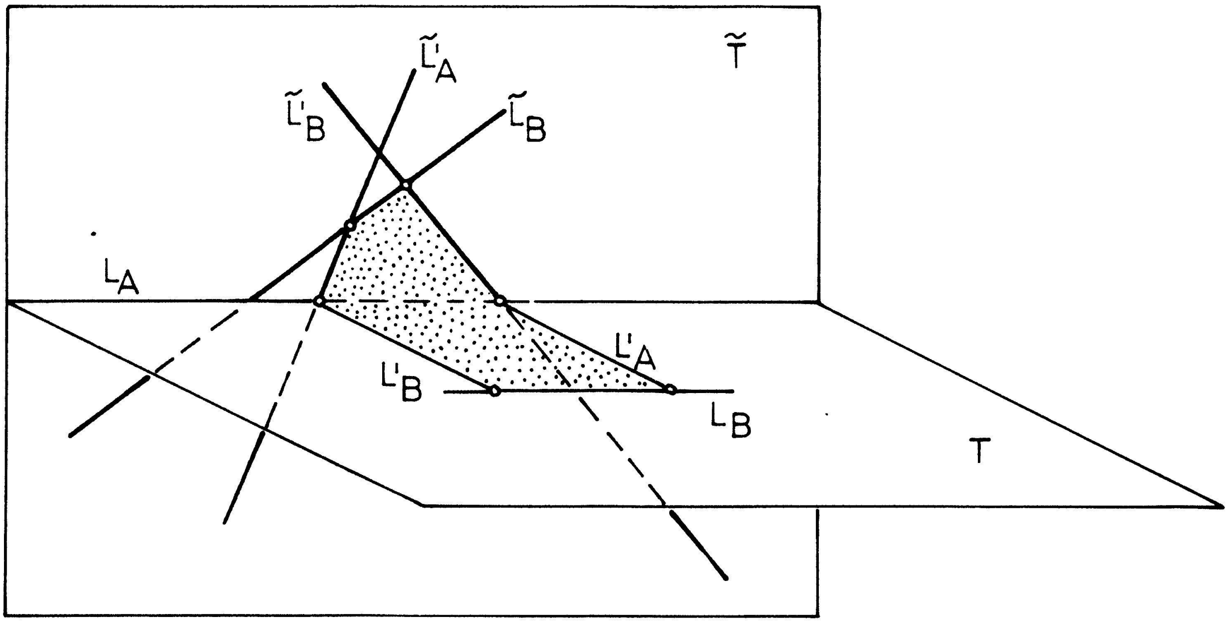

Figure 8

i_A, i_B, i'_A, i'_B on the curve E. We may think of these involutions as ***reflections*** in the four sides of the shaded quadrilateral in Fig. 7. If we begin with a fixed bitangent plane $T_0 \in E$ and successively apply all these reflections we generate a polyhedral figure $\Pi(T_0)$.

What makes this configuration so remarkable is that the polyhedron is both inscribed in and circumscribed about the pair of quadric surfaces S and S'. More precisely,

The planes $T \in \Pi(T_0)$ are all tangent to both S and S', and the vertices of $\Pi(T_0)$ are all points lying on $S \cap S'$.[9] *The edges of $\Pi(T_0)$ are lines lying alternately on S and S'.*

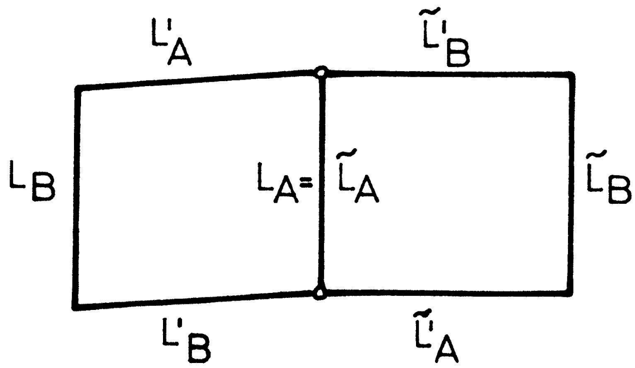

Figure 9

Naturally we now ask whether or not our configuration may be finite. By analogy with Poncelet in the plane we shall prove the

THEOREM. *For S and S' smooth quadric surfaces meeting transversely in $\mathbb{P}^3$, there exists a finite polyhedron both inscribing and circumscribing S and S' if, and only if, there exist infinitely many such.*

In other words, the question of whether or not $\Pi(T_0)$ is finite is independent of the initial bitangent plane T_0. Contrary to the plane case, it is not immediately apparent that there exist S and S' generating finite polyhedra, especially since at first glance this seems to impose three conditions on the 3-parameter family of pairs S, S'. This important point will be discussed following the proof of the theorem.

The proof is essentially the same as before. The four involutions generate a subgroup $G \subset \operatorname{Aut}(E)$ of the automorphism group of the Riemann surface E, and we are asking whether or not the *orbit* $G \cdot \{T_0\}$ is finite. If we apply the adjunction formula[(10)] to each of the two inclusions $E \subset S^* \subset \mathbb{P}^{3*}$, we find that the canonical bundle of E is trivial. Hence the genus of E is 1 and the argument proceeds as in the plane case.

For later use, it is of interest to prove this in a more elementary fashion. Instead of E we consider the curve $C = S \cap S'$ and the mapping $\pi_A : C \to \mathbb{P}^1$ sending a point $P \in C$ to the A-line L on S which contains P. Since L meets S' in two points, π_A is a 2-sheeted covering, and branching occurs exactly over those A-lines $L \subset S$ which are tangent to S'. To show that the genus of C is 1, it will suffice to prove that there are four such branch points. Dualizing, we then obtain that the genus of E is 1.

Now the other projection $\pi_B : C \to \mathbb{P}^1$ given by sending $P \in C$ to the B-line on S containing it is again a 2-sheeted covering, having by symmetry the same number of branch points as π_A. Thus, it will suffice to prove that there are exactly eight lines on S which are tangent to S'.

If a line $L \subset S$ is tangent to S' at P, then $P \in S \cap S'$ and $T_P(S') \in S^*$ by the second statement in (iii) above. Conversely, if $P \in S \cap S'$ and $T_P(S') = T_{P'}(S)$ for some $P' \in S$, then the line $\overrightarrow{P'P}$ lies in S and is tangent to S'. If $P \in S \cap S'$ has

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[9] The vertices are where three or more planes meet. In Fig. 8 these are the vertices of the shaded quadrilaterals.

[10] This is the formula

$$K_D = K_M \otimes [D]_D$$

for the canonical line bundle of a smooth divisor D on a complex manifold M.

coordinates x, then $T_P(S') \in S^*$ if, and only if, $Q'x = Qy$ for some y satisfying $(Qy, y) = 0$. This is the case if, and only if, $(Q'x, Q^{-1}Q'x) = 0$. It follows that the points $P \in S \cap S'$ for which $T_P(S') \in S^*$ are defined by

$$(Qx, x) = 0, \qquad (Q'x, x) = 0, \qquad (Q'x, Q^{-1}Q'x) = 0.$$

These are three quadrics meeting transversely[11] in $2 \cdot 2 \cdot 2 = 8$ points, which is what we wished to prove.

Returning to our curve E, the genus is 1 and the quotient by any of the four involutions is $\mathbb{P}^1$. Representing $E \simeq \mathbb{C}/\Lambda$ as before, we can write

$$i_A(u) = -u + \tau_1, \qquad i_B(u) = -u + \tau_2, \qquad i'_A(u) = -u + \tau_3, \qquad i'_B(u) = -u + \tau_4.$$

It is then clear that the orbit $G \cdot \{T_0\}$ is finite exactly when the differences

$$\tau_i - \tau_j \in \mathbb{Q} \cdot \Lambda. \tag{1}$$

Since this condition is independent of T_0, we conclude the proof of our theorem as before. Q.E.D.

We now discuss the conditions on S and S' which will insure that our polyhedron is finite. Although (1) seems to impose three conditions on the pair S, S', we shall prove the relation

$$i'_B \circ i_B \circ i'_A \circ i_A = \text{identity}, \tag{2}$$

or, equivalently,

$$\tau_1 + \tau_2 - \tau_3 - \tau_4 \in \Lambda, \tag{3}$$

which shows that (1) contains only two conditions. In fact, using (3) we may replace (1) by

$$\tau_1 - \tau_2 \in \frac{1}{m}\Lambda, \qquad \tau_3 - \tau_4 \in \frac{1}{n}\Lambda$$

for integers m, n; or, equivalently,

$$(i_A \circ i_B)^m = \text{identity} \tag{4}$$

$$(i'_A \circ i'_B)^n = \text{identity} \tag{4'}$$

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[11] That the intersection is transverse follows from the equations for S and S' given below.

To prove (2) we enlarge Fig. 9 to contain four quadrilaterals:

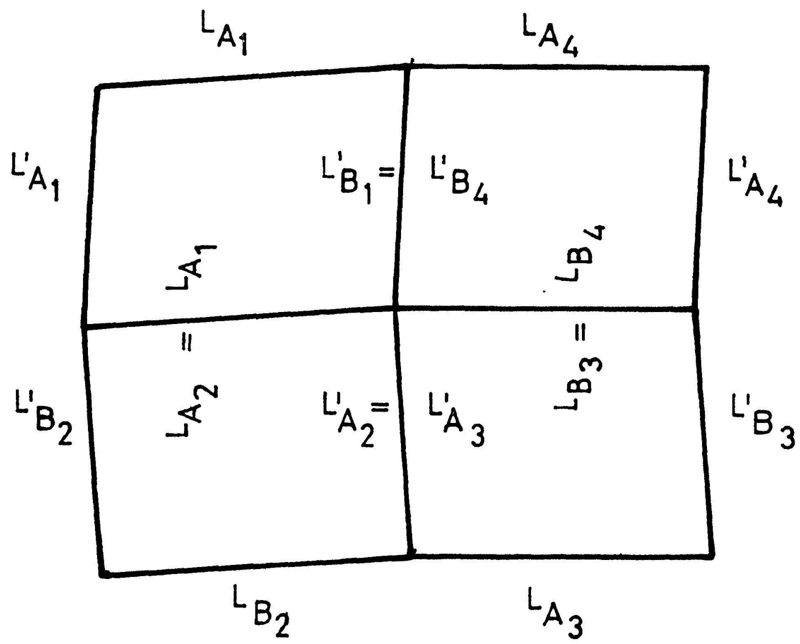

Figure 10

Each quadrilateral represents a different plane, all flattened out as in Fig. 9 to aid in visualization. The diagram of involutions is

$$\begin{tikzcd} \cdot \arrow[d, "i_A"'] & \cdot \arrow[l, "i_{B'}"'] \\ \cdot \arrow[r, "i_{A'}"'] & \cdot \arrow[u, "i_B"'] \end{tikzcd}$$

and (2) is equivalent to $L'_{B_1}=L'_{B_4}$. This, in turn, follows from the observation that if an A-line L_{A_1} and B-line L_{B_4} meet in a point $P\in S\cap S'$, then there is a unique B-line $L'_{B_1}=L'_{B_4}$ on S' passing through P.

To show that the condition (4) is nonvacuous, we use the following argument shown to us by Robert Steinberg. First, by simultaneously diagonalizing the pair of quadratic forms Q and Q', we may choose coordinates such that

$$S=\{x_0^2+x_1^2+x_2^2+x_3^2=0\}$$

and

$$S'=\{\beta_0x_0^2+\beta_1x_1^2+\beta_2x_2^2+\beta_3x_3^2=0\}.$$

The transversality condition is $\beta_i/\beta_j\neq 1$ $(i\neq j)$. Take $P=[0,0,0,1]$, H to be the hyperplane $x_3=0$, and $\pi:\mathbb{P}^3-\{P\}\to H$ given by $\pi(x)=[x_0/x_3, x_1/x_3, x_2/x_3]$ to be the linear projection. Set $C=S\cap H$, so that $C=\{x_0^2+x_1^2+x_2^2=0\}$. Let $C'=\pi(S\cap S')$, which is a curve with equation

$$C'=\{(\beta_0-\beta_3)x_0^2+(\beta_1-\beta_3)x_1^2+(\beta_2-\beta_3)x_2^2=0\}$$

obtained by eliminating the term containing x_3 from the equations of S and S'. Suppose that T_0 is a fixed bitangent plane and consider the sequence

$$T_{2n} = (i'_B \circ i'_A)^n T_0, \qquad T_{2n+1} = i'_A (i'_B \circ i'_A)^n T_0.$$

Letting $L_{2n} = L_{T_{2n,A}}$ and $L_{2n+1} = L_{T_{2n+1,B}}$, we obtain from Fig. 9 the picture

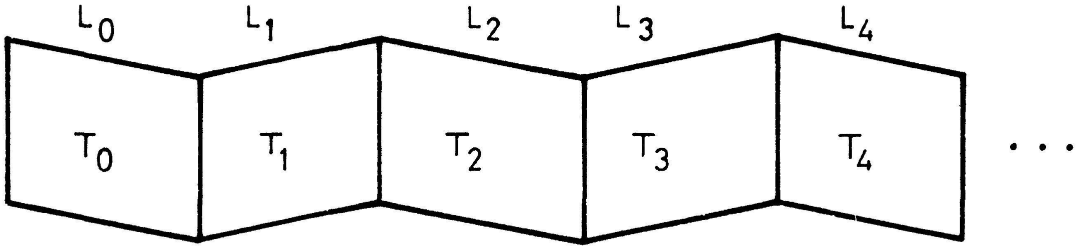

Figure 11

Each line L_i meets C at a point P_i and $L_i \subset T_{P_i}(S)$, from which it follows that $\pi(L_i)$ is tangent to C. Each intersection $L_i \cap L_{i+1}$ lies on $S \cap S'$, so that $\pi(L_i) \cap \pi(L_{i+1})$ is a point on C'. Summarizing:

The sequence $\pi(L_i)$ of lines in H forms a polygon inscribed in C' and circumscribed about C.

Now there are at most two lines in S lying over any line in H and at most two bitangent planes to S and S' containing any line of S. It follows that

$(i'_A \circ i'_B)^n$ = identity for some n if, and only if, the plane conics

$$\sum_{i=0}^{2} x_i^2 = 0 \qquad \sum_{i=0}^{2} (\beta_i - \beta_3) x_i^2 = 0 \tag{5}$$

satisfy the classical Poncelet condition. Similarly $(i_A \circ i_B)^m$ = identity for some m if, and only if, the plane conics

$$\sum_{i=0}^{2} \beta_i x_i^2 = 0 \qquad \sum_{i=0}^{2} (\beta_i - \beta_3) x_i^2 = 0 \tag{5'}$$

satisfy the classical Poncelet condition.

In particular, the conditions (4) and (4') are each satisfied on a countable union of algebraic surfaces in the parameter space $\mathbb{P}^3(\beta)$ for paris of quadric surfaces.

It remains to check that the conditions (4) and (4') are not mutually exclusive, i.e., the hypersurfaces they define meet somewhere in a pair of nonsingular quadrics in general position. For this we call S and S' *symmetric* if there exists a linear transformation of $\mathbb{P}^3$ carrying S to S' and S' to S. If S and S' are given by the above equations, then the transformation $y_i = \sqrt{\beta_i}\, x_i$ takes S' to S and S to the quadric $\sum_{i=0}^{3} \beta_i^{-1} y_i^2 = 0$. Consequently, S and S' are symmetric in case the sets of complex numbers $\{\beta_i\} = \{\lambda \beta_j^{-1}\}$ for some $\lambda \neq 0$, i.e., if $\beta_i = \beta_j \beta_k / \beta_l$ for some permutation (i, j, k, l) of $(0, 1, 2, 3)$. In particular, the symmetric pairs of quadrics form a hypersurface in the $\mathbb{P}^3(\beta)$ of all pairs of quadrics, and moreover, there exist symmetric pairs in general position.

Now let P, H, and $\pi : \mathbb{P}^3 - \{P\} \to H$ be as above. For a general pair of plane conics

$$C = \left\{ \sum_{i=0}^{2} x_i^2 = 0 \right\}, \qquad C' = \left\{ \sum_{i=0}^{2} \alpha_i x_i^2 = 0 \right\}$$

in H, we can find a symmetric pair S and S' of quadric surfaces such that $C = S \cap H$ and $C' = \pi(S \cap S')$. Namely, the conditions which must be met are

$$\begin{cases} \beta_i - \beta_3 = \lambda \alpha_i & i = 0, 1, 2 \\ \beta_0 \beta_2 = \beta_1 \beta_3 & \end{cases}$$

Writing the first equation as $\beta_i = \lambda \alpha_i + \beta_3$ and substituting in the second yields $\lambda^2 \alpha_0 \alpha_2 + \lambda(\alpha_0 \beta_3 + \alpha_2 \beta_3) + \beta_3^2 = \lambda \alpha_1 \beta_3 + \beta_3^2$. Canceling the β_3^2 and writing $\lambda = \gamma \beta_3$, both sides are divisible by β_3^2, and the equation becomes $\gamma^2 \alpha_0 \alpha_2 + \gamma(\alpha_0 + \alpha_2 - \alpha_1) = 0$, which has the solution $\gamma = (\alpha_0 + \alpha_2 - \alpha_1)/\alpha_0 \alpha_2$.

Now we are done. Choose C and C' for which the plane Poncelet condition is satisfied, and then choose a symmetric pair of quadrics S and S' associated to C and C' as above. Then (4') is satisfied for S, S', and by symmetry (4) is also satisfied. Summarizing:

The pairs of smooth quadric surfaces meeting transversely in $\mathbb{P}^3$ and which satisfy the Poncelet conditions of having a finite polyhedron both inscribed in and circumscribed about the pair of surfaces form a dense countable union of algebraic curves among all pairs of quadrics.

Finally, we would like to discuss the second Poncelet theorem over the real numbers. First, note that the remarks (i) and (ii) above about lines on a quadric hold for a real quadric $S \subset \mathbb{P}^3_{\mathbb{R}}$ if, and only if S is of hyperbolic type, as in Fig. 6, that is, given by a real quadratic form Q with two negative and two positive eigenvalues. For such a quadric, moreover, and any line $L \subset \mathbb{P}^3_{\mathbb{R}}$, there will be two tangent planes to S containing L if, and only if L meets S.

Thus, if S and S' are both of hyperbolic type and meet in a curve—so that every line lying on S meets S' and vice versa—we can carry out the Poncelet construction with S and S' in $\mathbb{P}^3_{\mathbb{R}}$ to obtain a configuration of real bitangent planes $\Pi(T_0)$ in $\mathbb{P}^3_{\mathbb{R}}$. Now for every bitangent plane T to S and S', let Δ_T denote the interior of the quadrilateral whose sides are the four lines of intersection of T with S and S', and whose vertices are the four points of intersection of T with S and S', as shaded in Fig. 7. Then, for any bitangent plane T_0, the regions $\{\Delta_T : T \in \Pi(T_0)\}$ *form a real polyhedron* $\Sigma(T_0)$ *in* $\mathbb{P}^3_{\mathbb{R}}$ *inscribed in and circumscribed about* S *and* S'.

Clearly, the figure $\Sigma(T_0)$ has a great deal of self-intersection, and so cannot be readily drawn in full. We can, however, say something about the abstract polyhedron $\Sigma(T_0)$ in case it is finite: as we have seen, every vertex of $\Sigma(T_0)$ lies on four faces of $\Sigma(T_0)$, and conversely, every face contains four vertices. Thus if we let V, E, and F denote the number of vertices, edges, and faces of the abstract complex $\Sigma(T_0)$, we have $V = F$ and $E = 2F$ so that the Euler characteristic $\chi(\Sigma(T_0)) = -E + F + V = 0$, i.e., *the abstract polyhedron* $\Sigma(T_0)$ *is a torus.* In particular, $\Sigma(T_0)$ cannot be a convex polyhedron in $\mathbb{R}^3$.

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[12] In this connection M. Berger pointed out two papers by Cayley giving explicit conditions for Poncelet polygon to be closed. The references are Philosophical Magazine, vol. VI (1853), 99–102, and Philosophical Trans. Royal Soc. London, vol. CLI (1861), 225–239.