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Harmonic analysis and centers of Beurling algebras

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A *weight function* on a locally compact group G is a measurable locally bounded function ω such that $\omega \geq 1$ everywhere and $\omega(xy) \leq \omega(x)\omega(y)$ for all $x, y \in G$. The corresponding *Beurling algebra* $L^1_\omega(G)$ is the Banach algebra under convolution of all functions f such that $f\omega \in L^1(G)$. In case G is locally compact abelian it is well known (cf. [3]) that the maximal ideal space $\Delta L^1_\omega(G)$ may be identified with the space of all complex characters $\chi : G \to (\mathbf{C} - \{0\}, \cdot)$ such that $|\chi(x)| \leq \omega(x)$ for all $x \in G$. The study of the harmonic analysis of such algebras was initiated in 1938 by A. Beurling [2] in the case $G = \mathbf{R}$, and carried on in the case of general l.c.a. G by Domar [3]. Among many other results, Beurling in the special case and Domar in the general case show that if ω is of non-quasianalytic type (i.e., if $\sum \log \omega(x^n)/n^2 < \infty$ for all $x \in G$), then $L^1_\omega(G)$ is (1) regular and (2) Tauberian: that is, (1) the family of Gelfand transforms separates points from closed sets in $\Delta L^1_\omega(G)$, and (2) $\{f \in L^1_\omega(G) \mid \hat{f}$ has compact support$\}$ is dense in $L^1_\omega(G)$. Thus the abstract version of Wiener's Tauberian Theorem (cf. [23, Ch. 2, §2]) holds in $L^1_\omega(G)$.

Now one can ask similar questions about the center $ZL^1_\omega(G)$ of an arbitrary Beurling algebra: what is its maximal ideal space; if $ZL^1_\omega(G)$ is involutive, when is it symmetric; under what conditions is $ZL^1_\omega(G)$ regular and Tauberian; and so on. In a previous paper [15], the authors have shown that to study the center $ZL^1(G)$ of the group algebra, it is indispensable to study the more general context of the algebras $Z^BL^1(G)$ of L^1 functions invariant under a compact group B of automorphisms, where G lies in the special class $[FIA]^-_B$. Similarly in this paper we consider the analogous algebras $Z^BL^1_\omega(G)$ of B-invariant L^1_ω functions. For most of our results we require that ω be B-invariant. This has the effect of requiring in the original context of $ZL^1_\omega(G)$ that ω be invariant under inner automorphisms. We first show (see §1) that the maximal ideal space of $Z^BL^1_\omega(G)$ can be identified with a space $\mathfrak{X}^B_\omega(G)$ of B-invariant, ω-bounded functions on G. In §2, we introduce the notion of a *rate of growth* of $\omega : \Omega(x) = \lim \omega(x^n)^{1/n}$ for all $x \in G$. We show that if ω_1 and ω_2 are B-invariant weight functions with corresponding growth rates Ω_1 and Ω_2, then $\mathfrak{X}^B_{\omega_1}(G) = \mathfrak{X}^B_{\omega_2}(G)$ iff $\Omega_1 = \Omega_2$. Thus the

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possible maximal ideal spaces are parameterized by the B-invariant growth rates; as a corollary we obtain that when ω is symmetric, $Z^B L^1_\omega(G)$ is symmetric iff $\Omega = 1$. In any case, the character space $\mathfrak{X}^B_\omega(G) = \mathfrak{X}^B(G)$, the space of positive definite characters, iff $\Omega = 1$. In a slightly different vein, we prove that when ω is symmetric, then the involutive Banach algebra $L^1_\omega(G)$ is symmetric iff $\Omega = 1$. In §3 we extend some of Domar's results: we show that if ω is of non-quasianalytic type and ω is B-invariant, then $Z^B L^1_\omega(G)$ is regular and Tauberian.

1. Centers and B-centers

We begin by recalling some terminology and results from [7, 13, 14, 15, 18, 19]. If G is a locally compact group, and $B \subset \mathrm{Aut}(G)$ a group of automorphisms, then G is said to be an:

$[IN]_B$ *group* if G contains a compact, B-invariant neighborhood of 1;
$[SIN]_B$ *group* if the B-invariant neighborhoods of 1 form a neighborhood basis at 1;
$[FC]^-_B$ *group* if each $x \in G$ has a precompact orbit under B;
$[FIA]^-_B$ *group* if B is precompact in the usual T_2-group topology on $\mathrm{Aut}(G)$.

In each case the subscript B is omitted if $B = I(G)$, the group of inner automorphisms. Grosser and Moskowitz have proved (cf. [7, 0.1]) that a group G is $[FIA]^-_B$ iff it is both $[FC]^-_B$ and $[SIN]_B$. One can prove that $[FC]^-_B$ groups are also $[IN]_B$ if $B \supset I(G)$ [14, 2.2].

We shall talk of *B-invariant functions* and *essentially B-invariant functions:* if $f^\beta(x) = f(\beta^{-1}x)$, then the former are characterized by the condition $f^\beta = f$ on G, for all $\beta \in B$, and the latter by the condition $f^\beta = f$ locally a.e. on G, for all $\beta \in B$ (where the exceptional set may depend on β). If B is a group of measure-preserving automorphisms, then B has a strongly continuous action on $L^p(G)$ ($1 \le p \le \infty$; for $p = \infty$ the topology in question is the weak-*-topology) induced by the map $(f, \beta) \mapsto f^\beta$ of $\mathscr{L}^p(G) \times B \to \mathscr{L}^p(G)$. (Here $\mathscr{L}^p$ denotes the actual functions, L^p the equivalence classes.) If we denote by $Z^B(L^p(G))$ the subspace of B-invariant elements, then it is clear that a function in $\mathscr{L}^p(G)$ is essentially B-invariant iff its equivalence class is in $Z^B(L^p(G))$.

The groups for which $Z^B(L^1(G)) \ne (0)$ are precisely the $[IN]_B$ groups. This one proves exactly as in [19], where the case $B = I(G)$ is dealt with. Moreover, it is shown there that if $Z(L^1(G))$, the center of $L^1(G)$, is not (0), then $Z(L^1(G)) = Z^{I(G)}(L^1(G))$, G is an $[IN]$ group, and in particular G is unimodular. When $B \supset I(G)$, $Z^B(L^1(G))$ is a closed *-subalgebra of the center $Z(L^1(G))$, and is a commutative, semisimple Banach algebra.

If G is an $[IN]_B$ group, $B \supset I(G)$, we let $G_0 = \{x \in G$: the orbit $B[x]$ is precompact$\}$. Then G_0 is an open, B-invariant subgroup of G, and G_0 is an $[FC]^-_{B_0}$ group, where $B_0 = B|_{G_0}$. Moreover, any essentially B-invariant $\mathcal{L}^1$-function f vanishes a.e. on $G - G_0$ (cf. [20, 1.2]), so the map $f \mapsto f|_{G_0}$ induces an isometric isomorphism of $Z^B(L^1(G))$ onto $Z^{B_0}(L^1(G_0))$. Consequently, in studying $Z^B(L^1(G))$, we can always assume that G is an $[FC]^-_B$ group. We have already observed in [15, 1.2] that we can in fact even assume that G is an $[FIA]^-_B$ group; we shall next prove a refinement of that result. For the proof, and for later use, it is convenient to recall that when G is an $[FIA]^-_B$ group, there is a way of averaging out continuous functions, or classes in L^p, over B-orbits in order to make them B-invariant: if g is continuous, we put $g^{\#}(x) = \int_{B^-} g(\beta^{-1}x)\, d\beta$, where $d\beta$ is normalized Haar measure on the compact group B^-, and $x \in G$; if $g \in L^p(G)$ $(1 \le p \le \infty)$, we put $g^{\#} = \int_{B^-} g^\beta\, d\beta$ (vector-valued integral), which clearly reduces to the same thing if g is continuous. Then $\#$ gives a linear projection of $C_c(G)$ onto $Z^B(C_c(G))$, and a continuous projection of $L^p(G)$ onto $Z^B(L^p(G))$ $(1 \le p \le \infty)$, satisfying

$$\langle f^{\#}, \phi \rangle = \langle f, \phi^{\#} \rangle \qquad \left(\langle f, \phi \rangle = \int f(x)\phi(x)\, dx \right)$$

for $f \in L^p(G)$, $\phi \in L^{p'}(G)$, or $f \in C_c(G)$ and ϕ continuous. These results are proved in [18, §1] for $1 \le p < \infty$, and for $p = \infty$ are easy to prove by transposition. Also, if $f \in L^1(G)$ and $g \in Z^B(L^1(G))$, then $(f * g)^{\#} = f^{\#} * g$.

LEMMA (1.1). *Let G be an $[FC]^-_B$ group, $B \supset I(G)$, and let K denote the intersection of the compact, B-invariant neighborhoods of 1 in G. Then K is a compact B-invariant (normal) subgroup of G, and $G' = G/K$ is an $[FIA]^-_{B'}$ group, where B' is the (precompact) group of automorphisms induced by B. Let $\pi : G \to G/K$ denote the canonical projection.*

(i) *If f is a measurable, essentially B-invariant function on G, then there is a measurable, essentially B'-invariant function f' on G' such that $f = f' \circ \pi$ l.a.e. If f is also continuous (resp. in $\mathcal{L}^1$), then f' may be chosen continuous (resp. in $\mathcal{L}^1$) so that $f' \circ \pi = f$ everywhere (resp. a.e.).*

(ii) *If f is continuous and B-invariant, or $f \in Z^B L^1(G)$, then for f' chosen as in* (i), *we have*

$$f'(\dot{x}) = \int_K f(xk)\, dk.$$

Equality holds everywhere if f is continuous, and a.e. if $f \in Z^B L^1(G)$.

(iii) *The map $f \mapsto f'$ induces an isometric isomorphism of $Z^B L^p(G)$ onto*

$Z^{B'}L^p(G')$ $(1 \le p \le \infty)$, *and a homeomorphism for the weak*-topologies* $\sigma(L^\infty, L^1)$ *if* $p = \infty$.

Proof. The assertions that K is a subgroup and B' is precompact are proved in [7, 2.5]. If f is continuous the remaining assertions are evident, and for $f \in Z^BL^1(G)$ the results follow from [15, 1.2] and its proof. Now without loss of generality assume f is uniformly bounded (replace f by $f/(1+|f|)$). For any compact subset C of G there is a $g \in Z^BL^1(G)$ such that $g|_{CK} = 1$: take $g(x) = h^{\#}(\pi(x))$, where h is the characteristic function of $\overline{B'}[\pi(C)]$. Thus $fg \in Z^BL^1(G)$ so the assertions hold for fg. Therefore f is locally almost everywhere K-periodic so assertion (i) follows from [23, Ch. 3, §6.5]. Assertion (ii) follows from the Weil formula [23, Ch. 3, §4.5] and the fact that K is compact. These last results also show that $\|f' \circ \pi\|_p = \|f'\|_p$ for $1 \le p < \infty$ and $f' \in \mathcal{L}^p(G/K)$, proving (iii) for this case; for $p = \infty$ one applies [23, Ch. 3, §3.9, Corollary].

We turn next to weight functions and Beurling algebras. We shall say that two weight functions ω_1 and ω_2 on G are *equivalent* if there are constants $c, d > 0$ such that for all $x \in G$,

$$c\omega_2(x) \le \omega_1(x) \le d\omega_2(x).$$

In this case ω_1 and ω_2 define the same Beurling algebra $L^1_\omega(G) = \{f \in L^1(G) : \int |f(x)|\, \omega(x)\, dx < \infty\}$ where $\omega = \omega_1$ or ω_2, and the norms $\|\ \|_{\omega_1}$ and $\|\ \|_{\omega_2}$ are equivalent, so for virtually all of our purposes ω_1 and ω_2 are interchangeable. In particular, as Reiter remarks [23, p. 83], if ω is a weight function and ω_1 is its upper semicontinuous envelope,

$$\omega_1(x) = \limsup_{y \to 1} \omega(xy),$$

then ω_1 is a weight function on G, and $\omega(x) \le \omega_1(x) \le \omega_1(1)\omega(x)$ for all $x \in G$, so ω and ω_1 are equivalent. On the other hand, if N is a normal subgroup of G, and ω_2 is defined by $\omega_2(x) = \inf\{\omega(xk) : k \in N\}$, then ω_2 is a locally bounded, submultiplicative function on G [23, p. 85], and thus is a weight function if it is measurable (in particular, if $\omega = \omega_1$ is upper semicontinuous). Now if N is compact, and $c = \sup\{\omega(k) : k \in N\}$ then for all $x \in G$ and $k \in N$ we have $\omega(x) \le \omega(xk)\omega(k^{-1}) \le c\omega(xk)$, so

$$\omega_2(x) \le \omega(x) \le c\omega_2(x). \tag{1}$$

In particular, if ω is upper semicontinuous then ω_2 is a weight function equivalent to ω. We collect a number of technical results on weight functions in the following lemma.

LEMMA (1.2). *Let G be a locally compact group, ω a weight function on G with upper semicontinuous envelope ω_1; if N is a normal subgroup let $\omega_2(x) = \inf\{\omega_1(xk): k \in N\}$.*

(i) *If ω is symmetric ($\omega(x) = \omega(x^{-1})$ for all $x \in G$), then so are ω_1 and ω_2.*

(ii) *If B is a subgroup of* Aut(G), *and ω is B-invariant ($\omega^\beta = \omega$ for all $\beta \in B$), then so is ω_1; if N is B-invariant, then ω_2 is also B-invariant.*

(iii) *If N is compact, and $\omega|_N = 1$, then ω and ω_1 are constant on cosets of N, and thus $\omega_1 = \omega_2$.*

Proof. Straightforward.

Because of the above lemma, *we assume henceforth without loss of generality that all weight functions we consider are upper semicontinuous.*

If G is a locally compact group, B a group of measure-preserving automorphisms, and ω a weight function on G, we can define the *B-invariant Beurling algebra* $Z^B(L^1_\omega(G))$ as $Z^B(L^1(G)) \cap L^1_\omega(G) = \{f \in L^1(G): f^\beta = f$ for all $\beta \in B$, and $\|f\|_\omega < \infty\}$. Then we have: $Z^B(L^1_\omega(G)) \neq (0)$ *if and only if G is an $[IN]_B$ group:* the necessity has already been noticed for $Z^B(L^1(G))$, while if G is an $[IN]_B$ group then the characteristic function of a compact, B-invariant neighborhood of 1 is in $Z^B(L^1_\omega(G))$. If $B = I(G)$, then $Z^B(L^1_\omega(G)) = Z(L^1_\omega(G))$ is just the center of $L^1_\omega(G)$; we call it a *central Beurling algebra.* If $B \supset I(G)$, $Z^B(L^1_\omega(G)) \subset Z(L^1_\omega(G))$, and thus $Z^B(L^1_\omega(G))$ is a commutative, semisimple Banach algebra.

PROPOSITION (1.3). *Let G be an $[IN]_B$ group, $B \supset I(G)$, and let ω be a weight function on G.*

(i) *There is an $[FIA]^-_{B'}$ group G', with $B' \supset I(G')$, and a weight function ω' on G', such that $Z^B(L^1_\omega(G))$ and $Z^{B'}(L^1_{\omega'}(G'))$ are canonically isomorphic as Banach algebras.*

(ii) *If ω is B-invariant, then $Z^B(C_c(G))$ is $\| \ \|_\omega$-dense in $Z^B(L^1_\omega(G))$.*

Proof. (i) is a straightforward modification of [15, Corollary 1.5]. Assuming $Z^B L^1_\omega(G) \neq (0)$ we set $G' = G_0/K$, where G_0 is the open subgroup $\{x \in G: B[x]$ is precompact$\}$ and K is the compact subgroup formed by the intersection of all compact B-invariant neighborhoods of 1. We take $\omega'(\dot{x}) = \inf\{\omega(xk): k \in K\}$ for each $x \in G_0$.

To prove (ii) we must observe first that if ω is invariant under B, then it is also invariant under its closure B^-. For if $x \in G$ is fixed, then by upper semicontinuity $\{y \in G: \omega(y) \geq \omega(x)\}$ is closed and contains $B[x]$. Since $\beta \mapsto \beta x$ is a continuous map of $\mathrm{Aut}(G) \to G$, we conclude that $\omega(\beta x) \geq \omega(x)$ for all $\beta \in B^-$ and $x \in G$. Now replacing x by $\beta^{-1}x$ we see that $\omega^\beta = \omega$ for all $\beta \in B^-$. Now to prove (ii) we may assume by (i) of the proposition that G is an $[FIA]_B^-$ group. Then for any $f \in C_c(G)$, we have easily that $\|f^\#\|_\omega \leq \|f\|_\omega$. Since $C_c(G)$ is $\| \ \|_\omega$-dense in $L^1_\omega(G)$ [23, p. 83], $\#$ extends uniquely to a norm-reducing linear map of $L^1_\omega(G)$ into $Z^B(L^1_\omega(G))$, and one proves that this extension is the identity on $Z^B(L^1_\omega(G))$ (so that $\#$ is a projection of $L^1_\omega(G)$ onto $Z^B(L^1_\omega(G))$) (cf. [18, 1.4]). Consequently if $g \in Z^B(L^1_\omega(G))$ and $f \in C_c(G)$ satisfies $\|g - f\|_\omega < \varepsilon$, then $f^\# \in Z^B(C_c(G))$ and $\|g - f^\#\| = \|g^\# - f^\#\| \leq \|g - f\| < \varepsilon$.

We shall prove next that the maximal ideal space of $Z^B(L^1_\omega(G))$ can be identified, for $[FC]_B^-$ groups and B-invariant weight functions, with a set of functions on G. The suggestive terminology in the next definition is chosen in light of Lemma (1.5).

DEFINITION (1.4). Let G be an $[FC]_B^-$ group, $B \supset I(G)$. A continuous, non-zero, B-invariant function ϕ on G is a *B-spherical function* (*central spherical function*, if $B = I(G)$) if the linear map

$$f \mapsto \int f(x)\phi(x)\,dx = L_\phi(f)$$

is multiplicative on $Z^B(C_c(G))$. If ω is a weight function on G, the set of *ω-bounded* B-spherical functions (those satisfying $|\phi(x)| \leq \omega(x)$ on G) will be denoted by $\mathfrak{X}^B_\omega(G)$, or simply $\mathfrak{X}_\omega(G)$ if $B = I(G)$.

We remark that although the definition is phrased for $[FC]_B^-$ groups instead of $[FIA]_B^-$ groups, the apparent generality gained is only formal, in view of (1.1).

LEMMA (1.5). *Let G be an $[FIA]_B^-$ group, $B \supset I(G)$. If ϕ is a continuous, non-zero function on G, then the following are equivalent:*

(i) *ϕ is a B-spherical function.*

(ii) *For all $x, y \in G$, $\phi(x)\phi(y) = \int_{B^-} \phi(x \cdot \beta y)\,d\beta$.*

(iii) *The function $\tilde{\phi}$, defined on the semidirect product $\tilde{G} = G \times_\eta B^-$ (holomorph) by $\tilde{\phi}(x, \beta) = \phi(x)$, is a spherical function with respect to the compact subgroup $\{1\} \times B^-$, in the sense of* [6].

Proof. See [18, 4.4].

If G is an abelian group, and $B = (1)$, then Lemma (1.5) shows that the "central spherical functions" in the sense of (1.4) are exactly the complex characters of G, that is, the continuous, non-zero homomorphisms of $G \to \mathbf{C}$.

These homomorphisms have been discussed by Mackey in [17]. On the other hand, if say $G=\mathbf{R}^n$ and $B=SO(n)$, then the B-spherical functions on G are exactly the functions

$$\phi_L(x)=\int_{SO(n)} e^{-iL(\beta x)}\,d\beta \qquad (x\in\mathbf{R}^n)$$

where $L\in\mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n,\mathbf{C})$ [9, p. 422].

In previous papers [15, 18] we have used $\mathfrak{X}^B(G)$ to denote the set of non-zero extreme points of the continuous, positive-definite, B-invariant functions ϕ on G satisfying $1\geq\phi(1)\ (=\|\phi\|_G)$. If G is an $[FC]_B^-$ group with $B\supset I(G)$, it follows from results of Hulanicki [11] that $\mathfrak{X}^B(G)$ is precisely the set of bounded (=1-bounded) B-spherical functions on G, $\mathfrak{X}^B(G)=\mathfrak{X}_1^B(G)$: one only needs to know that any $\phi\in\mathfrak{X}_1^B(G)$ is constant on cosets of K (Lemma 1.1) so that G can be assumed to be an $[FIA]_B^-$ group, and then [11, Th. 4.2] applies (cf. also [15, 1.3]). Later on (in §2) we shall consider conditions on ω which imply that $\mathfrak{X}_\omega^B(G)=\mathfrak{X}^B(G)$; this equality means, therefore, that all the ω-bounded B-spherical functions are actually bounded (hence positive-definite).

The next proposition is well-known when G is abelian and B is trivial (see, e.g. [8, Prop. 6.2]).

PROPOSITION (1.6). *Let G be an $[FC]_B^-$ group, $B\supset I(G)$, and let ω be a B-invariant weight function on G. Then the map $\phi\mapsto L_\phi$ is a bijective, bicontinuous map of $\mathfrak{X}_\omega^B(G)$ onto the maximal ideal space ΔA of $A=Z^B(L_\omega^1(G))$, where the former has the topology of uniform convergence on compacta, and the latter has the Gelfand topology.*

Proof. In view of 1.1 and 1.3 we may assume that G is an $[FIA]_B^-$ group. Now the Banach space dual of $L_\omega^1(G)$, for any weight function ω, is the space $L_\omega^\infty(G)$ of equivalence classes (modulo locally null functions) of functions ϕ such that $\|\phi\|_{\infty,\omega}=\|\phi/\omega\|_\infty<\infty$ [23, p. 84]. Using the projection $\#: L_\omega^1(G)\to A$ and the dual projection $\#: L_\omega^\infty(G)\to Z^BL_\omega^\infty(G)$ we can check easily that the dual space of A may be identified with $Z^BL_\omega^\infty(G)$. Now any ω-bounded B-spherical function ϕ is certainly in $Z^B(L_\omega^\infty(G))$, and by (1.3.ii) $L_\phi\in\Delta A$. Conversely, if $\psi\in Z^B(L_\omega^\infty(G))$ and $L_\psi\in\Delta A$, and if $f_0\in A$ satisfies $L_\psi(f_0)=1$, then for all $f\in A$

$$L_\psi(f)=L_\psi(f_0*f)=L_\phi(f),$$

where $\phi=f_0^\check{}*\psi$ (here $f_0^\check{}(x)=f_0(x^{-1})$). But ϕ is ω-bounded and continuous [23, p. 85], and one checks easily that ϕ is B-invariant, hence $\phi\in\mathfrak{X}_\omega^B(G)$. The bicontinuity of the map $\phi\mapsto L_\phi$ is proved as in [8, Th. 2.1], using (1.3.ii) together with the facts that for any $f \in L^1_\omega(G)$ the map $x \mapsto f^x$ is continuous from $G \to L^1_\omega(G)$ [23, p. 84], and that (in view of the properties of the #-operator) weak *-convergence of a net (L_{ϕ_ν}) on A, $\phi_\nu \in Z^B(L^\infty_\omega(G))$, is the same as weak *-convergence on $L^1_\omega(G)$.

LEMMA (1.7). *Let A be a commutative Banach algebra, and let A_0 be a closed subalgebra. Suppose that there is a continuous linear projection P of A onto A_0 satisfying $P(fg_0) = P(f)g_0$ for all $f \in A$, $g_0 \in A_0$. Then each multiplicative linear functional $h_0 \in \Delta(A_0)$ has the form $h_0 = h \mid A_0$ for some $h \in \Delta(A)$.*

Proof. Let $h_0 \in \Delta(A_0)$ and let $M_0 = \ker h_0$. Then M_0 is a regular maximal ideal of A_0. Let e be a relative identity for M_0 in A_0. Then it is routine to check that $M' = \{f \in A \mid P(fg) \in M_0 \text{ for all } g \in A\}$ is a closed ideal in A with relative identity e. It follows that there is a regular maximal ideal M in A with relative identity e such that $M \supset M' \supset M_0$ and $M \cap A_0 = M_0$. Let $h \in \Delta(A)$ correspond to M. Then $h \mid A_0 = h_0$, since $\ker(h \mid A_0) = M \cap A_0 = M_0$.

The next proposition generalizes [18, 5.8] where the case $\omega = 1$ is considered.

PROPOSITION (1.8). *Let G be an $[FIA]^-_B$ group, $B \supset I(G)$ and let ω be a B-invariant weight function on G. A function χ is in $\mathfrak{X}^B_\omega(G)$ iff*

$$\chi = \phi^\# = \int_{B^-} \phi^\beta \, d\beta$$

for some $\phi \in \mathfrak{X}_\omega(G)$, and the map $\phi \mapsto \phi^\#$ is a continuous and proper map of $\mathfrak{X}_\omega(G)$ onto $\mathfrak{X}^B_\omega(G)$.

Proof. For any $\phi \in \mathfrak{X}_\omega(G)$, it is clear that $\phi^\#$ is still ω-bounded and $L_{\phi^\#}$ agrees with L_ϕ on $Z^B(C_c(G))$. Now $\phi^\#(1) = \phi(1) = 1$, and so $\phi^\#$ is a B-spherical function. Conversely, consider $A = Z(L^1_\omega(G))$, $A_0 = Z^B(L^1_\omega(G))$ and the projection $P = \#: A \to A_0$. If $\psi \in \mathfrak{X}^B_\omega(G)$, then by 1.7 there is a $\phi \in \mathfrak{X}_\omega(G)$ such that $L_\psi = L_\phi \mid A_0$. But then $L_\psi = L_{\phi^\#} \mid A_0$ also, and $\phi^\# \in \mathfrak{X}^B_\omega(G)$, so $\psi = \phi^\#$. Furthermore, it is easy to see that $\phi \mapsto \phi^\#: \mathfrak{X}_\omega(G) \to \mathfrak{X}^B_\omega(G)$ is continuous in the respective Gelfand topologies, and extends to a continuous map at infinity if G is not discrete. (The one point compactifications are $\mathfrak{X}_\omega(G) \cup \{0\}$ and $\mathfrak{X}^B_\omega(G) \cup \{0\}$, respectively.) Hence # is a continuous, proper map of $\mathfrak{X}_\omega(G)$ onto $\mathfrak{X}^B_\omega(G)$.

EXAMPLE (1.9). Let $G = \mathbf{R}^n$, $B = SO(n)$ and let ω be a radial weight function on $\mathbf{R}^n$, i.e., one satisfying $\omega(\beta x) = \omega(x)$ for all $x \in \mathbf{R}^n$, $\beta \in SO(n)$. We shall determine the ω-bounded B-spherical functions on $\mathbf{R}^n$. First, any complex character of $\mathbf{R}^n$ is of the form

$$\phi_{u,v}(x)=e^{-i(u\cdot x+iv\cdot x)} \qquad (x\in\mathbf{R}^n),$$

where $u, v\in\mathbf{R}^n$, and $u\cdot x$ denotes the ordinary inner product. If we define ω_0 on $[0,\infty)$ by $\omega_0(\|x\|)=\omega(x)$ $(x\in\mathbf{R}^n)$, then ω_0 satisfies $\omega_0(r+s)\le\omega_0(r)\omega_0(s)$ for all $r, s\ge 0$, so $\alpha=\lim_{r\to\infty} r^{-1}\log\omega_0(r)$ exists and $\alpha=\inf\{r^{-1}\log\omega_0(r): r>0\}$ (cf. (2.1)). Now $\phi_{u,v}$ is ω-bounded iff $\|v\|\,\|x\|\,|\cos\theta|\le\log\omega_0(\|x\|)$ for all $x\in\mathbf{R}^n$, $0\le\theta\le 2\pi$, i.e. iff $\|v\|\le\alpha$. Thus $\mathfrak{X}_\omega(\mathbf{R}^n)$ is identified with $\mathbf{R}^n\times\{v\in\mathbf{R}^n:\|v\|\le\alpha\}\subset\mathbf{R}^n\times\mathbf{R}^n$. By (1.8), every $\chi\in\mathfrak{X}^B_\omega(\mathbf{R}^n)$ is of the form

$$\chi(x)=\chi_{u,v}(x)=\int_{SO(n)} e^{-i(u\cdot\beta x+iv\cdot\beta x)}\,d\beta$$

for $u, v\in\mathbf{R}^n$, $\|v\|\le\alpha$. Taking the Laplacian we get $\Delta\chi_{u,v}(x)=(\|v\|^2-2iu\cdot v-\|u\|^2)\chi_{u,v}(x)$, so that if $\chi_{u,v}=\chi_{u',v'}$ then $\|v\|^2-2iu\cdot v-\|u\|^2=\|v'\|^2-2iu'\cdot v'-\|u'\|^2$. It follows immediately from Corollary 3.3, p. 401 of [9], that the converse is true: if $\|v\|^2-2iu\cdot v-\|u\|^2=\|v'\|^2-2iu'\cdot v'-\|u'\|^2$ then $\chi_{u,v}=\chi_{u',v'}$. Thus $\chi_{u,v}=\chi_{u',v'}$ does not always imply $(u,v)=(\beta u',\beta v')$ for some β; in contrast, if $\omega=1$ (so $v=0$), then $\chi_u=\chi_{u'}$ iff $u=\beta u'$ for some β.

Other useful functorial properties of the set $\mathfrak{X}^B_\omega(G)$ are given in the next proposition.

PROPOSITION (1.10). *Let G be an $[FIA]_B^-$ group, $B\supset I(G)$, and let H be a closed B-invariant subgroup of G.*

(i) *Every B-spherical function on G restricts to a B_H spherical function on H, where B_H is the group of restrictions $\{\beta_H:\beta\in B\}$.*

(ii) *If H is open and ω is a B-invariant weight function on G, then the restriction map of $\mathfrak{X}^B_\omega(G)$ onto $\mathfrak{X}^{B_H}_{\omega_H}(G)$ is surjective, continuous, and proper.*

(iii) *Every B'-spherical function on G/H lifts to a B-spherical function on G. Here B' is the induced group of automorphisms $\{\beta'\in\operatorname{Aut}(G/H)\mid\beta\in B\}$.*

Proof. (i) and (iii) follow from the characterization (ii) of B-spherical functions in Lemma (1.5). For (ii), let $A=Z^BL^1_\omega(G)$, and let $A_0=Z^{B_H}(L^1_{\omega_H}(H))$. A_0 is isometrically embedded in A via the map $f\mapsto f^G$, where $f^G(x)=f(x)$ for $x\in H$ and $f^G(x)=0$ for $x\in G-H$. Also, the restriction map $P: A\to A_0$ given by $Pf=f_H$ satisfies $P(f*g)=Pf*g$ if g is supported in H. Now the result follows from 1.7, using the same proof as in 1.8.

2. Growth of B-spherical functions and symmetry

If G is an $[FC]_B^-$ group, $B \supset I(G)$, and ω is a weight function on G, then $Z^B(L^1_\omega(G)) \subset Z^B(L^1(G))$, so every multiplicative linear functional on the latter algebra restricts to one on the former. We may ask whether there are any other multiplicative linear functionals on $Z^B(L^1_\omega(G))$; i.e., whether every $\phi \in \mathfrak{X}^B_\omega(G)$ is bounded and hence positive definite by Hulanicki's result [11]. More generally, we seek to parameterize the family of character spaces $\{\mathfrak{X}^B_\omega(G)\}$; it turns out (see 2.3) that $\mathfrak{X}^B_\omega(G)$ is determined by the rate of growth Ω of ω. (See 2.1 below for the definition of Ω.) Thus $\mathfrak{X}^B_\omega(G) = \mathfrak{X}^B(G)$ iff $\Omega = 1$. Furthermore, if ω is symmetric, $L^1_\omega(G)$ and $Z^B L^1_\omega(G)$ are $*$ stable, under the usual involution. We will see in (2.5) and (2.6) that $L^1_\omega(G)$ and $Z^B L^1_\omega(G)$ are symmetric iff $\Omega = 1$.

PROPOSITION (2.1). *Let G be an $[FC]^-$ group, and let ω be a weight function on G. For $x \in G$, set $\Omega(x) = \lim_{n\to\infty} \omega(x^n)^{1/n}$, the rate of growth of ω. Then*

(i) *Ω is a continuous weight function which is homogeneous ($\Omega(x^n) = \Omega(x)^n$ for $n = 0, 1, 2, \ldots$). If ω is symmetric, so is Ω.*

(ii) *$1 \le \Omega \le \omega$, and $\Omega = \omega$ iff ω is homogeneous.*

(iii) *If P is the periodic subgroup of G (see [7]), then $\Omega(xk) = \Omega(x)$ for all $x \in G$ and $k \in P$.*

Proof. Since ω is submultiplicative, the limit defining Ω exists and equals $\inf\{\omega(x^n)^{1/n} \mid n \ge 1\}$ (see e.g., footnote 4 in [4, §4.2]); also $\Omega(x^k) = \Omega(x)^k$ for $k = 0, 1, \ldots$. If ω is homogeneous then clearly $\Omega = \omega$; if ω is symmetric so is Ω. To see that Ω is submultiplicative, fix $x \in G$. If C_x denotes the conjugacy class of x, then $x^{-1}C_x$ is a precompact subset of the commutator subgroup of G, and the latter is periodic (i.e., consists of elements which are contained in compact subgroups of G). Thus it follows from [7, Th. 3.11(2)] or [14, 2.1] that $x^{-1}C_x$ is contained in a compact normal subgroup K of G. Therefore every $y \in G$ commutes with x modulo K, so for each n $(xy)^n = x^n y^n k_n$ for some $k_n \in K$. Therefore

$$\omega((xy)^n)^{1/n} \le \omega(x^n)^{1/n}\omega(y^n)^{1/n}(\sup\{\omega(x) : x \in K\})^{1/n}$$

and in the limit we see that $\Omega(xy) \le \Omega(x)\Omega(y)$.

Now if ω_1 is the upper semicontinuous envelope of ω, we have noted (see remarks before (1.2)) that $\omega(x) \le \omega_1(x) \le \omega_1(1)\omega(x)$; thus $\Omega(x) = \inf \omega_1(x^n)^{1/n}$, which shows that Ω is upper semicontinuous. Since $\Omega(1) = 1$ and $\Omega \ge 1$, Ω is continuous at 1. Hence Ω is also continuous on G: for if $p = \log \Omega$, we have $p(1) = 0$ and

$$-p(yx^{-1}) \le p(x) - p(y) \le p(xy^{-1}).$$

Now for any compact subgroup K of G, ω is bounded on K, so $\Omega|_K = 1$. Thus by (1.2), $\Omega(xk) = \Omega(x)$ for all $x \in G$, $k \in K$. Since P is the union of the compact subgroups of G, Ω is constant on cosets of P.

PROPOSITION (2.2). *Suppose $G \in [FC]_B^-$, where $B \supset I(G)$, and ω is a weight function on G. Then*

(i) *$\mathfrak{X}_\omega(G) = \mathfrak{X}_\Omega(G)$, where Ω is the rate of growth of ω.*
(ii) *If ω is B-invariant, $\mathfrak{X}_\omega^B(G) = \mathfrak{X}_\Omega^B(G)$.*

Proof. For the proof, we may assume that $G \in [FIA]_B^-$.

(i) Since $\Omega \le \omega$, we know $\mathfrak{X}_\Omega(G) \subseteq \mathfrak{X}_\omega(G)$. Now suppose that $x \in G$, and that μ is the unique central probability measure supported in the closure C_x^- of the conjugacy class of x (so that $\mu(f) = f^{\#}(x)$ for all $f \in C_c(G)$). As in the proof of (2.1), $x^{-1}C_x^-$ is contained in a compact, normal subgroup K of G, so μ is supported in xK. Therefore the n-fold convolution power μ^n of μ is supported in x^nK for every n. It follows by Lemma 1.5(ii) that if $\phi \in \mathfrak{X}_\omega(G)$ then

$$|\phi(x)|^n = |\langle \phi, \mu^n \rangle| \le \langle |\phi|, \mu^n \rangle \le \langle \omega, \mu^n \rangle \le \omega(x^n) \sup\{\omega(x) : x \in K\}.$$

Thus $|\phi(x)| \le \omega(x^n)^{1/n} (\sup\{\omega(x) \mid x \in K\})^{1/n}$ for all n, and $|\phi(x)| \le \Omega(x)$. This shows $\phi \in \mathfrak{X}_\Omega(G)$.

(ii) Immediate from (i) and 1.8.

Proposition 2.2 and Theorem 2.3 below parameterize the possible character spaces $\mathfrak{X}_\omega^B(G)$, for B-invariant ω, by the B-invariant rates of growth Ω.

EXAMPLE. Suppose ω is a weight function on $\mathbf{R}$. Then $\Omega(x) = e^{\alpha x}$ for $x \ge 0$, and $\Omega(x) = e^{\beta x}$ for $x \le 0$, where $\alpha = \log \Omega(1)$, $\beta = \log \Omega(-1)$. The equality $\mathfrak{X}_\omega(\mathbf{R}) = \mathfrak{X}_\Omega(\mathbf{R})$ thus asserts that the complex character $\chi(x) = e^{-izx}$ is in $\mathfrak{X}_\omega(\mathbf{R})$ iff $-\beta \le \operatorname{Im} z \le \alpha$. In view of 1.6, this means that z is in the common domain of absolute convergence of all the bilateral Laplace transforms $\int f(x) e^{-izx}\,dx$ ($f \in L_\omega^1(\mathbf{R})$) iff $-\beta \le \operatorname{Im} z \le \alpha$, a fact which was already observed by Beurling in [2].

THEOREM (2.3). *Let G be an $[FC]_B^-$ group, $B \supset I(G)$, and ω_i be weight functions on G with rates of growth Ω_i $(i = 1, 2)$. Then*

(i) *$\mathfrak{X}_{\omega_1}(G) = \mathfrak{X}_{\omega_2}(G)$ iff $\Omega_1 = \Omega_2$.*
(ii) *If ω_1 and ω_2 are B-invariant, then $\mathfrak{X}_{\omega_1}^B(G) = \mathfrak{X}_{\omega_2}^B(G)$ iff $\Omega_1 = \Omega_2$.*

Proof. From (2.2 we see that it suffices to prove the following two assertions: (i) $\mathfrak{X}_{\Omega_1}(G) = \mathfrak{X}_{\Omega_2}(G)$ implies $\Omega_1 = \Omega_2$, and (ii) if Ω_1 and Ω_2 are B-invariant, then

$\mathfrak{X}^B_{\Omega_1}(G)=\mathfrak{X}^B_{\Omega_2}(G)$ implies $\mathfrak{X}_{\Omega_1}(G)=\mathfrak{X}_{\Omega_2}(G)$. If P is the periodic subgroup of G [7, 3.16] then P is B-invariant and Ω_1, Ω_2 are constant on P-cosets (2.1). Furthermore P contains the compact subgroup K, intersection of all the compact, B-invariant neighborhoods of 1 in G, so by the discussion at the beginning of §1, $G'=G/P$ is $[FIA]^-_{B'}$ for the induced group of automorphisms; G' is also abelian and aperiodic [7, 3.16]. Using (1.10(iii)) we see that we may replace G by G' and assume that G is an aperiodic abelian group (which we shall write additively) and B is precompact. We shall write $\mathfrak{X}_i(G)$, $\mathfrak{X}^B_i(G)$ $(i=1,2)$ for $\mathfrak{X}_{\Omega_i}(G)$, $\mathfrak{X}^B_{\Omega_i}(G)$, respectively.

Suppose now that Ω_1 and Ω_2 are B-invariant, and $\mathfrak{X}^B_1(G)=\mathfrak{X}^B_2(G)$; we shall prove that $\mathfrak{X}_1(G)=\mathfrak{X}_2(G)$. If not, then there exists ϕ in $\mathfrak{X}_1(G)$, say, but not in $\mathfrak{X}_2(G)$; so for some $x\in G$, $|\phi(x)|>\Omega_2(x)$. Choose c such that $|\phi(x)|>c>\Omega_2(x)$, and let $B_0=\{\beta\in B^-:|\phi(\beta x)|\geq c\}$. Then B_0 is a closed neighborhood of the identity in B^-, since the evaluation map $\beta\mapsto\beta x$ of $\mathrm{Aut}\,(G)\to G$ is continuous. Hence for any $n\in\mathbf{N}$,

$$\begin{aligned}|\phi|^{\#}(nx)&=\int_{B^-}|\phi(\beta(nx))|\,d\beta=\int_{B^-}|\phi(\beta x)|^n\,d\beta\\&\geq\int_{B_0}|\phi(\beta x)|^n\,d\beta\geq c^n\,|B_0|,\end{aligned}$$

where $|B_0|>0$ is the measure of B_0. But $|\phi|$ is also an ω-bounded complex (actually, a real-valued) character of G, so by (1.8) $|\phi|^{\#}\in\mathfrak{X}^B_1(G)=\mathfrak{X}^B_2(G)$; hence $|\phi|^{\#}(nx)\leq\Omega_2(nx)=\Omega_2(x)^n$. Therefore $\Omega_2(x)^n\geq c^n\,|B_0|$, so $\Omega_2(x)\geq c$, a contradiction. Thus $\mathfrak{X}^B_1(G)=\mathfrak{X}^B_2(G)$ implies $\mathfrak{X}_1(G)=\mathfrak{X}_2(G)$.

Now (without assuming B-invariance of ω_1 and ω_2) suppose $\mathfrak{X}_1(G)=\mathfrak{X}_2(G)$; we shall show that $\Omega_1(x)=\Omega_2(x)$ for each $x\in G$. In fact, if p is a continuous, non-negative, homogeneous subadditive function on G, then [5, B.3.1, p. 219] implies that for each $x\in G$ there is an algebraic homomorphism $L:G\to\mathbf{R}$ such that $L\leq p$ and $L(x)=p(x)$. Furthermore since $L\leq p$ it is easy to check that L is continuous. Setting $p=\log\Omega$ we get $\Omega(x)=\sup\{|\phi(x)|:\phi\in\mathfrak{X}_\Omega(G)\}$ for each continuous homogeneous weight function Ω on G. Since $\mathfrak{X}_1(G)=\mathfrak{X}_2(G)$ it follows that $\Omega_1=\Omega_2$.

LEMMA (2.4). *Let G be an $[FC]^-$ group, and let ω be a weight function on G with rate of growth $\Omega=1$. Then for any compact set $A\subset G$, $\lim_{n\to\infty}(\sup\{\omega(x):x\in A^n\})^{1/n}=1$.*

Proof. A is contained in a compact, $I(G)$-invariant subset with non-empty interior (cf. [14, 2.2]), so by passing to an open, normal subgroup we may assume that G is compactly generated. Let K be a compact subgroup of G such that G/K is abelian. If $\omega'(\dot{x}) = \inf\{\omega(xk) \mid k \in K\}$ $(x \in G)$, then by inequalities (1) before (1.2), $\omega'(\dot{x}) \leq \omega(x) \leq \sup\{\omega(k) : k \in K\}\omega'(\dot{x})$. Thus we may replace ω by ω' and G by G/K, and assume that G is abelian. There is a finite set $F = \{x_1, \ldots, x_k\} \subset G$ such that $A^2 \subset FA, \ldots, A^n \subset F^{n-1}A$. Since F is finite, given $c > 1$ there exists $b > 0$ such that $\omega(x^n) \leq bc^n$ for all $n \geq 1$ and all $x \in F$. Now $t \in F^{n-1}$ implies $t = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ where $\alpha_1, \ldots, \alpha_k \in \mathbf{N}$ and $\sum \alpha_i = n - 1$. For any $y \in A$ we then have

$$\begin{aligned} \omega(ty) &\leq \omega(x_1^{\alpha_1}) \cdots \omega(x_k^{\alpha_k})\omega(y) \\ &\leq b^k c^{n-1}(\sup\{\omega(x) : x \in A\}) \end{aligned}$$

Hence

$$(\sup\{\omega(x) : x \in A^n\})^{1/n} \leq b^{k/n} c^{(n-1)/n} (\sup\{\omega(x) : x \in A\})^{1/n} \to c,$$

which proves the lemma.

THEOREM (2.5). *Let G be an $[FC]_B^-$ group, $B \supset I(G)$, and let ω be a B-invariant weight function on G, with rate of growth Ω. Then the following are equivalent:*

(i) $\Omega = 1$
(ii) $\mathfrak{X}_\Omega^B(G) = \mathfrak{X}^B(G)$
(iii) *The maximal ideal space of $Z^B(L_\omega^1(G))$ equals that of $Z^B(L^1(G))$.*

If ω is symmetric, these conditions are equivalent to

(iv) *$Z^B(L_\omega^1(G))$ is a symmetric * algebra; i.e., every complex homomorphism of $Z^B L_\omega^1(G)$ respects the involution.*

Proof. The equivalence of (i), (ii), and (iii) follows from 1.6, 2.2, and 2.3. Now suppose that ω is symmetric so that $L_\omega^1(G)$ is involutive. For each $\phi \in \mathfrak{X}_\omega^B(G)$, let ϕ^* be defined by $\phi^*(x) = \overline{\phi(x^{-1})}$. Then it is easily verified that $Z^B L_\omega^1(G)$ is symmetric iff $\phi = \phi^*$ for each $\phi \in \mathfrak{X}_\omega^B(G) = \mathfrak{X}_\Omega^B(G)$ iff $Z^B L_\Omega^1(G)$ is symmetric. If (ii) holds, then by Hulanicki's result [11] each $\phi \in \mathfrak{X}_\omega^B(G)$ is positive definite, so equals ϕ^*; hence (iv) holds. Conversely, if (iv) holds, we may assume without loss of generality that $\omega = \Omega$. Now Ω is constant on P cosets, where P is the periodic subgroup of G; thus $\mathfrak{X}_\Omega^B(G/P) \subseteq \mathfrak{X}_\Omega^B(G)$ by 1.10(iii) and it is enough to prove (iv) $\Rightarrow$ (ii) in the case $G = G/P$ is locally compact abelian.

So assume G is abelian. By 1.8 it is enough to prove $\mathfrak{X}_\Omega(G) = \mathfrak{X}(G)$ $(= \hat{G})$. Fix $\psi \in \mathfrak{X}_\Omega(G)$; by taking $|\psi|$ we may assume ψ has positive values. For any $f \in Z^B(C_c(G))$, $f\psi \in L^1(G)$, so for all $\phi \in \hat{G}$, we have $(f^*\psi)\hat{}(\phi) = \langle f^*, \psi\phi \rangle = \langle f^*, (\psi\phi)^* \rangle$. But $\psi\phi \in \mathfrak{X}_\Omega(G)$, so $(\psi\phi)^* \in \mathfrak{X}_\Omega^B(G)$, so by the symmetry of

$Z^B(L^1_\Omega(G))$, we have $\langle f^*, (\psi\phi)^{\#}\rangle = \overline{\langle f, (\psi\phi)^{\#}\rangle} = \overline{\langle f, \psi\phi\rangle} = \overline{(f\psi)\hat{}(\phi)} = [(f\psi)^*]\hat{}(\phi)$. By the semisimplicity of $L^1(G)$ we have $f^*\psi = (f\psi)^*$ for all $f \in Z^B(C_c(G))$. Fixing x and choosing f so that $f^*(x) = 1$, we get $\psi(x)^{-1} = \psi(x^{-1}) = \psi(x)$ and $\psi(x) = 1$.

The following extends a result of Pytlik [22], in case $G \in [FC]^-$.

THEOREM (2.6). *Let $G \in [FC]^-$ and let ω be a symmetric weight function on G with rate of growth Ω. Then $L^1_\omega(G)$ is a symmetric *-algebra (see* [24, 4.7]*) iff $\Omega = 1$.*

Proof. Suppose that $\Omega = 1$. To prove that $L^1_\omega(G)$ is symmetric, we may show that whenever $f \in L^1_\omega(G)$, then the spectrum of $f^* * f$ lies in $[0, \infty)$. By [12, Prop. 2.5] this will follow if we show that whenever $f = f^* \in L^1_\omega(G)$, then $\nu_\omega(f) = \|T_f\|$, where $\nu_\omega(f) = \lim_{n\to\infty} \|f^n\|_\omega^{1/n}$ and T is the left regular representation of $L^1_\omega(G)$ on $L^2(G)$. Given $\varepsilon > 0$, there exist $g \in L^1_\omega(G)$ and $h \in C_c(G)$ such that $f = g + h$, $\|g\|_\omega < \varepsilon$. After [1], we make the following calculations:

$$f^{(n)} = (g+h)^{(n)} = \sum_{m=0}^{n} \sum_{p,q} g^{(p_1)} * h^{(q_1)} * \cdots * g^{(p_k)} * h^{(q_k)}$$

where $p = (p_1, \ldots, p_k)$ and $q = (q_1, \ldots, q_k)$ range over sequences of non-negative integers such that $\sum p_i = m$, $\sum q_i = n - m$, and no p_i or q_i is zero except possibly p_1, q_k, or both. Now if $q_k \neq 0$, then

$$\begin{aligned} & g^{(p_1)} * h^{(q_1)} * \cdots * g^{(p_k)} * h^{(q_k)}(s) \\ &\quad = \int \cdots \int g(t_1) \cdots g(t_m) h_{a_1} * h^{(q_1-1)} * \cdots * h_{a_k} * h^{(q_k-1)}(s)\, dt_1 \cdots dt_m, \end{aligned}$$

where $a_1 = t_1 \cdots t_{p_1}$, $a_2 = t_{p_1+1} \cdots t_{p_1+p_2}, \ldots, a_k = t_{p_1+\cdots+p_{k-1}+1} \cdots t_m$, and $h_a(t) = h(a^{-1}t)$. (If $p_1 = 0$, set $a_1 = 1$ and delete $g(t_i)$ and dt_i from the integral, for $i = 1, \ldots, p_1$.)

If $S = \operatorname{supp}(h)$, then $\operatorname{supp}(h_{a_1} * h^{q_1-1} * \cdots * h_{a_k} * h^{q_k-1}) \subset a_1 S^{q_1} \cdots a_k S^{q_k}$; if A is a compact, $I(G)$-invariant set containing S and 1, then $a_1 S^{q_1} \cdots a_k S^{q_k} \subset a_1 \cdots a_k A^{n-m} = t_1 \cdots t_m A^{n-m}$ since $Sa = a(a^{-1}Sa)$. Therefore if $q_k \neq 0$ then

$$\begin{aligned} & \|g^{(p_1)} * h^{(q_1)} * \cdots * g^{(p_k)} * h^{(q_k)}\|_\omega \\ &= \int_G |g^{(p_1)} * \cdots * h^{(q_k)}(s)|\, \omega(s)\, ds \leq \int_G \cdots \int_G |g(t_1)| \cdots |g(t_m)| \\ &\times \int_{t_1 \cdots t_m A^{n-m}} |h_{a_1} * h^{(q_1-1)} * \cdots * h_{a_k} * h^{(q_k-1)}(s)|\, \omega(s)\, ds\, dt_1 \cdots dt_m \end{aligned}$$

$$\leq \left(\sup_{A^{n-m}} \omega\right) \int \cdots \int |g(t_1)| \, \omega(t_1) \cdots |g(t_m)| \, \omega(t_m)$$

$$\times \|h_{a_1} * h^{(q_1-1)} \cdots h_{a_k} * h^{(q_k-1)}\|_1 \, dt_1 \cdots dt_m$$

$$\leq \left(\sup_{A^{n-m}} \omega\right) \mu(A^{n-m})^{1/2} \|T_h\|^{n-m} (\|h_2\| / \|T_h\|) \, \|g\|_\omega^m$$

where the last inequality uses the Schwarz inequality and μ denotes the Haar measure on G (the details of the computation are in [1]). If $q_k = 0$, we can obtain the same inequality by observing that

$$\|g^{(p_1)} * h^{(q_1)} * \cdots * h^{(q_{k-1})} * g^{(p_k)}\|_\omega \leq \|g^{(p_1)} * h^{(q_1)} * \cdots * h^{(q_{k-1})}\|_\omega \, \|g\|_\omega^{p_k}.$$

Since $1 \leq \sup_{A^{n-m}} \omega \leq \sup_{A^n} \omega$, and $\lim (\sup_{A^n} \omega)^{1/n} = 1$ (Lemma 2.4), the proof that $\nu_\omega(f) \leq \|T_f\|$ is completed as in [1]. But T is a * representation and so automatically $\|T_f\| \leq \nu_\omega(f)$.

Conversely, if $L^1_\omega(G)$ is symmetric, then so is $ZL^1_\omega(G)$ (see [24, Corollary 4.7.3]). We have $ZC_c(G) \subset ZL^1_\omega(G) \subset ZL^1_\Omega(G)$, and Ω is central, since it is defined on cosets of the periodic subgroup P and G/P is abelian (see (2.1)(iii).) Hence by (1.3) $ZL^1_\omega(G)$ is $\| \ \|_\Omega$-dense in $ZL^1_\Omega(G)$. Thus $ZL^1_\Omega(G)$ is symmetric, and $\Omega = 1$ by 2.5.

3. Regularity and non-quasianalyticity

When G is a locally compact abelian group, it is a classical fact that the Banach algebra $L^1(G)$ is a regular Tauberian algebra: that there are enough functions in $L^1(G)$ so that the Gelfand transforms separate points from closed sets in the maximal ideal space $\hat{G}$, and that $\{f \in L^1(G) : \hat{f}$ has compact support in $\hat{G}\}$ is dense in $L^1(G)$. When G is an $[IN]_B$ group, $B \supset I(G)$, then $Z^B(L^1(G))$ is also a regular Tauberian algebra; this follows from [15, 2.4 and 2.6], using (1.3). Now if ω is a weight function on G, then $Z^B(L^1_\omega G)) \subset Z^B(L^1(G))$, and we may ask for conditions on ω that ensure that the Gelfand transforms of functions in the B-invariant Beurling subalgebra still separate points from closed sets in $\mathfrak{X}^B(G)$ (the maximal ideal space of $Z^B(L^1(G))$). When G is a locally compact abelian group and $B = (1)$, Domar has proved [3, 2.11] that a necessary and sufficient condition is that ω be of *non-quasianalytic* type:

$$\sum_{n=1}^{\infty} \frac{\log \omega(x^n)}{n^2} < \infty$$

for all $x \in G$; furthermore, he proves that in this case, $L^1_\omega(G)$ is automatically a Tauberian algebra [3, 1.52]. We prove the direct parts of these assertions in the case of $Z^B(L^1_\omega(G))$, using his results.

THEOREM (3.1). *Let G be an $[FIA]_B^-$ group, $B \supset I(G)$, and let ω be a B-invariant weight function on G. If ω is of non-quasianalytic type, then*

(i) *If $\mathcal{K} \subset \mathcal{U} \subset \mathfrak{X}^B(G)$, $\mathcal{K}$ is compact, and $\mathcal{U}$ is open, then there exists $f \in Z^B(L^1_\omega(G))$ such that $0 \le \hat{f} \le 1$, $\hat{f} = 1$ on $\mathcal{K}$, and $\hat{f} = 0$ on $\mathfrak{X}^B(G) - \mathcal{U}$.*

(ii) *If $\mathcal{F} = \{f \in Z^B(L^1_\omega(G)) : \hat{f}$ has compact support on $\mathfrak{X}^B(G)\}$, then $\mathcal{F}$ is dense in $Z^B(L^1_\omega(G))$.*

Proof. The proofs of Propositions 2.4 and 2.6 in [15], replacing $\| \|_1$ by $\| \|_\omega$, show that it suffices to prove the proposition when G is a compactly generated group and $B = I(G)$. In this case, let K be a compact, normal subgroup such that G/K is abelian [7, Th. 3.20]. We recall some results from [15, 2.3]. $\mathfrak{X}(G)$ is the union of open and closed equivalence classes $\mathcal{W}(\phi) = \{\psi \in \mathfrak{X}(G) : \psi|_K = \phi|_K\}$. For each $\phi \in \mathfrak{X}(G)$, if $S = S(\phi) = \{x \in G : \phi(xk) \neq 0$ for some $k \in K\}$, then S is an open subgroup containing K (so S/K is abelian), and $\int_K |\phi(xk)|^2\, dk$ has a constant value $c \neq 0$ on S. Moreover, the map $\lambda \to \lambda^\sim \phi$ is a homeomorphism of the dual group $(S/K)\hat{}$ onto $\mathcal{W}(\phi)$, where $\lambda^\sim(x) = 0$ if $x \in G - S$, $\lambda^\sim(x) = \lambda(\dot{x})$ if $x \in S$ ($\dot{x} = xK$). We may assume therefore that $\mathcal{K} \subset \mathcal{U} \subset \mathcal{W}(\phi)$ for some $\phi \in \mathfrak{X}(G)$. Also, we may assume that ω is upper semicontinuous, so the equation $\omega'(\dot{x}) = \inf\{\omega(xk) \mid k \in K\}$ defines a weight function on G/K, and by restriction on S/K, which is also of non-quasianalytic type. By Domar's result in the case of abelian groups [3, 2.11] there exists $g \in L^1_{\omega'}(S/K)$ satisfying $0 \le \hat{g} \le 1$, $\hat{g}(\lambda) = 1$ if $\lambda^\sim \phi \in \mathcal{K}$, and $\hat{g}(\lambda) = 0$ if $\lambda^\sim \phi \in \mathcal{W}(\phi) - \mathcal{U}$. If we set $f = c^{-1} g^\sim \phi$, then f is central: for $g^\sim$ is constant on K-cosets and G/K is abelian, and ϕ is central. Furthermore $f \in L^1_\omega(G)$, for ϕ is bounded, and

$$\|g^\sim\|_\omega = \int_S |g(\dot{x})|\, \omega(x)\, dx \le \left(\sup_K \omega\right) \int_{S/K} |g(\dot{x})|\, \omega'(\dot{x})\, d\dot{x} < \infty. \tag{*}$$

Finally, $\hat{f}$ vanishes on $\mathfrak{X}(G) - \mathcal{W}(\phi)$, while for $\lambda \sim \phi \in \mathcal{W}(\phi)$ ($\lambda \in (S/K)\hat{}$) we have $\hat{f}(\lambda^\sim \phi) = \hat{g}(\lambda)$: these assertions are proved at the end of the proof of [15, 2.4]. This proves (i).

We have already noted (in (1.6)) that the Banach space dual of $Z(L^1_\omega(G))$ is $Z(L^\infty_\omega(G))$. To prove (ii) it therefore suffices to show that the only $\psi \in Z(L^\infty_\omega(G))$ satisfying $\psi \perp \mathcal{F}$ (under the pairing $\langle f, \psi \rangle = L_\psi(f)$) is $\psi = 0$. For each $\phi \in \mathfrak{X}(G)$, let $\mathcal{F}_\phi = \{g \in L^1_{\omega'}(S(\phi)/K) : \hat{g}$ has compact support in $(S/K)\hat{}\}$. If $g \in \mathcal{F}_\phi$, then $f = c^{-1} g^\sim \bar{\phi}$ is in $\mathcal{F}$, since as above $\hat{f}$ vanishes outside of $\mathcal{W}(\phi)$ and $\hat{f}(\lambda^\sim \phi) = \hat{g}(\lambda)$ for all $\lambda \in (S/K)^{\wedge}$. Therefore $g \in \mathscr{F}_{\phi}$ implies $\langle g^{\sim} \psi, \bar{\phi} \rangle = c\langle f, \psi \rangle = 0$. By Domar's result [3, 1.52] for the case of abelian groups, $\mathscr{F}_{\phi}$ is dense in $L^1_{\omega}(S/K)$, and inequalities (*) then imply that $\langle g^{\sim} \psi, \bar{\phi} \rangle = 0$ for all $g \in L^1_{\omega'}(S/K)$. Since ϕ vanishes indentically on $G - S$, $\langle g^{\sim} \psi, \bar{\phi} \rangle = 0$ for all $g \in L^1_{\omega'}(G/K)$. Now we can let ϕ vary over $\mathfrak{X}(G)$; since $Z(L^1(G))$ is semisimple with maximal ideal space $\mathfrak{X}(G)$, we conclude that $g^{\sim} \psi = 0$ (a.e.) for all $g \in L^1_{\omega'}(G/K)$. In particular, if $E \subset G$ is compact, we may choose $g \in C_c(G/K)$ such that $g(\dot{x}) = 1$ for all $x \in E$; then $\psi(x) = (g^{\sim} \psi(x) = 0$ a.e. on E, so $\psi = 0$ locally a.e. This completes the proof.

Remark. Let us assume that G is an $[FIA]^{-}$ group, and consider the conditions

(i) ω is of non-quasianalytic type;
(ii) $Z(L^1_{\omega}(G))$ is regular as an algebra of functions on $\mathfrak{X}(G)$.

As we have mentioned, Domar proves in [3, 2.11] that (i) and (ii) are equivalent if G is locally compact abelian (in that case $Z(L^1_{\omega}(G))$ is just $L^1_{\omega}(G)$, and $\mathfrak{X}(G)$ is just $\hat{G}$). He also proves [3, 1.41] that (ii) implies the condition

(iii) $\mathfrak{X}(G)$ is the full maximal ideal space of $Z(L^1_{\omega}(G))$.

We do not have a proof of the equivalence of (i) and (ii) in general, although we have just shown that (i) implies (ii). Nevertheless it is easy to see that either one implies (iii). For as we know already (2.5)(iii) is equivalent to the condition "$\Omega = 1$", where Ω is the rate of growth of ω. Now $\Omega \leq \omega$, and Ω is homogeneous, so (i) implies

$$\sum_{n=1}^{\infty} \frac{\log \Omega(x^n)}{n^2} = \sum \frac{\log \Omega(x)}{n} < \infty,$$

hence $\log \Omega = 0$. On the other hand, if P is the periodic subgroup of G, then G/P is abelian. Easy computations show that if (ii) holds then $L^1_{\omega'}(G/P)$ is regular on $(G/P)^{\wedge} \subset \mathfrak{X}(G)$, where $\omega'(x) = \inf\{\omega(xk) \mid k \in P\}$. So by Domar's result, (ii) implies that ω' is of non-quasianalytic type; but, $\Omega(x) \leq \omega'(\dot{x})$ for $x \in G$, since Ω is constant on cosets of P (2.1), so Ω is also of non-quasianalytic type. As before this implies $\Omega = 1$. At the same time we mention that a similar method also proves the equivalence of (i) and (ii) in case G contains a compact, normal subgroup K such that G/K is abelian.

COROLLARY (3.2). *Let G be an $[IN]_B$ group, $B \supset I(G)$, and let ω be a B-invariant weight function on G satisfying*

$$\sum_{n=1}^{\infty} \frac{\log \omega(x^n)}{n^2} < \infty$$

for all $x \in G$. Then $Z^B(L^1_\omega(G))$ is a regular, Tauberian Banach algebra, and in particular Wiener's Tauberian theorem holds: any closed, proper ideal is contained in a regular maximal ideal.

Proof. It is easy to see that the weight function ω' defined in (1.3) is of non-quasianalytic type when ω is, so we may assume by (1.3) that G is an $[FIA]_B^-$ group. In the remark we have shown that non-quasianalyticity of ω implies that its rate of growth $\Omega = 1$, so by (2.3) and (1.6) the maximal ideal space of $Z^B(L^1_\omega(G))$ is just $\mathfrak{X}^B(G)$. Now the corollary follows from (3.1).

We conclude by remarking that the condition of non-quasianalyticity is actually strictly stronger than that of having rate of growth $\Omega = 1$; the following example of a weight function satisfying the latter condition but not the former has been kindly suggested by the referee (another example is given in [16], p. 453). Consider the function u defined for $t \geq 0$ as follows: $u(t) = \exp(t/\log t)$ for $t \geq e$, $u(t) = u(e)$ for $0 \leq t \leq e$. Then $u(t+s) \leq u(t)u(s)$ for $s, t \geq 0$ (if $g(t) = \log u(t)$, then $g(t)/t$ is decreasing, so g is subadditive by [10, Th. 7.2.4]). From the definition $\omega(x) = u(|x|)$ one obtains a (radial) weight function on $G = \mathbf{R}^n$ with the desired properties.

REFERENCES

[1] Z. Anusiak, *Symmetry of L_1-group algebra of locally compact groups with relatively compact classes of conjugated elements*, Bull. Acad. Polon. Sci. *18* (1970), 329–332.

[2] A. Beurling, *Sur les integrales de Fourier absolument convergentes et leur application à une transformation fonctionelle*, IX congres des mathematiciens scandinaves, Helsinki, 1938.

[3] Y. Domar, *Harmonic analysis based on certain commutative Banach algebras*, Acta. Math. *96* (1956), 1–66.

[4] I. Gelfand, D. Raikov, and G. Shilov, *Commutative Normed Rings*, Chelsea, New York, 1964.

[5] R. Giles, *Mathematical Foundations of Thermodynamics*, Pergamon Press, Oxford, 1964.

[6] R. Godement, *Introduction aux travaux de A. Selberg*, Seminaire Bourbaki, 1956/1957, exposé 144.

[7] S. Grosser and M. Moskowitz, *Compactness conditions in topological groups*, J. Reine Angew. Math. *246* (1971), 1–40.

[8] A. Guichardet, *Analyse harmonique commutative*, Dunod, Paris, 1968.

[9] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

[10] E. Hille and R. Phillips, *Functional analysis and semigroups*, A.M.S. Colloquium Publication *#31*, 1957.

[11] A. Hulanicki, *On positive functionals on a group algebra multiplicative on a subalgebra*, Studia Math. *37* (1971), 163–171.

[12] A. Hulanicki, *On the spectrum of convolution operators on groups with polynomial growth*, Invent. Math. *17* (1972), 135–142.

[13] E. Kaniuth, *Zur harmonischen Analyse klassenkompakter Gruppen*, Math. Z. *110* (1969), 297–305.

[14] J. Liukkonen, *Dual spaces of groups with precompact conjugacy classes*, Trans. Amer. Math. Soc. *180* (1973), 85–108.

[15] J. Liukkonen and R. Mosak, *Harmonic analysis and centers of group algebras*, Trans. Amer. Math. Soc. *195* (1974), 147–163.
[16] Y. Lyubich and V. Matsaev, *On operators with a separable spectrum*, Matem. Sb. *56* (1962), 433–468.
[17] G. Mackey, *The Laplace transform for locally compact Abelian groups*, Proc. Nat. Acad. Sci. U.S.A. *34* (1948), 156–162.
[18] R. Mosak, *The L^1- and C^*-algebras of $[FIA]_B^-$ groups and their representations*, Trans. Amer. Math. Soc. *163* (1972), 85–108.
[19] R. Mosak, *Central functions in group algebras*, Proc. Amer. Math. Soc. *29* (1971), 613–616.
[20] R. Mosak and M. Moskowitz, *Central idempotents in measure algebras*, Math. Z. *122* (1971), 217–222.
[21] M. Naimark, *Normed Rings*, Noordhoff N.V., Groningen, 1964.
[22] T. Pytlik, *On the spectral radius of elements in group algebras*, Bull. Acad. Polon. Sci. *21* (1973), 899–902.
[23] H. Reiter, *Classical harmonic analysis and locally compact groups*, Oxford, 1968.
[24] C. Rickart, *General theory of Banach algebras*, Van Nostrand, Princeton, 1960.

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