Kähler metrics associated to a real hypersurface.

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Introduction

In their paper [2] Chern and Moser attach to a strongly pseudo-convex real hypersurface M in C^n, $n \geq 2$, a complete system of local invariants with respect to biholomorphic mappings. These invariants, which classically have been called pseudo-conformal invariants, include a curvature tensor and a family of curves called chains. Along these curves there are invariant notions of parallel translation and projective parameter.

From a different approach, using approximate solutions to a Monge–Ampère equation, Fefferman [4] has also derived invariants for M. The method is to construct a defining function for M which is a solution to second order at M and use it to define a Kähler metric. From this Kähler metric one constructs an invariant conformal family of Lorentz metrics on a circle bundle over M. Using the connection form of [2] Burns and Shnider [1] have constructed a similar family of Lorentz metrics. In both cases the null geodesics of such a metric project to chains on M.

The aim of this paper is to relate all the invariants derived from approximate solutions to the Monge–Ampère equation to the pseudo-conformal invariants. The approach here is to work directly with the Kähler metric. The main tool is the method of moving frames. In section one we associate to any defining function for M an indefinite Kähler metric as in [4]. In section two we consider those curves in $C \times M$ which have null velocity and null acceleration vectors. Along such "doubly-null" curves there are parallel translation of vectors and a projective parameter.

In sections three and four we compare the connection forms of the Kähler metric to the pseudo-conformal connection forms. The main results are as follows. If the defining function satisfies the Monge–Ampère equation to second order at M, then some of the components of the two curvature tensors can be identified. Also, the doubly null curves project to chains, and the two parallel translations agree. If the equation is satisfied to third order at M, then the two curvature forms are equal and the projective parameters agree. As a corollary we obtain a simple proof that the null geodesics of [4] project to chains.

Throughout this paper we use the notation of tensor calculus. Small Greek indices always run from 1 to $n-1$, while small latin indices run from zero to n, except where indicated otherwise. Repeated indices are summed over their respective ranges. The hermitian matrices $h_{\alpha\bar\beta}$ of sections 1 and 2 and $g_{\alpha\bar\beta}$ of sections 3 and 4 are used to raise and lower indices. Bars over indices indicate complex conjugation, e.g.

$$\bar{A}_{\bar\alpha\beta} = A_{\alpha\bar\beta} = A_{\alpha}^{\gamma}{}_{\cdot}\, g_{\gamma\bar\beta}, \quad \text{etc.}$$

Finally, I wish to acknowledge that various conversations with C. Fefferman on this subject have been very helpful to me in writing this paper.

1. The family of Kähler metrics

Let M be a real hypersurface of dimension $2n-1$ in complex n-space C^n. We introduce complex coordinates $Z=(z^1, \ldots, z^n)$ and express M as the zero set of a real valued function r

$$r(Z, \bar{Z}) = 0, \qquad dr \neq 0.$$

We also assume that the domain $\{r<0\}$ bounded by M is strongly pseudo-convex, so that the function r is determined up to multiplication by a positive function.

Given such a function r we define an auxiliary function

$$R = r(Z, \bar{Z})(z^0 z^{\bar 0})^p \tag{1.1}$$

on $C\times C^n$ as in [4], where z^0 is the complex coordinate on C and p is a positive power (later we take $p=(n+1)^{-1}$). We now define a Kähler metric H by

$$H = \sum R_{i\bar j}\, dz^i \otimes dz^{\bar j}, \tag{1.2}$$

where i and j are summed from 0 to n. We use subscripts on r and R to denote partial derivatives:

$$r_\alpha = \partial r/\partial z^\alpha, \qquad R_{i\bar j} = \partial^2 R/\partial z^i\, \partial z^{\bar j}, \quad \text{etc.}$$

We wish to put H into a more convenient form. If we put

$$\omega^n = -iu\, \partial r = -iu(r_\alpha\, dz^\alpha + r_n\, dz^n), \qquad u = (z^0 z^{\bar 0})^p, \tag{1.3}$$

(α is summed from 1 to $n-1$) then H can be written

$$H = up^2(z^0 z^{\bar{0}})^{-1} r\, dz^0 \otimes dz^{\bar{0}} - ip(z^0)^{-1}\, dz^0 \otimes \omega^{\bar{n}} + i\omega^n \otimes p(z^{\bar{0}})^{-1}\, dz^{\bar{0}} + ur_{i\bar{j}}\, dz^i \otimes dz^{\bar{j}}, \quad (1.4)$$

where i and j are summed from 1 to n in the last term.

For purposes of local computation we assume that $r_n \neq 0$. Further use of (1.3) then gives

$$H = -i\omega^0 \otimes \omega^{\bar{n}} + i\omega^n \otimes \omega^{\bar{0}} + h_{\alpha\bar{\beta}}\omega^\alpha \otimes \omega^{\bar{\beta}} + rp^2u(z^0 z^{\bar{0}})^{-1}\, dz^0 \otimes dz^{\bar{0}}, \quad (1.5)$$

where

$$\omega^0 = p(z^0)^{-1}\, dz^0 - \eta_\alpha\, dz^\alpha + iQ\omega^n, \qquad \omega^\alpha = dz^\alpha, \quad (1.6)$$

and

$$h_{\alpha\bar{\beta}} = u\{r_{\alpha\bar{\beta}} - r_\alpha r_{n\bar{\beta}}(r_n)^{-1} - r_{\alpha\bar{n}} r_{\bar{\beta}}(r_{\bar{n}})^{-1} + r_\alpha r_{\bar{\beta}} r_{n\bar{n}}(r_n r_{\bar{n}})^{-1}\}, \quad (1.7a)$$

$$\eta_\alpha = -r_{\alpha\bar{n}}(r_{\bar{n}})^{-1} + r_\alpha r_{n\bar{n}}(r_n r_{\bar{n}})^{-1}, \quad (1.7b)$$

$$Q = r_{n\bar{n}}(2ur_n r_{\bar{n}})^{-1}. \quad (1.7c)$$

Equation (1.7a) defines the Levi form of M, which is hermitian and positive definite, since M is strongly pseudo-convex. Thus near $C \times M$ (where $r = 0$), excluding $z^0 = 0$, H is a non-degenerate hermitian form of signature $(n, -1)$.

We next recall the local formulas of Kähler geometry as in [2]. Relative to a frame e_j of type $(1,0)$ and the dual coframe ω^j of type $(1,0)$

$$H = h_{i\bar{j}}\omega^i \otimes \omega^{\bar{j}}. \quad (1.8)$$

The covariant derivative of e_j is

$$De_j = \omega_j{}^i{}_\cdot e_i, \quad (1.9)$$

where the connection forms $\omega_j{}^i{}_\cdot$ are uniquely determined by the conditions

$$d\omega^i = \omega^j \wedge \omega_j{}^i{}_\cdot, \quad (1.10)$$

$$dh_{i\bar{j}} = \omega_i{}^k{}_\cdot h_{k\bar{j}} + h_{i\bar{k}}\omega_{\bar{j}}{}^{\bar{k}}{}_\cdot, \qquad (\omega_{\bar{j}}{}^{\bar{k}}{}_\cdot = \bar{\omega}_j{}^k{}_\cdot). \quad (1.11)$$

The curvature forms $\Omega_i{}^j{}_\cdot$ and curvature tensor $R_i{}^j{}_{\cdot kl}$ are given by

$$d\omega_i{}^j{}_\cdot = \omega_i{}^k{}_\cdot \wedge \omega_k{}^j{}_\cdot + \Omega_i{}^j{}_\cdot, \quad (1.12)$$

and

$$\Omega_{i\cdot}^{j} = R_{i\cdot k\bar{l}}^{j}\omega^{k} \wedge \omega^{\bar{l}}, \tag{1.13}$$

respectively. Differentiating (1.10) and (1.11) gives

$$0 = \omega^{j} \wedge \Omega_{j\cdot}^{i}, \tag{1.14a}$$

and

$$0 = \Omega_{i\cdot}^{k} h_{k\bar{j}} + h_{i\bar{k}}\Omega_{\bar{j}\cdot}^{\bar{k}} \equiv \Omega_{i\bar{j}} + \Omega_{\bar{j}i}, \tag{1.14b}$$

from which follows

$$R_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}} = R_{i\bar{l}k\bar{j}}, \qquad R_{i\bar{j}k\bar{l}} = \bar{R}_{j\bar{i}l\bar{k}} \equiv R_{\bar{j}i\bar{l}k}. \tag{1.14c}$$

Given another frame of type $(1, 0)$

$$\tilde{e}_{i} = U_{i\cdot}^{j} e_{j}, \qquad \omega^{i} = \tilde{\omega}^{j} U_{j\cdot}^{i}, \qquad U \in GL(n+1, C), \tag{1.15}$$

with connection form $\tilde{\omega}_{i\cdot}^{j}$, we have the relations

$$dU_{i\cdot}^{j} + U_{i\cdot}^{k}\omega_{k\cdot}^{j} = \tilde{\omega}_{i\cdot}^{k} U_{k\cdot}^{j}, \tag{1.16a}$$

and

$$U_{i\cdot}^{k}\Omega_{k\cdot}^{j} = \tilde{\Omega}_{i\cdot}^{k} U_{k\cdot}^{j}. \tag{1.16b}$$

LEMMA (1.1). *Relative to the coframe ω^{0}, ω^{α}, ω^{n} defined by* (1.3) *and* (1.6) *we have*

$$\omega_{0\cdot}^{j} = \omega^{j}, \quad \text{and} \quad \Omega_{0\cdot}^{j} = 0.$$

Proof. Let $\tilde{\theta}^{j} = dz^{j}$, so that $\tilde{h}_{i\bar{j}} = R_{i\bar{j}}$. Equations (1.10) and (1.11) imply that

$$\tilde{\omega}_{0\cdot}^{j} = \partial\tilde{h}_{0\bar{k}}\tilde{h}^{\bar{k}j}.$$

Since $z^{0}R_{0} = pR$, we get

$$\partial\tilde{h}_{0\bar{k}} = p(z^{0})^{-1}R_{\bar{k}i}\, dz^{i} - (z^{0})^{-1}R_{0\bar{k}}\, dz^{0},$$

so that

$$\tilde{\omega}_{0\cdot}^{j} = p(z^{0})^{-1}\, dz^{j} - (z^{0})^{-1}\delta_{0\cdot}^{j}\, dz^{0},$$

δ_i^j being the Kronecker delta. By the formula (1.16a) and the fact that

$$U^{-1}{}_{\cdot 0 \cdot}{}^{0} = z^0 p^{-1}, \qquad U^{-1}{}_{\cdot 0 \cdot}{}^{\alpha} = U^{-1}{}_{\cdot 0 \cdot}{}^{n} = 0,$$

we see that $\omega^j_{0\cdot} = \omega^j$. $\Omega^j_{0\cdot} = 0$ now follows from (1.10) and (1.12).

2. Adapted frames and special curves in $C \times M$

Let e_j be a frame of type (1, 0) in $C \times C^n$. The inner product relative to the hermitian form H is given by $(e_i, e_j) = h_{i\bar{j}}$, $h_{i\bar{j}}$ given by (1.8), and is linear in the first and conjugate in the second argument. The frame e_j will be called hermitian if

$$h_{0\bar{0}} = h_{0\bar{\alpha}} = h_{0\bar{n}} = h_{n\bar{\alpha}} = h_{n\bar{n}} = 0, \qquad h_{0\bar{n}} = -i, \qquad h_{n\bar{0}} = i. \tag{2.1}$$

These conditions in conjunction with (1.11) imply the following

$$\begin{aligned} &\omega^n_{0\cdot} = \bar{\omega}^n_{0\cdot}, \qquad \omega^0_{n\cdot} = \bar{\omega}^0_{n\cdot}, \qquad \omega^0_{0\cdot} + \bar{\omega}^n_{n\cdot} = 0, \\ &\omega^n_{\alpha\cdot} = i h_{\alpha\bar{\gamma}} \omega^{\bar{\gamma}}_{0\cdot}, \qquad \omega^0_{\beta\cdot} = -i \omega^{\bar{\alpha}}_{\bar{n}\cdot} h_{\bar{\alpha}\beta}, \qquad dh_{\alpha\bar{\beta}} = \omega^{\gamma}_{\alpha\cdot} h_{\gamma\bar{\beta}} + h_{\alpha\bar{\gamma}} \omega^{\bar{\gamma}}_{\bar{\beta}\cdot\cdot} \end{aligned} \tag{2.2}$$

An hermitian frame e_j will be called an adapted frame if the following conditions are satisfied: over the complex number field the n vectors e_α, e_0 span the vector subspace $H(C \times M) = T(C \times M) \cap iT(C \times M)$ of the real tangent space to $C \times M$; the first vector e_0 of the frame is the distinguished vector $(p^{-1}z^0, 0, \ldots, 0)$ tangent to the factor C of $C \times M$; the last vector e_n is tangent to $C \times M$ while ie_n is transverse.

From (1.15) and (2.1) we see that two adapted frames $\tilde{e}_j$ and e_j at a point are related by

$$\tilde{e}_0 = e_0, \qquad \tilde{e}_\alpha = U^0_{\alpha\cdot} e_0 + U^\beta_{\alpha\cdot} e_\beta, \qquad \tilde{e}_n = U^0_{n\cdot} e_0 + U^\beta_{n\cdot} e_\beta + e_n, \tag{2.3a}$$

where

$$\tilde{h}_{\alpha\bar{\beta}} = h_{\rho\bar{\gamma}} U^\rho_{\alpha\cdot} U^{\bar{\gamma}}_{\bar{\beta}\cdot}, \qquad U^0_{\alpha\cdot} = -i U^\beta_{\alpha\cdot} h_{\beta\bar{\gamma}} U^{\bar{\gamma}}_{\bar{n}\cdot}, \qquad \operatorname{Im}(U^0_{n\cdot}) = -\tfrac{1}{2} h_{\beta\bar{\gamma}} U^\beta_{n\cdot} U^{\bar{\gamma}}_{\bar{n}\cdot\cdot} \tag{2.3b}$$

From the dual transformation we see that the one-form ω^n and the system $(\omega^\alpha, \omega^n)$ are invariant. Also, upon restricting to $C \times M$ we have $\omega^n = \omega^{\bar{n}}$. The frame dual to (1.3), (1.6) at $r = 0$ is clearly an adapted frame.

LEMMA (2.1). *Relative to any adapted frame e_j, the dual coframe ω^j, connection forms $\omega^j_{0}\cdot$ and $\omega^n_{j}\cdot$, and curvature forms $\Omega^j_{0}\cdot$ and $\Omega^n_{j}\cdot$ satisfy*

a. $\omega^j_{0}\cdot = \omega^j, \qquad \omega^n_{j}\cdot = ih_{j\bar{k}}\omega^{\bar{k}} \equiv i\omega_j,$

b. $\Omega^j_{0}\cdot = \Omega^n_{j}\cdot = 0,$

for $0 \le j \le n$.

The proof follows immediately from Lemma (1.1), (1.16a–b), (2.3), and (1.14b). Notice that since Lemma (1.1) is true without restricting the forms to $C \times M$ (i.e. without assuming $\omega^n = \omega^{\bar{n}}$) the same is true for Lemma (2.1)b. The curvature condition can be expressed by saying that $R_{i\bar{j}k\bar{l}} = 0$ if one of the indices i, j, k, or l is 0.

We will denote by P the bundle of adapted frames over $C \times M$ and by P_1 those adapted frames for which $h_{\alpha\bar{\beta}}$ is the identity matrix. Then P_1 is a principal fibre bundle with structure group K of the matrices U defined in (2.3a–b) where $\tilde{h} = h = id$.

Using the hermitian connection D restricted to adapted frames we can define a special class of curves in $C \times M$. A curve $t \to Z(t)$ will be called a doubly null curve if its velocity and acceleration vectors are both null. Since

$$dZ = \omega^0 e_0 + \omega^\alpha e_\alpha + \omega^n e_n,$$

a null curve is characterized by

$$0 = (dZ, dZ) = -i\omega^n(\omega^0 - \omega^{\bar{0}}) + h_{\alpha\bar{\beta}}\omega^\alpha\omega^{\bar{\beta}}.$$

We are interested in curves which have a non-trivial projection from $C \times M$ to M, so we require $\omega^n \neq 0$. Along such a null curve we can choose a frame e_j for which

$$\omega^0 = \omega^{\bar{0}}, \qquad \omega^\alpha = \omega^{\bar{\alpha}} = 0. \tag{2.4a}$$

The acceleration is then given by

$$Z'' = D(dZ) = (\omega^{0'} + \omega^0\omega^0_{0}\cdot + \omega^n\omega^0_{n}\cdot)e_0 + \omega^n\omega^\alpha_{n}\cdot e_\alpha + (\omega^{n'} + \omega^n\omega^n_{n}\cdot)e_n.$$

Since ω^0_n, $\omega^0_0 = \omega^0$, and ω^n_n are real

$$(Z'', Z'') = (\omega^n)^2 h_{\alpha\bar{\beta}}\omega^\alpha_{n}\cdot\omega^{\bar{\beta}}_{\bar{n}}\cdot\cdot$$

This leads to the additional equation

$$\omega_{n}^{\alpha}\cdot = \omega_{\bar{n}}^{\bar{\alpha}}\cdot = 0 \tag{2.4b}$$

for doubly null curves. It follows from the structure equations (1.10) and (1.12) that the system (2.4a–b) satisfies the Frobenius integrability condition. These are to be viewed as equations on the bundle P of adapted frames, solution curves of which project to doubly null curves in $C \times M$. Also, it is easy to see that any vertical curve is a doubly null curve.

As a consequence of (2.4a–b) and (2.2) we see that the vectors e_α, $1 \le \alpha \le n-1$, are parallel along a doubly null curve if

$$\omega_{\alpha}^{\beta}\cdot = \omega_{\bar{\alpha}}^{\bar{\beta}}\cdot = 0. \tag{2.5}$$

Also, (2.4a–b), the structure equations, and the remarks following Lemma (2.1) imply

$$d\omega^n = \omega^n \wedge (-2\omega^0), \qquad d\omega^0 = \omega^n \wedge \omega_n^0\cdot, \qquad d\omega_n^0\cdot = -2\omega^0 \wedge \omega_n^0\cdot. \tag{2.6}$$

The real differential one-forms ω^n, $-2\omega^0$, $-2\omega_n^0\cdot$ satisfy the structure equations of the projective transformation group of the real line. Hence, we have a preferred parameter on a doubly null curve which is defined up to a projective transformation.

We can define a Riemannian metric on $C \times C^n$ by $ds^2 = \mathrm{Re}\, H$; i.e. the inner product of two vectors is given by $\langle v, w\rangle = \mathrm{Re}\,(v, w)$. When restricted to $C \times M$, ds^2 becomes degenerate, since the vector e_0 is perpendicular to $T(C \times M)$ at each point. In [4] this problem is gotten around by the following construction. Let f be a strictly positive function on M and define

$$M_f = \{(z^0, x) \in C \times M \mid (z^0 z^{\bar{0}})^p = f(x)\}.$$

M_f is a trivial circle bundle over M, and ds^2 restricted to M_f is a non-degenerate Lorentz metric of signature $(2n-1, -1)$.

LEMMA (2.2). *The null geodesics of (M_f, ds^2) are doubly null curves.*

Proof. A manifold M_f is characterized in terms of adapted frames by the condition that its intersection with $C \times \{\text{point}\}$ be a curve tangent to ie_0. Therefore restricting to M_f we have

$$\mathrm{Re}\,\omega^0 = a_n\omega^n + a_\alpha\omega^\alpha + a_{\bar{\alpha}}\omega^{\bar{\alpha}}, \qquad a_n = \bar{a}_n,$$

where the a's are some functions. If we choose the frame so that the null vector e_n is tangent to M_f, then $a_n = 0$. Now we define new vectors

$$u_\alpha = e_\alpha + (a_\alpha + a_{\bar\alpha})e_0, \qquad v_\alpha = ie_\alpha + i(a_\alpha - a_{\bar\alpha})e_0,$$

so that $(u_\alpha, v_\alpha, e_n, ie_0)$ span the tangent space of M_f. The orthogonality conditions

$$0 = \langle dZ, ie_0\rangle = \langle dZ, e_n\rangle = \langle dZ, u_\alpha\rangle = \langle dZ, v_\alpha\rangle$$

can be written as

$$0 = \operatorname{Re}\omega^n = \operatorname{Im}\omega^0, \qquad 0 = i(a_\alpha + a_{\bar\alpha})\omega^n + \operatorname{Re} h_{\alpha\bar\beta}\omega^{\bar\beta},$$
$$0 = (a_\alpha - a_{\bar\alpha})\omega^n + \operatorname{Im} h_{\alpha\bar\beta}\omega^{\bar\beta}.$$

Now given a null curve $t \to Z(t)$ in M_f which is nowhere verticle, we choose our frame so that in addition to the above

$$dZ = \omega^n e_n, \qquad \omega^n = \omega^{\bar n} \neq 0.$$

This results in

$$\omega^\alpha = \omega^{\bar\alpha} = \omega^0 = \omega^{\bar 0} = 0.$$

This null curve is a geodesic on M_f if and only if its acceleration vector

$$Z'' = D(dZ) = \omega^n \omega_n^0 . e_0 + \omega^n \omega_n^\alpha . e_\alpha + \omega^{n'} e_n$$

is perpendicular to $T(M_f)$. The orthogonality conditions for Z'' imply $\omega_n^\alpha . = \omega_n^{\bar\alpha} . = 0$. Hence, $Z(t)$ is a doubly null curve.

3. Pseudo-conformal structure

We now discuss the structure M inherits as a submanifold of C^n [3]. The one-form $\theta = -i\,\partial r$ is real when restricted to M and annihilates the complex tangent bundle $H(M)$. Let E be the line bundle of positive multiples $u\theta$ of θ. On E we have an intrinsic real one-form $\theta^n = u\theta$. Let B^* be the bundle of coframes $\{\theta^0, \operatorname{Re}\theta^\alpha, \operatorname{Im}\theta^\alpha, \theta^n\}$ which satisfy

$$d\theta^n = ig_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta} + \theta^n \wedge \theta^0, \tag{3.1a}$$

$$\{\theta^\alpha, \theta^n\} \equiv 0, \qquad \operatorname{mod}\{dz^\alpha, dz^n\}, \qquad 1 \le \alpha \le n-1, \tag{3.1b}$$

and let B be the bundle of dual frames over E. If $r_n \neq 0$ then one such frame field is given by

$$\theta^0 = -u^{-1}\, du + \eta_\alpha\, dz^\alpha + \eta_{\bar\alpha}\, dz^{\bar\alpha}, \qquad \theta^\alpha = dz^\alpha, \tag{3.2}$$

where $g_{\alpha\bar\beta} = h_{\alpha\bar\beta}$ and η_α are given by (1.7a–b). The hermitian matrix $g_{\alpha\bar\beta}$ in (3.1a) is always positive definite.

The forms θ^0, θ^α, θ^n are determined up to the transformation

$$\begin{aligned} &\theta^0 = \tilde\theta^0 + \tilde\theta^\alpha V^0_{\alpha\cdot} + \tilde\theta^{\bar\alpha} V^0_{\bar\alpha\cdot} + \tilde\theta^n V^0_{n\cdot}, \\ &\theta^\alpha = \tilde\theta^\beta V^\alpha_{\beta\cdot} + \tilde\theta^n V^\alpha_{n\cdot}, \qquad \theta^n = \tilde\theta^n, \end{aligned} \tag{3.3a}$$

where

$$\tilde g_{\alpha\bar\beta} = g_{\rho\bar\gamma} V^\rho_{\alpha\cdot} V^{\bar\gamma}_{\bar\beta\cdot}, \qquad V^0_{\alpha\cdot} = i V^\gamma_{\alpha\cdot} g_{\gamma\bar\beta} V^{\bar\beta}_{\bar n\cdot}, \qquad V^0_{n\cdot} = \bar V^0_{n\cdot}. \tag{3.3b}$$

Let B_1 be the sub-bundle of frames for which $g_{\alpha\bar\beta}$ is the identity matrix, and let G be the group of matrices V defined by (3.3a–b) where $\tilde g = g = I$. Then B_1 is a principal fibre bundle over E with structure group G.

We want to compare the bundle B with the adapted frame bundle P of section 2. Recall that P depends on a fixed choice of defining function r for M. For this r we put $\theta = -i\,\partial r$ and let u be the fibre coordinate of E relative to θ. We define a map f from $C \times M$ to E by

$$f(z^0, x) = (u, x), \qquad u = (z^0, z^{\bar 0})^p, \tag{3.4}$$

where $x = (x^1, \ldots, x^{2n-1})$ is the coordinate on M. For the coframes (1.6) and (3.2) we see that

$$f^*\theta^0 = -2\,\mathrm{Re}\,\omega^0, \qquad f^*\theta^\alpha = \omega^\alpha, \qquad f^*\theta^{\bar\alpha} = \omega^{\bar\alpha}, \qquad f^*\theta^n = \omega^n. \tag{3.5}$$

On the frame level we define a map $\hat f$ by

$$\hat f(e_0, e_\alpha, e_n) = (-\tfrac{1}{2} f_*(e_0), f_*(e_\alpha), f_*(ie_\alpha), f_*(e_n)). \tag{3.6}$$

Note that $f_*(ie_0) = 0$.

If we choose a point (z^0, x) in $C \times M$ and another frame at $f(z^0, x)$ by (3.3a–b), then (3.5) determines a unique adapted frame at (z^0, x) via (2.3a–b). It is given by

$$V^\alpha_{\beta\cdot} = U^\alpha_{\beta\cdot}, \qquad V^0_{\beta\cdot} = -U^0_{\beta\cdot}, \qquad V^\alpha_{n\cdot} = U^\alpha_{n\cdot}, \qquad V^0_{n\cdot} = -2\,\mathrm{Re}\,V^0_{n\cdot}. \tag{3.7}$$

Thus the map $\hat{f}$ makes P a circle bundle over B. If we take ω^j and θ^j as the canonical forms on P and B, respectively, then it follows from (3.4) and (3.7) that

$$\begin{aligned} &\hat{f}^*\theta^0 = -2\,\mathrm{Re}\,\omega^0, \qquad \hat{f}^*\theta^\alpha = \omega^\alpha, \qquad \hat{f}^*\theta^{\bar{\alpha}} = \omega^{\bar{\alpha}}, \\ &\hat{f}^*\theta^n = \omega^n, \qquad h_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} \circ \hat{f}. \end{aligned} \tag{3.8}$$

Also, (3.7) gives a group isomorphism from G to K.

One of the main theorems of [3] is that the principal bundle B_1, ($g_{\alpha\bar{\beta}} = \mathrm{const} = \delta_{\alpha\bar{\beta}}$) admits an invariant connection. To state this theorem we first extend the matrix $g_{\alpha\bar{\beta}}$ to an $(n+1)\times(n+1)$ matrix $g_{i\bar{j}}$ by defining

$$g_{0\bar{0}} = g_{0\bar{\beta}} = g_{\alpha\bar{n}} = g_{n\bar{n}} = 0, \qquad g_{0\bar{n}} = -i,$$

and requiring it to be hermitian. The connection will be given by a matrix of one-forms π_i^j. Its curvature matrix and tensor are given by

$$d\pi_{i\cdot}^{j} = \pi_{i\cdot}^{k} \wedge \pi_{k\cdot}^{j} + \Pi_{i\cdot}^{j}, \tag{3.9}$$

and

$$\Pi_{i\cdot}^{j} = S_{i\cdot k\bar{l}}^{j}\,\pi_{0\cdot}^{k} \wedge \pi_{0\cdot}^{\bar{l}}. \tag{3.10}$$

THEOREM (3.1) [3]. *There exists a unique connection matrix π on B_1 satisfying*

a. $-2\,\mathrm{Re}\,\pi_{0\cdot}^{0} = \theta^0$, $\pi_{0\cdot}^{\alpha} = \theta^\alpha$, $\pi_{0\cdot}^{n} = \theta^n$,

b. *π is $su(n,1)$-valued:*

$$\pi_{i\cdot}^{k} g_{k\bar{j}} + g_{i\bar{k}}\pi_{\bar{j}\cdot}^{\bar{k}} = 0, \qquad \mathrm{tr}\,\pi = \pi_{i\cdot}^{i} = 0,$$

c. *π is torsion free:*

$$S_{i\bar{j}k\bar{l}} \equiv g_{\bar{j}s} S_{i\cdot k\bar{l}}^{s} = 0 \quad \textit{if one of the indices } i, j, k, l \textit{ is zero,}$$

d. $S_{i\cdot k\bar{l}}^{i} = 0$, $0 \le k, l \le n$.

The curvature tensor S also satisfies the relations (1.14c). The (non-trivial) components of S can be expressed in terms of the defining function r of M and its derivatives of order less than or equal to

4 if all indices are less than n,
5 if only one index is n,

6 if only two indices are n,
7 if three indices are n.

The component $S_{n\bar{n}n\bar{n}}$ is undefined since $\theta^n = \theta^{\bar{n}}$.

Chains on M are defined in terms of the connection by the equations

$$\pi_0^\alpha\cdot = \pi_0^{\bar{\alpha}}\cdot = \pi_n^\alpha\cdot = \pi_n^{\bar{\alpha}}\cdot = 0. \tag{3.11}$$

Parallel translation of the complex tangent space along a chain is defined by the additional equations

$$\pi_\alpha^\beta\cdot = \pi_{\bar{\alpha}}^{\bar{\beta}}\cdot = 0. \tag{3.12}$$

Equation (3.12) leads to

$$d\theta^n = \theta^n \wedge \theta^0, \qquad d\theta^0 = \theta^n \wedge (-2\pi_n^0\cdot), \qquad d\pi_n^0\cdot = \theta^0 \wedge \pi_n^0\cdot. \tag{3.13}$$

It follows that θ^n, θ^0, $-2\pi_n^0\cdot$ satisfy the structure equations of the real projective group on the real line and so give rise to a projective parameter along the chain.

4. Comparison of connections – the Monge–Ampère equation

In order to compare the pseudo-conformal connection π and the hermitian connection ω, we use the map $\hat{f}$ defined by (3.6) to pull the forms π and Π on B_1 back to forms on the bundle P_1. Omitting the notation $\hat{f}^*$ we write (3.8) as

$$\theta^0 = -2 \operatorname{Re} \omega^0, \qquad \theta^\alpha = \omega^\alpha, \qquad \theta^n = \omega^n, \qquad h_{i\bar{j}} = g_{i\bar{j}}.$$

By Lemma (2.1), Theorem (3.1), and the equation (2.2), which hold for π also, we have

$$\operatorname{Re} \omega_0^0\cdot = \operatorname{Re} \pi_0^0\cdot, \qquad \omega_0^\alpha\cdot = \pi_0^\alpha\cdot, \qquad \omega_0^n\cdot = \pi_0^n\cdot, \tag{4.1}$$

and

$$\omega_\alpha^n\cdot = \pi_\alpha^n\cdot, \qquad \operatorname{Re} \omega_n^n\cdot = \operatorname{Re} \pi_n^n\cdot.$$

LEMMA (4.1). *For an arbitrary defining function r and the corresponding* *hermitian metric, the following relations hold:*

$$\omega_{\beta}^{\alpha}{\cdot} = \pi_{\beta}^{\alpha}{\cdot} + \mu\,\delta_{\beta}^{\alpha}{\cdot} + B_{\beta}^{\alpha}{\cdot}\omega^{n},$$
$$\omega_{n}^{\alpha}{\cdot} = \pi_{n}^{\alpha}{\cdot} + B_{\beta}^{\alpha}{\cdot}\omega^{\beta} + B^{\alpha}\omega^{n},$$
$$\omega_{n}^{0}{\cdot} = \pi_{n}^{0}{\cdot} + \operatorname{Im}(B_{\alpha}\omega^{\alpha}) + E\omega^{n}.$$

The B_{β}^{α}, B^{α}, and E are certain functions satisfying

$$B_{\beta}^{\gamma}{\cdot}g_{\gamma\bar{\alpha}} + g_{\beta\bar{\gamma}}B_{\bar{\alpha}}^{\bar{\gamma}}{\cdot} = 0, \qquad E = \bar{E},$$

and

$$\mu = -\bar{\mu} = \omega_{0}^{0}{\cdot} - \pi_{0}^{0}{\cdot}.$$

Proof. We first differentiate the relation $\omega^{\alpha} = \theta^{\alpha}$ and make use of the structure equations (1.10) and (3.9) and the fact that $\Omega_{0}^{j}{\cdot} = \Pi_{0}^{j}{\cdot} = 0$. This results in

$$0 = \omega^{\beta}\wedge(\omega_{\beta}^{\alpha}{\cdot} - \pi_{\beta}^{\alpha}{\cdot} - \mu\,\delta_{\beta}^{\alpha}{\cdot}) + \omega^{n}\wedge(\omega_{n}^{\alpha}{\cdot} - \pi_{n}^{\alpha}{\cdot}).$$

By Cartan's lemma

$$\omega_{\beta}^{\alpha}{\cdot} - \pi_{\beta}^{\alpha}{\cdot} - \mu\,\delta_{\beta}^{\alpha}{\cdot} = A_{\beta}^{\alpha}{\cdot}_{\gamma}\omega^{\gamma} + B_{\beta}^{\alpha}{\cdot}\omega^{n}, \qquad A_{\beta}^{\alpha}{\cdot}_{\gamma} = A_{\gamma}^{\alpha}{\cdot}_{\beta}, \qquad \omega_{n}^{\alpha}{\cdot} - \pi_{n}^{\alpha}{\cdot} = B_{\beta}^{\alpha}{\cdot}\omega^{\beta} + B^{\alpha}\omega^{n},$$

for certain functions A and B. The relation

$$\pi_{\alpha}^{\gamma}{\cdot}g_{\gamma\bar{\beta}} + g_{\alpha\bar{\gamma}}\pi_{\bar{\beta}}^{\bar{\gamma}}{\cdot} = 0$$

from (2.2), which also holds for the $\omega_{\alpha}^{\beta}{\cdot}$, when applied to the first equation gives

$$A_{\beta}^{\alpha}{\cdot}_{\gamma} = 0, \qquad B_{\beta\bar{\alpha}} + B_{\bar{\alpha}\beta} = 0.$$

We next differentiate the relation $\operatorname{Re}\omega_{0}^{0}{\cdot} = \operatorname{Re}\pi_{0}^{0}{\cdot}$ to get

$$0 = \operatorname{Re}(\omega^{\alpha}\wedge(\omega_{\alpha}^{0}{\cdot} - \pi_{\alpha}^{0}{\cdot})) + \omega^{n}\wedge(\omega_{n}^{0}{\cdot} - \pi_{n}^{0}{\cdot}).$$

But

$$\omega_{\alpha}^{0}{\cdot} - \pi_{\alpha}^{0}{\cdot} = -ig_{\alpha\bar{\beta}}(\omega_{\bar{n}}^{\bar{\beta}}{\cdot} - \pi_{\bar{n}}^{\bar{\beta}}{\cdot}) = iB_{\alpha\bar{\beta}}\omega^{\bar{\beta}} - iB_{\alpha}\omega^{n},$$

so that

$$0 = \omega^{n}\wedge(\omega_{n}^{0}{\cdot} - \pi_{n}^{0}{\cdot} - \operatorname{Im}(B_{\alpha}\omega^{\alpha})).$$

Since this is a real equation, there is a real function E making the last equation of the lemma true. This finishes the proof.

The error terms $B_{\beta\bar{\alpha}}$, B_α, E in Lemma (4.1) won't vanish in general. Condition (d) of Theorem (3.1) suggests that we should require that the trace of the curvature tensor R vanish:

$$R^{i}_{i\cdot k\bar{l}}=0.$$

This leads to Fefferman's Monge–Ampère equation [4], which was derived from other considerations. To see this we compute the Ricci form relative to the coordinate frame dz^j.

By (1.10) and (1.11)

$$\partial h_{i\bar{j}}=\omega^{k}_{i\cdot}h_{k\bar{j}},$$

so that

$$\Omega^{i}_{i\cdot}=d\omega^{i}_{i\cdot}=d(h^{i\bar{j}}h_{i\bar{j}})=\bar{\partial}\partial\log\det(h_{i\bar{j}}), \tag{4.2}$$

where

$$h_{i\bar{j}}=R_{i\bar{j}}=\partial^2R/\partial z^i\,\partial z^{\bar{j}}.$$

Since, for $i, j\geq 1$,

$$R_{i\bar{j}}=(z^0z^{\bar{0}})^p r_{i\bar{j}},\qquad R_{0\bar{j}}=p(z^0)^{-1}(z^0z^{\bar{0}})^p r_{\bar{j}},$$
$$R_{0\bar{0}}=p^2(z^0z^{\bar{0}})^{p-1}r,$$

the determinant in (4.2) is given by

$$\det(h_{i\bar{j}})=p^2(z^0z^{\bar{0}})^{p(n+1)-1}J(r),$$

where J is the operator

$$J(r)=\det\begin{bmatrix} r & r_{\bar{j}}\\ r_i & r_{i\bar{j}}\end{bmatrix}\qquad(1\leq i,\ j\leq n). \tag{4.3a}$$

If we choose $p=(n+1)^{-1}$, then the equation

$$J(r)=\text{const.} \tag{4.3b}$$

makes the Ricci form (4.2) vanish. A less restrictive condition will do. If (4.3ab) holds to second order at the hypersurface $M:(r=0)$, then one sees from (4.2) that

$\Omega_{i\cdot}^{i}$ will vanish when restricted to $C\times M$ ($\theta^n=\theta^{\bar n}$). If (4.3ab) holds to third or higher order then $\Omega_{i\cdot}^{i}$ will vanish for $r=0$, $\theta^n\neq\theta^{\bar n}$. Summarizing, we have

LEMMA (4.2). *If the defining function r for M satisfies the Monge–Ampère equation* (4.3ab) *to second order at M, then*

$$R_{i\cdot j\bar k}^{i}=0, \quad \textit{unless} \quad j=k=n.$$

If r satisfies the equation to third or higher order at M, then

$$R_{i\cdot j\bar k}^{i}=0, \quad \textit{for all} \quad j,k=0,1,\ldots,n.$$

LEMMA (4.3). *If the defining function r for M satisfies* (4.3ab) *to second order at M, then the coefficients $B_{\alpha\bar\beta}$ and B_α in Lemma* (4.1) *vanish. If r satisfies* (4.3ab) *to third or higher order at M, then the coefficient E also vanishes.*

Proof. We first take the exterior derivative of the trace

$$\omega_{\alpha\cdot}^{\alpha}=\pi_{\alpha\cdot}^{\alpha}+(n-1)\mu+B_{\alpha\cdot}^{\alpha}\omega^n$$

of the first equation of Lemma (4.1). If we utilize the structure equations (1.12) and (3.9), no curvature terms appear, since by Lemma (4.2)

$$\Omega_{\alpha\cdot}^{\alpha}=\Omega_{i\cdot}^{i}=0$$

on P_1. With the aid of (2.2), (3.1a) and (4.1) this exterior derivative becomes

$$\begin{aligned} n(\omega_{\alpha\cdot}^{0}-\pi_{\alpha\cdot}^{0})\wedge\omega^\alpha+i\omega_\alpha\wedge(\omega_{n\cdot}^{\alpha}-\pi_{n\cdot}^{\alpha})&=(n-1)\omega^n\wedge(\omega_{n\cdot}^{0}-\pi_{n\cdot}^{0})+dB_{\gamma\cdot}^{\gamma}\wedge\omega^n\\ &\quad+B_{\gamma\cdot}^{\gamma}(ig_{\alpha\bar\beta}\omega^\alpha\wedge\omega^{\bar\beta}-\omega^n\wedge(\omega^0+\omega^{\bar 0})). \end{aligned} \tag{4.4}$$

Using Lemma (4.1) we get

$$-i(n+1)B_{\alpha\bar\beta}\omega^\alpha\wedge\omega^{\bar\beta}\equiv iB_{\gamma\cdot}^{\gamma}g_{\alpha\bar\beta}\omega^\alpha\wedge\omega^{\bar\beta}, \quad \mathrm{mod}\ \omega^n,$$

so that

$$(n+1)B_{\alpha\bar\beta}=-B_{\gamma\cdot}^{\gamma}g_{\alpha\bar\beta}.$$

The trace of this equation yields

$$B_{\alpha\bar\beta}=0.$$

Equation (4.4) now reduces to

$$\frac{i}{2}(n+1)(B_\alpha\omega^\alpha - B_{\bar\alpha}\omega^{\bar\alpha})\wedge\omega^n = 0,$$

so that

$$B_\alpha = 0.$$

Now if we differentiate $\omega_n^\alpha\cdot$ and use the structure equation (1.12) and Lemma (4.1) we get

$$\Omega_n^\alpha\cdot = \Pi_n^\alpha\cdot + E\omega^\alpha\wedge\omega^n,$$

so that

$$R_{n\cdot\rho\bar n}^{\alpha} = S_{n\cdot\rho\bar n}^{\alpha} + E\,\delta_\rho^\alpha\cdot\cdot$$

The trace of this last equation is

$$R_{i\cdot n\bar n}^{i} = R_{n\cdot\alpha\bar n}^{\alpha} = S_{n\cdot\alpha\bar n}^{\alpha} + (n-1)E = (n-1)E,$$

by Theorem (3.1)(d). The second part of Lemma (4.2) gives the final conclusion of Lemma (4.3), finishing the proof.

THEOREM (4.4).

a) *Suppose that the defining function r of the strongly pseudo-convex real hypersurface M satisfies the Monge–Ampère equation* (4.3ab) *to second order at M. Then between the hermitian curvature tensor R and the pseudo-conformal curvature tensor S the following relations hold:*

$$R_{\alpha\bar\beta\rho\bar\sigma} = S_{\alpha\bar\beta\rho\bar\sigma},\qquad R_{\alpha\bar\beta\rho\bar n} = S_{\alpha\bar\beta\rho\bar n},\qquad R_{\alpha\bar n\beta\bar n} = S_{\alpha\bar n\beta\bar n}.$$

The doubly null curves (sec. 2) *in $C\times M$ project to chains in M. Parallel complex frames along such a curve project to parallel frames along the chain.*

b) *Suppose r satisfies* (4.3ab) *to third or higher order at M. Then the hermitian connection ω and the pseudo-conformal connection π satisfy*

$$\omega_i^j\cdot = \pi_i^j\cdot + \mu\delta_i^j\cdot,\qquad -\mu = \bar\mu,\qquad d\mu = 0.$$

The curvature forms are equal

$$\Omega_i^j\cdot = \Pi_i^j\cdot\cdot$$

The projective parameter on a doubly null curve agrees with the projective parameter on the corresponding chain.

Proof. The proof follows from Lemmas (4.1), (4.2), (4.3) and the structure equations (1.12) and (3.9). That doubly null curves project to chains follows from equations (2.4ab) and (3.11). The statement about parallel translation follows from (2.5) and (3.12). The equality of the parameters follows from (2.6) and (3.13).

COROLLARY (4.5) (Burns–Fefferman–Shnider). *The chains on a strongly pseudo-convex real hypersurface M may be realized as the projections of the null geodesics of a conformal family of Lorentz metrics on a circle bundle over M.*

Proof. If we take as our circle bundle M_f as in section 2, then the proof follows immediately from Theorem (4.4)(a) and Lemma (2.2).

The Lorentz structures (M_f, ds^2) of section 2 are the same as those introduced in [4].

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