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On the inverse limit of free nilpotent groups

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0. Introduction

A group P is called *parafree* (see [B1], [B2]) if it is residually nilpotent and if there exists a free group F and a homomorphism $\varphi : F \to P$ such that φ induces isomorphisms $\varphi_i : F/F_i \xrightarrow{\sim} P/P_i$, $i \geq 2$ modulo the terms F_i, P_i of the lower central series. Since F is residually nilpotent, the map φ is injective, so that F may be thought of as a subgroup of P. If $F = F(X)$ is free on the set X, then P is called parafree on X. It is plain (see [B2]) that a parafree group on X can be embedded in $\hat{F} = \lim_{\leftarrow} F/F_i$. The group $\hat{F}$ thus certainly merits some interest. This paper is a contribution to the study of the group $\hat{F}$.

In section 1 we introduce some notation and prove a basic lemma which enables us to identify the subgroups of $\hat{F} = \hat{F}(X)$ which are parafree on X. In section 2 we deal with the case of finitely generated free groups $F = F(X)$. It turns out (Corollary 2.2) that in this case the group $\hat{F}$ is parafree on X. This result contrasts with the case where X is infinite. In section 3 we deal in detail with the case where X is countably infinite. We prove the following results: $\hat{F}$ is *not* parafree on X (Corollary 3.5; see also [BK]); $\hat{F}_{ab}$ contains uncountably many linearly independent divisible elements (Theorem 3.9); $\hat{F}$ contains a free subgroup of uncountable rank with a generating set which is linearly independent mod $\hat{F}_2$ (Theorem 3.6); the 2-generator subgroups of $\hat{F}$ are free (Theorem 3.11). We note that the restriction to the countable case in all of the main results of this section is not decisive. The conclusions remain true if X is allowed to be uncountable.

In section 4 we define two subgroups $\bar{F}, \tilde{F}$ of $\hat{F} = \hat{F}(X)$. The group $\bar{F}$ is the union of the subgroups of $\hat{F}$ which are parafree on X. It is shown that $\bar{F}$ is parafree on X, so that $\bar{F}$ is the universal parafree group on X in the sense that it contains all groups which are parafree on X (Theorem 4.1). The subgroup $\tilde{F}$ of $\hat{F}$ consists of all elements of $\hat{F}$ which can be expressed by finitely many elements of X. It is shown that $\tilde{F}$ too is parafree on X (Proposition 4.4). Hence clearly $\tilde{F} \subseteq \bar{F}$. But we show that $\tilde{F} \neq \bar{F}$ if X is countably infinite (Proposition 4.5).

We also show that the group $\bar{F}$ is freely indecomposable (Corollary 4.3). We were however unable to settle the question whether $\tilde{F}$ and $\hat{F}$ are freely indecomposable in case X is infinite.

1. The inverse limit

Let G be a group and let $\{G_i\}$ denote its lower central series, i.e.

$$G_1 = G, \quad G_i = [G, G_{i-1}], \qquad i \geq 2. \tag{1.1}$$

As usual we shall denote G/G_2 by G_{ab}. We consider the inverse system of the canonical projections

$$\{G/G_i \to G/G_{i-1}\}. \tag{1.2}$$

Its inverse limit is denoted by

$$\hat{G} = \varprojlim G/G_i \tag{1.3}$$

and its canonical maps by $\tau_i : \hat{G} \to G/G_i$. We may regard $\hat{G}$ as the subgroup of the (categorical) product $\prod_{i=2}^{\infty} G/G_i$ consisting of the elements

$$\Lambda = (\lambda_1 G_2, \lambda_2 G_3, \ldots) \tag{1.4}$$

with $\lambda_i \in G$ and $\lambda_{i+1} \equiv \lambda_i$ modulo G_{i+1}. Then clearly $\tau_i(\Lambda) = \lambda_{i-1} G_i \in G/G_i$. By universality of the inverse limit the family $\pi_i : G \to G/G_i$ of canonical projections induces a homomorphism $h : G \to \hat{G}$ such that $\pi_i = \tau_i h$. Plainly it is given by

$$h(x) = (xG_2, xG_3, \ldots), \qquad x \in G. \tag{1.5}$$

The homomorphism h is injective if and only if G is residually nilpotent (i.e. if $G_\omega = \bigcap_{i \geq 2} G_i = e$).

In the sequel we shall be interested in subgroups P of $\hat{G}$ with $hG \subseteq P \subseteq \hat{G}$ and with the property that h induces isomorphisms $h_i : G/G_i \xrightarrow{\sim} P/P_i$, $i \geq 2$. The following lemma characterizes these subgroups.

LEMMA 1.1. *Let P be a group with $hG \subseteq P \subseteq \hat{G}$. Then the following statements are equivalent*

(i) $h : G \to P$ *induces isomorphisms* $h_i : G/G_i \xrightarrow{\sim} P/P_i$, $i \geq 2$;
(ii) $\tau_i : P \to \hat{G} \to G/G_i$ *induces isomorphisms* $\sigma_i : P/P_i \xrightarrow{\sim} G/G_i$, $i \geq 2$;
(iii) $h : G \to P$ *induces an epimorphism* $h_2 : G_{ab} \twoheadrightarrow P_{ab}$;
(iv) $\tau_2 : P \to G_{ab}$ *induces a monomorphism* $\sigma_2 : P_{ab} \rightarrowtail G_{ab}$.

Proof. We consider the map

$$\pi_i = \tau_i h : G \to P \subseteq \hat{G} \to G/G_i, \qquad i \geq 2. \tag{1.6}$$

It induces the identity $G/G_i \xrightarrow{h_i} P/P_i \xrightarrow{\sigma_i} G/G_i$, $i \geq 2$. Hence h_i is always injective and σ_i surjective. Moreover h_i is surjective if and only if σ_i is injective. This proves the equivalence of (i) and (ii). Also, it is well known that h_i is surjective if and only if h_2 is, proving the equivalence of (i) and (iii). Finally h_2 is surjective if and only if σ_2 is injective, proving the equivalence of (iii) and (iv).

COROLLARY 1.2. *Let $F = F(X)$ be the free group on the set X. Then a group P with $F \subseteq P \subseteq \hat{F}$ is parafree on X if and only if $h_2 : F_{ab} \to P_{ab}$ is surjective.*

Proof. This is immediate from Lemma 1.1, since the groups F and $\hat{F}$ are clearly residually nilpotent.

2. The case of finitely generated free groups

Let $F = F(X)$ be the free group on the set $X = \{x_1, x_2, \ldots, x_n\}$. The following result is due to Bousfield–Kan [BK]. Since its proof to be found in [BK] uses topological methods we shall include, for completeness, a purely algebraic proof; it is also due to Bousfield–Kan.

THEOREM 2.1. *Let F be a finitely generated free group. Then $h : F \to \hat{F}$ induces isomorphisms $h_i : F/F_i \xrightarrow{\simeq} \hat{F}/\hat{F}_i$, $i \geq 2$.*

COROLLARY 2.2. *If $F = F(x_1, \ldots, x_n)$, then $\hat{F}$ is parafree on $X = \{x_1, \ldots, x_n\}$.*

Proof. By Lemma 1.1 we only have to show that $\sigma_2 : \hat{F}_{ab} \to F_{ab}$ induced by $\tau_2 : \hat{F} \to F/F_2$ is injective. We thus have to show that an element

$$\Lambda^* = (\lambda_1 F_2, \lambda_2 F_3, \ldots) \in \hat{F} \tag{2.1}$$

with $\lambda_1 \in F_2$ is in $\hat{F}_2$. In the course of the proof we shall need the following two wellknown results which we mention without proof.

LEMMA 2.3.

$$[ab, c] = [a, c]^b [b, c],$$
$$[c, ab] = [c, b][c, a]^b.$$

LEMMA 2.4. *Let $F = F(x_1, \ldots, x_n)$. Then given $a \in F_k$, $k \geq 2$ there exist*

$u_1, \ldots, u_n \in F_{k-1}$ *such that*

$$a \equiv [u_1, x_1][u_2, x_2] \cdots [u_n, x_n] \bmod F_{k+1}.$$

We shall construct elements

$$\Gamma^{(i)} = (\gamma_1^{(i)} F_2, \gamma_2^{(i)} F_3, \ldots) \in \hat{F}, \qquad 1 \leq i \leq n \tag{2.2}$$

such that in $\hat{F}$

$$\Lambda^* = [\Gamma^{(1)}, h(x_1)][\Gamma^{(2)}, h(x_2)] \cdots [\Gamma^{(n)}, h(x_n)]. \tag{2.3}$$

In order to find $\gamma_k^{(i)}$ we proceed by induction on k. Since $\lambda_1 \in F_2$ and hence $\lambda_2 \in F_2$ there are, by Lemma 2.4, elements $u_1, \ldots, u_n \in F_1$ such that

$$\lambda_2 \equiv [u_1, x_1] \cdots [u_n, x_n] \bmod F_3. \tag{2.4}$$

Set $\gamma_1^{(i)} = u_1$. We then have

$$\lambda_1 \equiv [\gamma_1^{(1)}, x_1] \cdots [\gamma_1^{(n)}, x_n] \bmod F_2 \tag{2.5}$$

and also

$$\lambda_2 \equiv [\gamma_1^{(1)}, x_1] \cdots [\gamma_1^{(n)}, x_n] \bmod F_3. \tag{2.6}$$

Suppose now that $\gamma_1^{(i)}, \ldots, \gamma_k^{(i)}$, $1 \leq i \leq n$ have already been determined such that

$$\gamma_{l+1}^{(i)} \equiv \gamma_l^{(i)} \bmod F_{l+1}, \qquad 1 \leq l \leq k-1, \tag{2.7}$$

$$\lambda_l \equiv [\gamma_l^{(1)}, x_1] \cdots [\gamma_l^{(n)}, x_n] \bmod F_{l+1}, \qquad 1 \leq l \leq k, \tag{2.8}$$

and in addition

$$\lambda_{k+1} \equiv [\gamma_k^{(1)}, x_1] \cdots [\gamma_k^{(n)}, x_n] \bmod F_{k+2}. \tag{2.9}$$

Since $\lambda_{k+2} \equiv \lambda_{k+1} \bmod F_{k+2}$ there exists $r_{k+2} \in F_{k+2}$ such that

$$\lambda_{k+2} = ([\gamma_k^{(1)}, x_1] \cdots [\gamma_k^{(n)}, x_n]) r_{k+2}. \tag{2.10}$$

By Lemma 2.4 we can find $v_i \in F_{k+1}$ such that

$$r_{k+2} \equiv [v_1, x_1] \cdots [v_n, x_n] \bmod F_{k+3}. \tag{2.11}$$

We may thus set

$$\gamma_{k+1}^{(i)} = \gamma_k^{(i)} \cdot v_i, \qquad 1 \leqslant i \leqslant n. \tag{2.12}$$

We then clearly have

$$\gamma_{k+1}^{(i)} \equiv \gamma_k^{i} \bmod F_{k+1}. \tag{2.13}$$

Moreover

$$\begin{aligned}
[\gamma_{k+1}^{(1)}, x_1] \cdots [\gamma_{k+1}^{(n)}, x_n] &= [\gamma_k^{(1)} v_1, x_1] \cdots [\gamma_k^{(n)} v_n, x_n] \\
&= [\gamma_k^{(1)}, x_1]^{v_1}[v_1, x_1] \cdots [\gamma_k^{n}, x_n]^{v_n}[v_n, x_n], \text{ by Lemma 2.3} \\
&\equiv ([\gamma_k^{(1)}, x_1] \cdots [\gamma_k^{(n)}, x_n])([v_1, x_1] \cdots [v_n, x_n]) \bmod F_{k+3} \\
&\equiv \lambda_{k+2} \bmod F_{k+3}, \text{ by (2.10), (2.11).}
\end{aligned} \tag{2.14}$$

A fortiori we have

$$[\gamma_{k+1}^{(1)}, x_1] \cdots [\gamma_{k+1}^{(n)}, x_n] \equiv \lambda_{k+1} \bmod F_{k+2}. \tag{2.15}$$

This completes the proof of Theorem 2.1.

For the proof of Corollary 2.2 we only have to remark that $\hat{F}$, being a subgroup of $\prod_{i \geqslant 2} F/F_i$, is residually nilpotent. Next we recall that the group $\hat{F}$ contains any parafree group on X (see [B2]). Since, by Corollary 2.2, the group $\hat{F}$ is itself parafree on X if X is finite, we see that $\hat{F}$ is the biggest parafree group on the (finite) set X.

We note that the proof of Theorem 2.1 works equally well if the free group F is replaced by an arbitrary group G generated by the set $X = \{x_1, \ldots x_n\}$. We may thus state

COROLLARY 2.5. *If G is finitely generated, then $\hat{\hat{G}} = \hat{G}$.*

Finally we note

COROLLARY 2.6. *If F is finitely generated, then $\hat{F}$ is freely indecomposable.*

Proof. Suppose $\hat{F} = A * B$ with $A \neq \{e\} \neq B$. Then there are surjective maps $A \twoheadrightarrow C_1$, $B \twoheadrightarrow C_2$ with C_i, $i = 1, 2$ infinite cyclic. Thus we obtain an epimorphism $f: \hat{F} \to C_1 * C_2 = F(x, y)$ onto the free group on two generators. Take elements $a, b \in \hat{F}$ with $fa = x$, $fb = y$. Since $F(x, y)$ is free there exists $g: F(x, y) \to \hat{F}$ with $gx = a$, $gy = b$ and $fg = Id_F$. Using Corollary 2.5 we obtain an epimorphism $p: \hat{F}(x, y) \to \hat{\hat{F}} \cong \hat{F} \to F(x, y)$. But, by Corollary 2.2 the groups $\hat{F}(x, y)$ and $F(x, y)$

are parafree of the same rank, so that by Theorem 1.1 of [B2] p is an isomorphism and $\hat{F}(x, y)$ is free. This is a contradiction.

3. The case of free groups of countably infinite rank

Let $F=F(x_1, x_2 \cdots)$ be a free group on the countably infinite set $X=\{x_1, x_2, \ldots\}$. In order to obtain results on $\hat{F}$ we shall first construct a metabelian group W.

Let A be the free abelian group on $X=\{x_1, x_2, \ldots\}$ and let IA be the augmentation ideal of the integral group ring of A. Clearly IA/IA^2 is free abelian on $\{x_1-1, x_2-1, \ldots\}$. The following lemma is a generalization of this fact (see [BG]).

LEMMA 3.1. *IA^n/IA^{n+1} is the free abelian group on the set* $\{\prod_{j=1}^n (x_{i(j)}-1)\}$.

We now define W as the semi-direct product $W=IA \rtimes A$ where IA is regarded as right A-module in the usual way. We prove

LEMMA 3.2. $W_n=IA^n$, $n \geqslant 2$.

Proof. Let $u, v \in IA$, $x, y \in A$; then using Lemma 2.3 we obtain

$$[ux, vy]=[u, vy]^x[x, vy]=[u, y]^x[u, v]^{yx}[x, y][x, v]^y=[u, y]^x[x, v]^y,$$

since both IA and A are abelian. Thus we have $W_2=[W, W]=[IA, A]=IA^2$. Using induction it is now easy to prove $W_n=IA^n$ for $n>2$. We leave the details to the reader. We now consider $\hat{W}=\lim_{\leftarrow} W/W_n=\lim_{\leftarrow}(IA/IA^n \rtimes A)=(\lim_{\leftarrow} IA/IA^n) \rtimes A$. By Lemma 3.1 the group IA^n/IA^{n+1} is free abelian on the n-fold products $\prod_{i=1}^n (x_{i(j)}-1)$. We may thus identify IA/IA^n, as abelian group, with the augmentation ideal of the quotient of the polynomial ring on $y_i=x_i-1$, $i=1, 2, \ldots$ modulo the ideal generated by the n-fold products. As a consequence we see that $\lim_{\leftarrow} IA/IA^n$, as an abelian group, is isomorphic to the augmentation ideal J of the power series ring on $y_i=x_i-1$, $i=1, 2, \ldots$. The operation of x_j, x_j^{-1} on $y_i=x_i-1$ is given by

$$(x_i-1)\circ x_j=(x_i-1)(x_j-1)+(x_i-1)=y_iy_j+y_i, \tag{3.1}$$

$$(x_i-1)\circ x_j^{-1}=y_i(1-y_j+y_j^2-y_j^3+\cdots). \tag{3.2}$$

We note for further reference that in $\hat{W} = J \wr A$

$$[x_i - 1, x_j] = -(x_i - 1) + (x_i - 1) \circ x_j = y_i y_j, \tag{3.3}$$

$$[x_i - 1, x_j^{-1}] = -y_i y_j + y_i y_j^2 - y_i y_j^3 + \cdots. \tag{3.4}$$

Here we have used the fact that conjugation of $x_i - 1$ by x_j, x_j^{-1} in $\hat{W}$ is just operation of x_j, x_j^{-1} on y_i in J. Since

$$\hat{W}_2 = [J, A] \qquad \hat{W}_3 = [J, A, A], \text{ etc.} \tag{3.5}$$

We obtain from (3.3), (3.4) the following key result.

LEMMA 3.3. *$\hat{W}_k = J \cdot IA^{k-1}$, $k \geq 2$. In particular, an element $v \in J$ is in $\hat{W}_2$ if and only if it can be written as a finite linear combination*

$$v = \sum_{i=1}^{n} w_i y_i, \qquad w_i \in J. \tag{3.6}$$

With this result it is now possible to settle various questions about our group $\hat{F}$. We first reprove a result of Bousfield–Kan [BK, p. 114].

THEOREM 3.4. *Let $F = F(X)$ be a free group where X is countably infinite. Then $h_2 : F_{ab} \to \hat{F}_{ab}$ is not surjective.*

COROLLARY 3.5. *Let $F = F(X)$ be a free group where X is countably infinite. Then $\hat{F}$ is not parafree on X.*

Proof. We enumerate the elements of X as follows

$$X = \{x_{11}, x_{21}, x_{22}, x_{31}, x_{32}, x_{33}, x_{41}, \ldots\} \tag{3.7}$$

and consider (see [BK]) the element $\Lambda = (\lambda_1 F_2, \lambda_2 F_3, \ldots) \in \hat{F}$ where

$$\lambda_1 = e, \lambda_k = [x_{21}, x_{22}][x_{31}, x_{32}, x_{33}] \cdots [x_{k1}, \ldots, x_{kk}], k \geq 2. \tag{3.8}$$

We shall show that $\Lambda \notin \hat{F}_2$ but $\tau_2(\Lambda) = e \in F_{ab}$; hence τ_2 is not injective. This implies, by Lemma 1.1, that h_2 is not surjective.

Consider the free abelian group A on X and the group $W = IA \wr A$. Define a map $f:F\to W$ by

$$f(x_{kj})=\begin{cases}x_{kj}-1\in IA\subseteq W & \text{for } j=1,\\ x_{kj}\in A\subseteq W & \text{for } 2\leqslant j\leqslant k.\end{cases} \tag{3.9}$$

We then obtain a map $\hat{f}:\hat{F}\to\hat{W}$ with

$$\hat{f}(\Lambda)=(x_{21}-1)(x_{22}-1)+(x_{31}-1)(x_{32}-1)(x_{33}-1)+\cdots \tag{3.10}$$

(see (3.3)). It is then clear from Lemma 3.3 that $\hat{f}(\Lambda)\notin\hat{W}_2$, so that $\Lambda\notin\hat{F}_2$. This completes the proof of Theorem 3.4.

We note that it might conceivably be the case that $\hat{F}$ is parafree on some set other than X. That this is not the case follows from Corollary 3.9 where the existence of non-trivial divisible elements in $\hat{F}_{ab}$ is proved. We first state

THEOREM 3.6. *Let $F=F(X)$ be a free group where X is countably infinite. Then $\hat{F}$ contains a subgroup which is free on an uncountable set Y of elements which are linearly independent* mod $\hat{F}_2$.

For the proof of this result we shall need the following

LEMMA 3.7. *There exists an uncountable set Σ of sequences $\sigma=(\sigma_0,\sigma_1,\ldots)$ of natural numbers σ_i with the following properties:*

(i) $\sigma_0\geqslant 2$, $\sigma_{i+1}>\sigma_i$, $i\geqslant 0$;
(ii) *if $\{\sigma^{(1)},\ldots,\sigma^{(n)}\}$ is a finite subset of Σ then there exists $i\geqslant 0$ such that for every $k\geqslant i$ the entries $\sigma_k^{(1)},\ldots,\sigma_k^{(n)}$ are different.*

Proof. Let Ω be an uncountable set of sequences $\omega=(\omega_0,\omega_1,\ldots)$ of numbers $0,1,2,\ldots,9$ with $\omega_0\geqslant 2$. Define, for any such ω, a sequence

$$\sigma(\omega)=(\sigma_0(\omega),\sigma_1(\omega),\ldots) \tag{3.11}$$

by setting

$$\sigma_i(\omega)=\omega_0\cdot 10^i+\omega_1\cdot 10^{i-1}+\cdots\omega_{i-1}\cdot 10^1+\omega_i\cdot 10^0. \tag{3.12}$$

It is plain that $\omega\neq\omega'$ implies $\sigma(\omega)\neq\sigma(\omega')$. Also, it is clear that $\sigma_0(\omega)=\omega_0\geqslant 2$ and that $\sigma_{i+1}(\omega)>\sigma_i(\omega)$, $i\geqslant 0$. Moreover, if $\{\sigma^{(1)},\ldots,\sigma^{(n)}\}$ is a finite subset of Σ with $\sigma^{(l)}=\sigma(\omega^{(l)})$, then there exists $i\geqslant 0$ such that for $k\geqslant i$ the elements $\sigma_k^{(1)},\ldots,\sigma_k^{(n)}$ are different. We may thus set $\Sigma=\{\sigma(\omega)\mid\omega\in\Omega\}$.

Proof (of Theorem 3.6). Let $F=F(X)$ where

$$X=\{x_{01}, x_{02}, \ldots, x_{11}, x_{12}, \ldots, x_{21}, x_{22}, \ldots\} \tag{3.13}$$

Define for any sequence $\sigma=(\sigma_0, \sigma_1, \ldots)\in\Sigma$ an element

$$\Lambda(\sigma)=(\lambda_1F_2, \lambda_2F_3, \ldots)\in\hat{F} \tag{3.14}$$

by setting

$$\begin{cases}\lambda_i=e \quad \text{for} \quad i<\sigma_0,\\ \lambda_i=[x_{01},\ldots,x_{0\sigma_0}][x_{11},\ldots,x_{1\sigma_1}]\cdots[x_{l1},\ldots,x_{l\sigma_l}] \quad \text{for} \quad \sigma_l\leqslant i<\sigma_{l+1}.\end{cases} \tag{3.15}$$

Note that $\Lambda(\sigma)$ is an element of $\hat{F}$ since σ is strictly increasing. Next we shall show that the elements $\Lambda(\sigma)$, $\sigma\in\Sigma$ generate a free subgroup of $\hat{F}$. For this it is enough to show that any *finite* set of elements

$$\Lambda^{(l)}=\Lambda(\sigma^{(l)}), \qquad 1\leqslant l\leqslant n$$

freely generate a free subgroup. By Lemma 3.7 we may conclude that there exists an $i\geqslant 0$ such that the entries $\sigma_i^{(1)},\ldots,\sigma_i^{(n)}$ are all different. We now consider the projection $p:F(X)\to F(x_{i1}, x_{i2}, x_{i3},\ldots)$; then p induces a map $\hat{p}:\hat{F}(X)\to \hat{F}(x_{i1}, x_{i2}, x_{i3},\ldots)$ with

$$\hat{p}(\Lambda(\sigma^{(l)}))=[x_{i1},\ldots,x_{i\sigma_i^{(l)}}]\in F(x_{i1}, x_{i2}, x_{i3},\ldots)\subseteq\hat{F}(x_{i1}, x_{i2}, x_{i3},\ldots) \tag{3.16}$$

It follows at once that $\hat{p}(\Lambda(\sigma^{(l)}))$ freely generate a free subgroup of $F(x_{i1}, x_{i2},\ldots)$. Hence the elements $\Lambda(\sigma^{(l)})$, $1\leqslant l\leqslant n$ freely generate a free subgroup of $\hat{F}(X)$.

It remains to show that the elements $\Lambda(\sigma), \sigma\in\Sigma$ are linearly independent mod $\hat{F}_2$. For this we consider the group $W=IA \wr A$ where A is the free abelian group on X and the map $f:F\to W$ defined by

$$f(x_{ik})=\begin{cases}(x_{i1}-1)\in IA\subseteq W & \text{for} \quad k=1, \quad i\geqslant 0,\\ x_{ik}\in A\subseteq W & \text{for} \quad k\geqslant 2, \quad i\geqslant 0.\end{cases} \tag{3.17}$$

For the induced map $\hat{f}:\hat{F}\to\hat{W}$ we then obtain

$$\hat{f}(\Lambda(\sigma))=(x_{01}-1)\cdots(x_{0\sigma_0}-1)+(x_{11}-1)\cdots(x_{1\sigma_1}-1)+\cdots. \tag{3.18}$$

It follows from Lemma 3.3 that $\hat{f}(\Lambda(\sigma))\notin\hat{W}_2$. Moreover no non-trivial linear combination of elements $\hat{f}(\Lambda(\sigma))$ lies in $\hat{W}_2$. For, if $\sigma^{(1)}, \ldots, \sigma^{(n)}$ are different elements of Σ then, by Lemma 3.7, there exists $i \geq 0$ such that for all $k \geq i$ the entries $\sigma_k^{(1)}, \ldots, \sigma_k^{(n)}$ are different. Hence a nontrivial linear combination of $\hat{f}(\Lambda(\sigma^{(1)})), \ldots, \hat{f}(\Lambda(\sigma^{(n)}))$ cannot lie in $\hat{W}_2$.

We note that in the course of the above proof we also have proved the following result

COROLLARY 3.8. *Let $F = F(X)$ be a free group where X is countably infinite. Then $\hat{F}_{ab}$ is uncountably infinite.*

We next prove

THEOREM 3.9. *Let $F = F(X)$ be a free group where X is countably infinite. Then $\hat{F}_{ab}$ contains uncountably many linearly independent divisible elements.*

Proof. Let $X = \{x_{01}, x_{02}, \ldots, x_{11}, x_{12}, \ldots, x_{21}, x_{22}, \ldots\}$ and consider for $\sigma \in \Sigma$ (Lemma 3.7) the element

$$\Gamma(\sigma) = (\gamma_1 F_2, \gamma_2 F_3, \ldots) \in \hat{F} \tag{3.19}$$

defined by

$$\begin{cases} \gamma_i = e \quad \text{for} \quad i < \sigma_2, \\ \gamma_i = ([x_{21}, \ldots, x_{2\sigma_2}]([x_{31}, \ldots, x_{3\sigma_3}] \cdots [x_{l1}, \ldots, x_{l\sigma_l}]^{l \cdots})^3)^2 \quad \text{for} \quad \sigma_l \leq i < \sigma_{l+1}. \end{cases} \tag{3.20}$$

(The fact that we start with σ_2 is merely a notational convenience.) We shall show that each $\Gamma(\sigma)$ gives rise to a divisible element in $\hat{F}_{ab}$. We first recall that F_{ab} is a direct summand of $\hat{F}_{ab}$. Thus in order to exhibit divisible elements in $\hat{F}_{ab}$ we may consider the quotient of $\hat{F}_{ab}$ by F_{ab}, in other words we may consider the quotient of $\hat{F}$ by the normal subgroup N generated by F and $\hat{F}_2$. In order to show that for $k \geq 2$ the element $\Gamma(\sigma)$ is a k-th power modulo N we consider the element

$$\Delta = \Delta(\sigma) = (\delta_1 F_2, \delta_2 F_3, \ldots) \in \hat{F} \tag{3.21}$$

where

$$\begin{cases} \delta_i = e \quad \text{for} \quad i < \sigma_k \\ \delta_i = ([x_{k1}, \ldots, x_{k\sigma_k}]([x_{l1}, \ldots, x_{l\sigma_l}]^{l \cdots})^{k+1})^{k!} \quad \text{for} \quad \sigma_i \leq i < \sigma_{l \times 1}, \; l \geq k. \end{cases} \tag{3.22}$$

Clearly, modulo N the elements $\Gamma(\sigma)$ and $\Delta(\sigma)$ are equivalent. But $\Delta(\sigma)$ is a k-th power. Hence $\Gamma(\sigma)$ is a k-th power modulo N. An argument similar to the one used in the proof of Theorem 3.6 shows that the elements $\Gamma(\sigma), \sigma \in \Sigma$ are linearly independent in $\hat{F}_{ab}$.

COROLLARY 3.10. *Let $F = F(X)$ be a free group, where X is countably infinite. Then $\hat{F}_k/\hat{F}_{k+1}$, $k \geq 1$ contains non-trivial divisible elements.*

Proof. Let $X = \{x_{01}, x_{02}, \ldots, x_{11}, x_{12}, \ldots, x_{21}, x_{22}, \ldots\}$ and let $\Gamma = \Gamma(\sigma) \in \hat{F}$ be one of the elements defined in (3.19), (3.20). We consider the element

$$\Gamma^* = [\Gamma, x_{01}, x_{02}, \ldots, x_{0,k-1}] \in \hat{F}_k. \tag{3.23}$$

Since the k-fold commutator is a linear map from the k-fold tensor product of $\hat{F}_{ab}$ into $\hat{F}_k/\hat{F}_{k+1}$

$$[\cdot\cdot\cdot]: \hat{F}_{ab} \otimes \cdots \otimes \hat{F}_{ab} \to \hat{F}_k/\hat{F}_{k+1} \tag{3.24}$$

the element Γ^* gives certainly rise to a divisible element in $\hat{F}_k/\hat{F}_{k+1}$. It remains to prove that Γ^* is non-trivial. We use the map $f: F \to W$ defined by

$$\begin{cases} f(x_{0l}) = x_{0l} \in A \subseteq W, & l \geq 1. \\ f(x_{il}) = x_{il} \in A \subseteq W, & i \geq 1, \quad l \geq 2. \\ f(x_{i1}) = (x_{i1} - 1) \in IA \subseteq W, & i \geq 1. \end{cases} \tag{3.25}$$

For the induced map $\hat{f}: \hat{F} \to \hat{W}$ we obtain

$$\hat{f}(\Gamma^*) = \\ [2(x_{21}-1)\cdots(x_{2\sigma_2}-1) + 3!(x_{31}-1)\cdots(x_{3\sigma_3}-1) + \cdots](x_{01}-1)\cdots(x_{0,k-1}-1) \tag{3.26}$$

By Lemma 3.3, $\hat{f}(\Gamma^*) \notin \hat{W}_{k+1}$ and hence $\Gamma^* \notin \hat{F}_{k+1}$.

The following result should be compared with Theorem 4.2 of [B2].

THEOREM 3.11. *The 2-generator subgroups of $\hat{F}$ are free.*

Proof. Let $a, b \in \hat{F} = \hat{F}(X)$. We have to consider two cases.

(i) Let $[a, b] \neq e$. In this case we shall deduce our result from Theorem 4.2 of [B2]. Clearly there exists $i \geq 1$ such that for $\tau_i: \hat{F} \to F/F_i$ we have $\tau_i[a, b] \neq e$. Consider then a finite subset $Y \subseteq X$ and the projection $p: F(X) \to F(Y)$, such that for $p\tau_i: \hat{F}(X) \to F(Y)/F_i(Y)$ we have $[p\tau_i a, p\tau_i b] \neq e$. We may thus consider $\hat{p}: \hat{F}(X) \to \hat{F}(Y)$. Since $\hat{F}(Y)$ is parafree and $[pa, pb] \neq e$ we may conclude from

Theorem 4.2 of [B2] that $\hat{p}a, \hat{p}b$ generate a free subgroup. Hence a, b generate a free subgroup in $\hat{F}(X)$.

(ii) Let $[a, b]=e$. We shall proceed as in the proof of Theorem 4.2 of [B2]. By Theorem 3.1 of [B2] the group $\hat{F}=\hat{F}(X)$ can be embedded in the power series ring $\mathbf{Z}[[X]]$ and hence in $\mathbb{Q}[[X]]$. Under that embedding let

$$\begin{cases} a=1+a_i+\cdots, & a_i\neq 0 \\ b=1+b_j+\cdots, & b_j\neq 0 \end{cases} \tag{3.27}$$

i.e. a_i, b_j are the first non-zero terms in the power series corresponding to a, b. The elements a_i, b_j are Lie elements in $\mathscr{L}[X]$, the Lie algebra over $\mathbb{Q}$ generated by X in $\mathbb{Q}[X]$. Since $\mathscr{L}[X]$ is a free Lie algebra over $\mathbb{Q}$ it follows that a_i, b_j generate a free sub-$\mathbb{Q}$-Lie algebra (see Sirsov [S], Witt [W]). Since $[a, b]=e$ we have in $\mathbb{Q}[[X]]$

$$ab=1+a_i+b_j+a_ib_j+\cdots=1+a_i+b_j+b_ja_i+\cdots=ba. \tag{3.28}$$

Hence in $\mathscr{L}[X]$ we have $[a_i, b_j]=0$, so that the Lie algebra generated by a_i, b_j is abelian. Since it is free it must be isomorphic to $\mathbb{Q}$. It follows that there exist integers $m, n>0$ such that

$$ma_i=nb_j. \tag{3.29}$$

In particular $i=\deg a_i=\deg b_j=j$. We now compute $c=a^{-m}\cdot b^n$ in $\mathbb{Z}[[X]]$.

$$\begin{aligned} c=a^{-m}b^n&=(1-ma_i+\cdots)(1+nb_j+\cdots) \\ &=1-ma_i+nb_j+\{\text{terms of degree}\geqslant i+1\} \\ &=1+\{\text{terms of degree}\geqslant i+1\}. \end{aligned} \tag{3.30}$$

But c, being an element in the subgroup generated by a, b commutes with a and b and hence satisfies $[a, c]=e\cdot A$ repetition of the above argument with c at the place of b shows that the power series expansion of c has the form

$$c=1+c_k+\cdots,\qquad c_k\neq 0 \tag{3.31}$$

with $k=i$. This is a contradiction to (3.30) so that $c=1$, i.e. $a^m=b^n$. Since $\hat{F}$ is torsion-free it follows that the subgroup generated by a, b is infinite cyclic, hence free.

4. Two subgroups of $\hat{F}$

Let $F = F(X)$ be the free group on the set X. Here we shall exhibit two subgroups $\bar{F}, \tilde{F}$ of $\hat{F} = \hat{F}(X)$ which are parafree on X. Note that $\hat{F}$ is not itself parafree on X if X is at least countably infinite.

THEOREM 4.1. *There exists a subgroup $\bar{F} \subseteq \hat{F} = \hat{F}(X)$ which is parafree on X and has the property that it contains all subgroups of $\hat{F}$ which are parafree on X.*

Proof. We first show that the system of subgroups of $\hat{F}$ which are parafree on X is directed. Thus let U, V be two subgroups of $\hat{F}$ which are parafree on X. Let W be the subgroup of $\hat{F}$ generated by U, V. We claim that W is parafree on X. Since $F \subseteq W \subseteq \hat{F}$ we may apply Lemma 1.1, so that we have to show that $F_{ab} \to W_{ab}$ is surjective. Consider the obvious epimorphism $U *_F V \twoheadrightarrow W$. We then have the series of maps

$$F_{ab} \xrightarrow{\sim} U_{ab} \to (U *_F V)_{ab} \twoheadrightarrow W_{ab} \tag{4.1}$$

so that we only have to show that $U_{ab} \to (U *_F V)_{ab}$ is surjective. But this is trivial since U, V are parafree on X. We may thus define $\bar{F}$ by $\bar{F} = \lim_{\rightarrow} U$, where U is a subgroup of $\hat{F}$ which is parafree on X. It is then clear that $F_{ab} \to \bar{F}_{ab}$ is surjective, so that $\bar{F}$ is parafree on X, by Lemma 1.1.

It is obvious that the above construction is independent of the fact that the group we start with is free. Thus if G is an arbitrary group we may find in $\hat{G}$ a group $\bar{G}$ with $hG \subseteq \bar{G} \subseteq \hat{G}$ such that h induces isomorphisms $h_i : G/G_i \xrightarrow{\sim} \bar{G}/\bar{G}_i$, $i \geq 2$, and with the following universal property. If $f : G \to H$ is a homomorphism such that $f_i : G/G_i \xrightarrow{\sim} H/H_i$, $i \geq 2$ and H is residually nilpotent, then there exists a unique $\bar{f} : H \to \bar{G}$ such that $\bar{f} \circ f = h : G \to \bar{G}$. By construction of $\bar{G}$ we have

COROLLARY 4.2. $\bar{\bar{G}} = \bar{G}$.

As in Corollary 2.6 we obtain from Corollary 4.2

COROLLARY 4.3. *Let F be free, then $\bar{F}$ is freely indecomposable.*

We shall now construct another subgroup $\tilde{F}$ of $\hat{F} = \hat{F}(X)$ which is parafree on X. Thus we have $\tilde{F} \subseteq \bar{F}$, by Theorem 4.1, but we shall later show that $\tilde{F} \neq \bar{F}$ if X is (at least) countably infinite.

Consider the directed system of finite subsets Y of X and the associated directed system of groups $\hat{F}(Y)$. Define

$$\tilde{F} = \varinjlim \hat{F}(Y), \qquad Y \subseteq X,\ Y \text{ finite} \tag{4.2}$$

If we consider $\hat{F}(X)$ as a subgroup of the power series ring $\mathbb{Z}[[X]]$ (see Theorem 3.1 of [B2]), then $\tilde{F}$ may be described as the subgroup of those power series of $\hat{F}$ which involve only finitely many elements of X.

PROPOSITION 4.4. *The group $\tilde{F}$ is parafree on X.*

Proof. Clearly $F \subseteq \tilde{F} \subseteq \hat{F}$. Since $\tilde{F}_{ab} = \lim_{\rightarrow} (\hat{F}(Y))_{ab} = \lim_{\rightarrow} (F(Y))_{ab}$ is free abelian on X, the map $F_{ab} \to \tilde{F}_{ab}$ is surjective. By Lemma 1.1 we conclude that $\tilde{F}$ is parafree on X.

PROPOSITION 4.5. *Let $F = F(X)$ where X is countably infinite. Then $\tilde{F}$ is a proper subgroup of $\bar{F}$.*

Proof. We shall exhibit a subgroup U of $\hat{F}$ which is parafree on X, but not contained in $\tilde{F}$. Since $U \subseteq \bar{F}$ by Theorem 4.1 it then follows that $\tilde{F} \neq \bar{F}$.

Let $X = \{x_1, x_2, \ldots, y_1, y_2, \ldots\}$. Define elements

$$Z^{(i)} = (\zeta_1^{(i)} F_2, \zeta_2^{(i)} F_3, \ldots) \in \hat{F}, \qquad i = 0, 1, \ldots \tag{4.3}$$

by setting

$$\begin{cases} \zeta_1^{(i)} = x_i, \\ \zeta_2^{(i)} = [x_{i+1}, y_{i+1}]x_i, \\ \zeta_3^{(i)} = [[x_{i+2}, y_{i+2}]x_{i+1}, y_{i+1}]x_i, \text{ etc.} \end{cases} \tag{4.4}$$

where $x_0 = e$. Modulo any F_k and hence in $\hat{F}$ we have

$$Z^{(i)} = [Z^{(i+1)}, y_{i+1}]x_i, \qquad i = 0, 1, \ldots. \tag{4.5}$$

Consider now the subgroup U of $\hat{F}$ generated by $x_1, x_2, \ldots, y_1, y_2, \ldots, Z^{(0)}, Z^{(1)}, Z^{(2)}, \ldots$. We claim that U is parafree on X. By Lemma 1.1 we only have to show that $F_{ab} \to U_{ab}$ is surjective. But it is clear that

$$\begin{cases} Z^{(0)} \equiv e \bmod U_2, \\ Z^{(i)} \equiv x_i \bmod U_2, \qquad i \geq 1. \end{cases} \tag{4.6}$$

Finally it is plain that none of the elements $Z^{(i)}$ is contained in $\tilde{F}$.

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