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Autor(en): **Smillie, John**

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Flat manifolds with non-zero Euler characteristics

By John Smillie

A flat structure on a smooth vector bundle ξ can be given either by a connection on ξ with zero curvature or by a reduction of the structure group of ξ from $Gl(\mathbb{R}^n)$ to $Gl(\mathbb{R}^n)$ with the discrete topology. Using either definition we see that if ξ and ξ' admit flat structures and f is a map then $\xi+\xi'$ and $f^*\xi$ admit flat structures. A flat structure on a manifold M is a flat structure on $T(M)$. We say a manifold is flat if it admits a flat structure.

Let M_g denote the surface of genus g for $g \geq 1$. Milnor (1) proves that an oriented bundle ξ over M_g admits a flat structure if and only if

$$|e(\xi)[M_g]| \leq -\tfrac{1}{2}\chi(M_g) \tag{1}$$

where $e(\xi)$ is the Euler class of ξ and $\chi(M)$ is the Euler characteristic of M. If M_g is a flat manifold this formula implies $\chi(M_g)=0$. According to Hirsch-Thurston (2) it is an old conjecture that the Euler characteristic of any flat manifold is zero. The question is raised by Milnor (1) and Kamber-Tondeur (3, p. 47). In this paper we prove the following.

THEOREM. *There are flat manifolds M^{2n} with non-zero Euler characteristic for all $n>1$.*

Starting with surfaces we construct manifolds M^{2n} by taking products and connected sums. To prove the existence of a flat structure on $T(M^{2n})$ we establish that $T(M^{2n})$ is isomorphic to a second bundle ζ. ζ admits a flat structure because it is the sum of pullbacks of 2-plane bundles which admit flat structures. A bundle isomorphism allows us to transfer the flat structure from ζ to $T(M)$.

ALMOST TRIVIAL BUNDLES. We will call a bundle ξ^n almost trivial if $\xi^n+\varepsilon^1=\varepsilon^{n+1}$ where ε^n denotes the trivial bundle of dimension n.* Sums and pullbacks of almost trivial bundles are almost trivial. If $T(M)$ is almost trivial then M is said to be almost parallelizable.

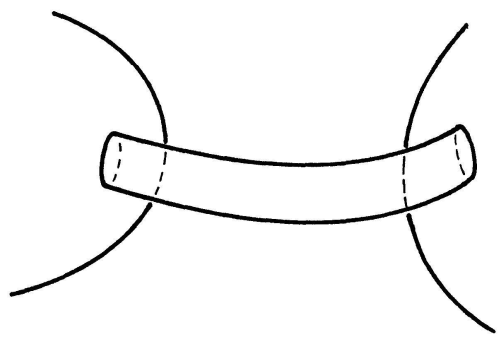
* If the base space of ξ^n is n-dimensional then ξ^n is almost trivial if and only if ξ^n is stably trivial.

PROPOSITION 2. *An oriented 2-plane bundle ξ over a surface M_g is almost trivial if $e(\xi)[M_g]$ is even.*

Proof. $\xi+\varepsilon^1$ admits a two-frame over M_g if a certain obstruction class $\mathcal{O}_2(\xi+\varepsilon^1)$ vanishes. $\mathcal{O}_2$ can be identified with $w_2(\xi+\varepsilon^1)=w_2(\xi)$ which vanishes when $e(\xi)[M_g]$ is even. Assuming $\mathcal{O}_2=0$, $\xi+\varepsilon^1=\eta+\varepsilon^2$ but $w_1(\eta)=w_1(\xi)=0$ so $\eta=\varepsilon^1$.

PROPOSITION 3. *If n dimensional manifolds N and M are almost parallelizable then $N\#M$ is almost parallelizable.*

Proof. We can immerse N and M in $\mathbb{R}^{n+1}$. We can connect them by a tube. Smoothing the corners we see that we have constructed an immersion of $N\#M$ in $\mathbb{R}^{n+1}$ hence $N\#M$ is almost parallelizable.



LEMMA. *If ξ is an almost trivial bundle over M then there is a map $g:M\to S^n$ so that $\xi=g^*T(S^n)$.*

Proof. Choose a bundle map $f:\xi+\varepsilon^1\to\mathbb{R}^{n+1}$. We have a non-zero section $s:M\to\xi+\varepsilon^1$ associated to ε^1. Using a Gram-Schmidt procedure we may assume that $fs(x)$ is of unit length and perpendicular to $f(\xi_x)$.

$$\begin{tikzcd} \xi \arrow[r, "fsp\times f|\xi"] \arrow[d, "p"] & S^n\times\mathbb{R}^{n+1} \arrow[d, "\pi"] \\ M \arrow[r, "fs"] & S^n \end{tikzcd}$$

Set $g=fs$. ξ is the pullback by g of the sub-bundle $T(S^n)\subset S^n\times\mathbb{R}^{n+1}$ consisting of pairs (x, y) where y is perpendicular to x.

PROPOSITION 4. *Let η_1 and η_2 be almost trivial oriented $2n$-plane bundles over an oriented $2n$-manifold. If $e(\eta_1)=e(\eta_2)$ then η_1 and η_2 are isomorphic.*

Proof. The lemma yields maps g_i such that $\eta_i=g_i^*T(S^{2n})$. $e(\eta_i)=2g_i^*u$ where $u\in H^{2n}(S^{2n};\mathbb{Z})$ is dual to the orientation class. The homotopy class of $g_i:M^{2n}\to S^{2n}$ is determined by the cohomology class g_i^*u hence g_1 and g_2 are homotopic. The isomorphism type of $g_i^*T(S^{2n})$ depends only on the homotopy type of g_1 hence η_1 and η_2 are isomorphic.

PROOF OF THEOREM. We construct a flat 4-manifold N with $\chi(N)=4$ and a flat 6-manifold Q with $\chi(Q)=8$. Products of these two manifolds with themselves give examples of flat manifolds with non-zero Euler characteristics of any even dimension greater than 2.

Let P be a parallelizable 4-manifold such as $S^1\times S^3$. Let $N=(M_3\times M_3)\#\underbrace{P\#\cdots\#P}_{6\text{-copies}}$ where M_3 is the surface of genus 3. Clearly $M_3\times M_3$ is almost parallelizable. N is almost parallelizable by proposition 3. We have the formulas

$$\chi(M_g)=2-2g$$
$$\chi(M^{2n}\#N^{2n})=\chi(M^{2n})+\chi(N^{2n})-2.$$

We calculate that $\chi(N)=4$.

Let ξ be the oriented bundle over M_3 with $e(\xi)[M_3]=2$. Proposition 2 implies that ξ is almost trivial. $\xi\times\xi$ is a bundle over $M_3\times M_3$. Let $f:N\to M_3$ be a degree one map that, for example, sends the six P summands to a point. $f^*\xi\times\xi$ is almost trivial and $e(f^*\xi\times\xi)[N]=4$. $T(N)$ is almost trivial and $e(T(N))[N]=\chi(N)=4$. It follows from proposition 4 that $T(N)$ and $f^*\xi\times\xi$ are isomorphic. Milnor's formula implies that ξ admits a flat structure hence $f^*\xi\times\xi$ admits a flat structure. Choosing an isomorphism we transfer the flat structure from $f^*\xi\times\xi$ to $T(N)$.

Let Q be the 6-manifold $\{(M_3\times M_3)\#\underbrace{P\#\cdots\#P}_{9\text{-copies}}\}\times M_3$. We calculate that $\chi(Q)=8$. We have a flat bundle $\eta=\{h^*(\xi\times\xi)\}\times\xi$ over Q where h is the appropriate degree one map. $e(\eta)[Q]=8$. $T(Q)$ and η are almost trivial and have the same Euler class. By proposition 4 they are isomorphic hence $T(Q)$ admits a flat structure.

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Dept. of Mathematics
University of Chicago,
Chicago, Ill. 60637 U.S.A.

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