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On Products in a Family of Cohomology Theories Associated to the Invariant Prime Ideals of $\pi_*(BP)$

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1. Introduction

Let $I_n \subset \pi_*(BP)$ be a $BP^*(BP)$-invariant prime ideal. Using Baas–Sullivan technique one may construct a BP-module spectrum $P(n)$ with the property that $\pi_*(P(n)) = BP_*/I_n$ ([5]). For different n, the $P(n)$ are related by exact triangles in the stable homotopy category and the whole tower of the homology theories $P(n)_*(-)$ provides a good tool for describing the structure of $BP_*(-)$ (see [5]). If one is willing to work with $P(n)_*(-)$, however, a source of pain arises in the lack of usable information about multiplicative structures in these theories. The purpose of this paper is to fill this gap, at least for $p > 2$.

Let p be an *odd* prime. Our main result (Theorem 5.1) implies in particular that the spectra $P(n)$ are perfectly good ring spectra and that there is a *canonical* choice for the product map $m_n : P(n) \wedge P(n) \to P(n)$ (see 2.14). As a first application of this we show that $P(n)_*(P(n))$ is a Hopf algebra and determine its structure (2.13). It turns out that $P(n)_*(P(n))$ interpolates between $BP_*(BP)$ and $H\mathbb{Z}_{p^*}(H\mathbb{Z}_p)$. We then prove that the members of a family of exotic K-theories $K(n)^*(-)$ studied by Morava and others are also multiplicative (in a canonical way, see 2.14).

The method we use is entirely based on [9] and uses no geometry. We remark that if $k \geq \max\{m, n\}$, $P(k)^*(P(m) \wedge P(n))$ is a comodule over the coalgebra $P(k)^*(BP \wedge BP)$ (where everything is endowed with a profinite topology) and our main result is proved by an analysis of primitive elements in $P(k)^*(P(m) \wedge P(n))$.

Section 2 contains some preliminaries and a collection of our results. In 3 we adapt some methods and results from [9] to our present situation and use this in Section 4 to give a description of the primitive elements in the $P(k)^*(BP \wedge BP)$-comodule $P(k)^*(P(m) \wedge P(n))$. Using this we prove in 5 our main result. In Section 6 we determine the structure of the Hopf algebra $P(n)_*(P(n))$ and in 7 we show that the theories $K(n)^*(-)$ are multiplicative. We close the paper with some remarks.

After this paper has been written, the author became aware of a preprint of Shimada and Yagita on "Multiplications in the complex bordism theory with singularities." Whereas there seems to be some overlap with parts of the results (concerning existence of products in the theories $P(n)^*(-)$), the methods used are completely different.

It is a pleasure for us to thank Peter Landweber for his careful reading of the manuscript and his helpful comments.

Some of the results of this paper have been announced in [18].

2. Statement of the main results

Let BP denote the Brown–Peterson spectrum for the fixed prime p. Recall that

2.1. $BP_*\otimes\mathbb{Q}=\mathbb{Q}[m_1, m_2, \ldots]$

where the m_i are the coefficients of the logarithm of the formal group law $F_{BP}(X, Y)$ of $BP^*(-)$:

2.2. $\log_{BP}(T)=\sum_{i\geq 0} m_i \cdot T^{p^i}, \qquad m_0=1.$

In particular, degree $m_i=2(p^i-1)$. From the construction of $F_{BP}(X, Y)$ and Miscenko's calculation of $\log_{MU}(T)$ it follows that

2.3. $m_i=p^{-i}\cdot[\mathbb{C}P_{p^i-1}]$.

Following Hazewinkel [4] we may (and we will) chose polynomial generators v_i of BP_* which are related to the m_i by the formula

2.4. $v_n=p\cdot m_n-\sum_{i=1}^{n-1} m_{n-i}\cdot (v_i)^{p^{n-i}}.$

We agree to put $v_0=p$. Note that $BP^*=BP_{-*}$ and that v_i, considered as an element of BP^*, has degree $-2(p^i-1)$. For all $n>0$ let $I_n\subset BP^*$ be the prime ideal $I_n=(v_0, v_1, \ldots, v_{n-1})$ and set $I_0=(0)$, $I_\infty=\bigcup_{n>0} I_n$. A result of Landweber (see also [5] for a simple proof) asserts that the I_n are exactly the finitely generated prime ideals of BP^* which are invariant under the action of $BP^*(BP)$.

Following Johnson and Wilson [5] there is a tower of BP-module spectra (one for each invariant prime ideal I_n) $P(n)$ and $k(n)$ and maps of BP-module spectra

2.5.
$$
\begin{tikzcd}
BP \simeq P(0) \arrow[d, "\eta_0"'] & & \\
& P(0) \arrow[ul, "\theta_0"'] \arrow[r, "\lambda_0'"] & H\mathbb{Q} \\
BP\mathbb{Z}_p \simeq P(1) \arrow[ur] \arrow[d, "\eta_1"'] & & \\
& P(1) \arrow[ul, "\theta_1"'] \arrow[r, "\lambda_1'"] & k(1) \simeq \widetilde{G\mathbb{Z}}_p \\
P(2) \arrow[ur, "\partial_1"'] \arrow[d] & & \\
\vdots \arrow[d] & & \\
P(n) \arrow[d, "\eta_n"'] & & \\
& P(n) \arrow[ul, "\theta_n"'] \arrow[r, "\lambda_n'"] & k(n) \\
P(n+1) \arrow[ur, "\partial_n"'] \arrow[d] & & \\
\vdots \arrow[d] & & \\
H\mathbb{Z}_p \simeq P(\infty) = \varinjlim P(m) & &
\end{tikzcd}
$$

with the following properties:

2.6. $P(n)^* = P(n)^*(S^0) \cong BP^*/I_n \cong \mathbb{Z}_p[v_n, v_{n+1}, \ldots]$
2.7. θ_n is induced by multiplication with v_n, i.e.

$$\theta_n : S^{|v_n|} \wedge P(n) \xrightarrow{\varphi \wedge id} BP \wedge P(n) \xrightarrow{\nu_n} P(n)$$

where φ represents v_n and ν_n denotes the module map $BP \wedge P(n) \to P(n)$

2.8. ∂_n has degree $2p^n - 1$, η_n has degree 0.
2.9. The triangles in 2.5 are exact in the stable category **S**.
2.10. $k(n)^*(S^0) \cong \mathbb{Z}_p[v_n]$
2.11. The ring structure on $P(n)^*$ and $k(n)^*$ is induced by the respective BP-module maps (see [5], appendix) and $P(n)^*(-)$ (resp. $k(n)^*(-)$) takes

values in the category of $P(n)^*$-modules (resp. $k(n)^*$-modules). I.e. $P(n)^*(-)$ and $k(n)^*(-)$ are cohomology modules in the sense of Section 3.

Remark. $\tilde{G}$ denotes the (-1)-connected cover of the spectrum G corresponding to the multiplicative summand in the Adams splitting of $K^*(-, \mathbb{Z}_{(p)})$ [1]. 6.15 of [9] implies that there is a canonical equivalence of BP-module spectra $k(1) \simeq \tilde{G}\mathbb{Z}_p$. See also [11].

Let $\mu_n : BP \to P(n)$ be the map of BP-module spectra $\mu_n = \eta_{n-1} \circ \cdots \circ \eta_0$ and denote by $m : BP \wedge BP \to BP$ the product map of the ring spectrum BP. m induces an exterior product

$$\wedge : BP^*(X) \otimes BP^*(Y) \to BP^*(X \wedge Y).$$

The following is an immediate consequence of the more general Theorem 5.1: (for the case $p=2$, compare the Remark 5.7).

2.12. THEOREM. *Let p be an odd prime. There is one and only one natural bilinear pairing*

$$\wedge_n : P(n)^*(X) \otimes P(n)^*(Y) \to P(n)^*(X \wedge Y)$$

over the category of CW-complexes (or spectra) such that

(1) $\wedge_n(\Sigma(x) \otimes y) = \Sigma\left(\wedge_n(x \otimes y)\right) = (-1)^{|x|} \wedge_n(x \otimes \Sigma y)$

(2) $\mu_n : BP^*(-) \to P(n)^*(-)$ *is multiplicative*

(3) *For all* $u \in BP^*(X)$, $v \in BP^*(Y)$, $x \in P(n)^*(X)$ *and* $y \in P(n)^*(Y)$ *one has*

$$(u \cdot x) \wedge_n (v \cdot y) = (-1)^{|v||x|}(u \wedge v) \cdot \left(x \wedge_n y\right)$$

Moreover, $\wedge_n$ *is associative, commutative and has a unit and the transformations* η_n ; $P(n)^*(-) \to P(n+1)^*(-)$ *are multiplicative.*

As a first application of this theorem we show in §6 that $P(n)_*(P(n))$ is a Hopf algebra (in the sense of Adams), it then follows that the homology theory $P(n)_*(-)$ takes values in the category of $P(n)_*(P(n))$-comodules. Recall that $BP_*(BP) = BP_*[t_1, t_2, \ldots]$ where $|t_i| = 2(p^i - 1)$. There is a canonical map of ring spectra $T_n: P(n) \to H\mathbb{Z}_p$ and μ_n and T_n induce homomorphisms of Hopf algebras $(\mu_n \wedge \mu_n)_*: BP_*(BP) \to P(n)_*(P(n))$ resp. $(T_n \wedge T_n)_*: P(n)_*(P(n)) \to H\mathbb{Z}_{p*}(H\mathbb{Z}_p) \cong \mathbb{Z}_p[\xi_1, \xi_2, \ldots] \otimes E(\tau_0, \tau_1, \ldots)$. Modulo our knowledge of the structure of $BP_*(BP)/I_n$ which has been studied in [17], the following theorem describes $P(n)_*(P(n))$ (ψ_n resp. c_n denote the diagonal resp. the canonical antiautomorphism):

2.13. THEOREM. *Let p be an odd prime, n a positive integer. Then*

(1) *There is an isomorphism of (left) $P(n)_*$-algebras*

$$P(n)_*(P(n)) \cong P(n)_* \otimes \mathbb{Z}_p[s_1, s_2, \ldots] \otimes E(a_0, \ldots, a_{n-1})$$

where the generators s_i and a_k may be chosen such that

$$(\mu_n \wedge \mu_n)_*(t_i) = s_i \text{ resp. } (T_n \wedge T_n)_*(a_k) = -\tau_k.$$

(2) *$C(n)_* = P(n)_* \otimes \mathbb{Z}_p[s_1, s_2, \ldots]$ is a sub-Hopf-algebra of $P(n)_*(P(n))$ and $(\mu_n \wedge \mu_n)_*$ induces an isomorphism $BP_*(BP)/I_n \cong C(n)_*$.*

(3) *On the exterior generators a_k we have*

$$\Psi_n(a_k) = \sum_{i=0}^{i=k} a_i \otimes s_{k-i}^{p^i} + 1 \otimes a_k$$

$$c_n(a_k) = -a_k.$$

(4) *The right unit $\eta_R^n: P(n)_* \to P(n)_*(P(n))$ factors through the right unit of $BP_*(BP)/I_n$.*

Let $T_n \subset k(n)^*(S^0) = \mathbb{Z}_p[v_n]$ denote the multiplicatively closed set $\{1, v_n, v_n^2, \ldots\}$. If we localize $k(n)^*(-)$ with respect to T_n we get a periodic cohomology theory $K(n)^*(-) = T_n^{-1}k(n)^*(-)$ with coefficient object $\mathbb{Z}_p[v_n, v_n^{-1}]$. These exotic K-theories have been studied by Morava (see for example [8]) and Johnson and Wilson [5] (see also [9]). At first glance, $K(n)^*(-)$ is only defined on the category of finite complexes, but because $K(n)^*(S^0)$ is finite in each dimension, it admits a unique additive extension on the category of all complexes. There is a canonical transformation $\lambda_n: P(n)^*(-) \to K(n)^*(-)$ which is simply the composite of λ_n' with the localization map.

2.14. THEOREM. *Let p be an odd prime. For all $n > 0$, $K(n)^*(-)$ admits a unique associative and commutative product with unit such that the transformation λ_n is multiplicative.*

Let $F(X, Y)$ be a commutative one-dimensional formal group law over a graded ring R of positive characteristic p. The *height* of $F(X, Y)$, $ht(F)$, is the greatest integer r such that $[p]_F(X)$ may be written as a power series of the form

$$2.15.\quad [p]_F(X) = a_r X^{p^r} + a_{r+1}(X^{p^r})^2 + \dots, \qquad a_r \neq 0,$$

see [16] for details. If $[p]_F(X) = 0$, we set $ht(F) = \infty$. As an application of 2.13 we show (6.8) that the theories $P(n)^*(-)$ are universal for complex-oriented ring theories with formal group of height n and coefficient ring h^* satisfying $h^{1-2p^k} = 0$ for $k = 0, 1, \dots, n-1$.

3. *BP*-module theories with locally finite coefficients

Let $\mathbf{W}$ be the category of pointed spaces of the homotopy type of a CW-complex, $\mathbf{W}^f$ the full subcategory of finite CW-complexes. All cohomology theories $h^*(-)$ we consider are supposed to be representable (if they are defined on $\mathbf{W}$), $h^*(-)$ may then be extended on the stable category $\mathbf{S}$. The representing spectrum of $h^*(-)$ is denoted by h.

Suppose the coefficient object $h^* = h^*(S^0)$ of the cohomology theory $h^*(-)$ is a commutative (graded) ring with unit. Following the notation of [9] $h^*(-)$ is called a *cohomology module*, if it takes values in the category of h^*-modules. A transformation of cohomology modules is a transformation of cohomology theories $\tau: h^*(-) \to k^*(-)$ which commutes with the respective module maps and such that $\tau_{S^0}: h^* \to k^*$ is a ring homomorphism.

Let E be a ring spectrum, $h^*(-)$ a cohomology module. By an *E-orientation* for $h^*(-)$ we mean a map of spectra of degree 0 $\nu_h: E \wedge h \to h$ which turns h into a module spectrum over E and which has the property that the natural transformation $\mu_h: E^*(-) \to h^*(-)$ defined by $\mu_h(u) = \nu_h(u \otimes c_X^*(1))$ is a transformation of cohomology modules ($c_X: X \to S^0$ denotes the trivial map). A transformation of E-oriented cohomology modules is a transformation of cohomology modules τ such that $\tau \circ \nu_h = \nu_k(id \wedge \tau)$. An E-oriented cohomology module will also be called an E-module theory. Obviously, the canonical map μ_h is a transformation of E-module theories. Moreover, if θ is any transformation of E-module theories, one has $\theta\mu_h = \mu_k$.

EXAMPLES. The theories $MUQ^*(-)$ considered in [9] are MU-module theories, the $P(n)^*(-)$ and $k(n)^*(-)$ are for all primes p and for all n BP-module theories. Using Baas–Sullivan technique one may construct a lot of further examples (for different E's).

Let R be a locally finite graded ring (i.e. R^q is a finite abelian group for all q). Let $\mathbf{Mod}_R^{\mathrm{prof}}$ denote the category whose objects are inverse limits (over directed sets) of finitely generated graded R-modules and whose morphisms are homomorphisms of graded R-modules which are continuous in each dimension (with respect to the limit topology). $\mathbf{Mod}_R^{\mathrm{prof}}$ is abelian ([9], 3.4).

Remark. The definition of $\mathbf{Mod}_R^{\mathrm{prof}}$ given in [9] is slightly different from the present one. Both categories have the same properties and the proofs are word for word the same (in fact they are equivalent). However, the definition of $\mathbf{Mod}_R^{\mathrm{prof}}$ given in [9] may cause some difficulties with the tensor product $\boxtimes$ introduced there, which are avoided if one restricts to finitely generated modules (which is sufficient for the purposes of both [9] and the present paper).

If $M = \varprojlim M_\alpha$ is an object of $\mathbf{Mod}_R^{\mathrm{prof}}$, the system $\bar{M}_\alpha = \ker\{M \to M_\alpha\}$ is a basis of open neighborhoods of 0 and $M \cong \varprojlim M/\bar{M}_\alpha$. If $\{a_\alpha\}_{\alpha \in A}$ is a set of indeterminates of degree $|a_\alpha|$, we denote by $R[[a_\alpha]]$ the R-module whose elements of degree q are the infinite sums $\sum \lambda_\alpha a_\alpha$ where $\lambda_\alpha \in R$ and $|\lambda_\alpha + a_\alpha| = q$. $R[[a_\alpha]]$ is an object of $\mathbf{Mod}_R^{\mathrm{prof}}$ and isomorphic (in $\mathbf{Mod}_R^{\mathrm{prof}}$) to the product $\Pi R \cdot a_\alpha$.

Let M, N be objects of $\mathbf{Mod}_R^{\mathrm{prof}}$. We define their tensor product $M \hat{\otimes}_R N$ (this has been written $M \boxtimes_R N$ in [9]) by

3.1. $$M \hat{\otimes}_R N = \varprojlim_{\alpha,\beta} (M \otimes_R N/[im(\bar{M}_\alpha \otimes_R N) + im(M \otimes_R \bar{N}_\beta)]),$$

see [9], 3.6. Note that $M \hat{\otimes}_R N = M \otimes_R N$ if M, N are finitely generated R-modules.

Remark. 3.1. is clearly an ad hoc construction. In a systematic treatment one would first define a completion functor $\hat{M}$ (for *all* R-modules) by $\hat{M} = \varprojlim M/L$ where L runs over all submodules of M such that M/L is a locally finite R-module and then define $M \hat{\otimes}_R N = (M \otimes_R N)^\wedge$. In our present situation this reduces to 3.1.

If $h^*(-)$ is a cohomology module with locally finite coefficient ring, it takes values in the category $\mathbf{Mod}_{h^*}^{\mathrm{prof}}$ ([9], 3.9). If $h^*(-)$ is a BP-module theory, the same argument as in [9], 2.8 shows that

3.2. $$h^*(BP) \cong h^*[[\mu_h(r^E)]] \qquad (\text{in } \mathbf{Mod}_{h^*}^{\mathrm{prof}})$$

where the $r^E \in BP^*(BP)$ are the Quillen-operations. For any BP-module theory $h^*(-)$ with locally finite coefficients, the module map $\nu_h: BP \wedge h \to h$ induces a natural isomorphism (in $\mathbf{Mod}_{h^*}^{\text{prof}}$)

3.3. $t_X: h^*(BP)\hat{\otimes}_{h^*} h^*(X) \xrightarrow{\simeq} h^*(BP\wedge X)$

over the category $\mathbf{S}(-1)$ of (-1)-connected CW-spectra. This is the BP-analog of [9], 3.14 and is proved in the same manner (see also the proof of 3.8). Let $m: BP\wedge BP \to BP$ denote the product map of the ring spectrum BP. From 3.3 it follows that m induces a map

3.4. $m^*: h^*(BP) \to h^*(BP)\hat{\otimes}_{h^*} h^*(BP)$

for any BP-module theory with locally finite coefficients and one sees that $h^*(BP)$ is a coalgebra in $\mathbf{Mod}_{h^*}^{\text{prof}}$ with coaction map m^* and counit induced by $i: S^0 \to BP$. Note that the category $\mathbf{Com}_{h^*(BP)}$ of $h^*(BP)$-comodules which are objects of $\mathbf{Mod}_{h^*}^{\text{prof}}$ is abelian ([9], 4.3.).

The following BP-analog of [9], proposition 4.12, is proved exactly as there:

3.5. PROPOSITION. *Let $h^*(-)$ be a BP-module theory with locally finite coefficients and suppose $\mu_h(I_n)=0$. Then for all $m\le n$ there is an isomorphism of $h^*(BP)$-comodules*

$$h^*(P(m)) \cong h^*(BP)\hat{\otimes}_{h^*} E_{h^*}(a_0, \ldots, a_{m-1})$$

where degree $a_i = 2p^i - 1$. Moreover, the morphisms $(\eta_m)^*$ are split epic in $\mathbf{Com}_{h^*(BP)}$ for $0\le m \le n$.

We will need the following variant of 3.3. By a *BP-module theory with product* we mean a BP-module theory $h^*(-)$ together with a map of spectra $m_h: h\wedge h \to h$ of degree 0 such that
3.6. the homomorphism $\mu_h: BP^*(S^0) \to h^*(S^0)$ is epic and
3.7. the diagram

$$\begin{tikzcd} BP\wedge h \arrow[r, "\nu_h"] \arrow[d, "\mu_h\wedge id"'] & h \\ h\wedge h \arrow[ur, "m_h"'] & \end{tikzcd}$$

commutes.

3.8. LEMMA. *Let $h^*(-)$ be a BP-module theory with product and locally finite coefficients. Suppose $\mu_h(I_n)=0$. Then m_h induces for all $m\leq n$ a natural isomorphism*

$$t_m: h^*(P(m))\hat{\otimes}_{h^*} h^*(X) \xrightarrow{\cong} h^*(P(m)\wedge X)$$

over $\mathbf{S}(-1)$. *Moreover,* $t_0=t$.

3.9. *Remark.* Observe that it follows from 3.6 and 3.7 that the product induced by m_h on h^* agrees with the given one and that the module map ν_h may be described by the formula

$$\nu_h(u\otimes x)=m_h(\mu_h(u)\otimes x).$$

Proof of 3.8. The product m_h induces a natural transformation

$$t'_m: h^*(P(m))\otimes_{h^*} h^*(X)\rightarrow h^*(P(m)\wedge X)$$

over the category $\mathbf{W}$ (note, however, that the left hand side is not an additive cohomology theory over this category). From 3.5, 3.1 one sees that there is an isomorphism of profinite h^*-modules

$$h^*(P(m))\hat{\otimes}_{h^*} h^*(X)\cong \prod_{C\in\mathbf{C}_n} \varprojlim_{\beta,W}\left(\frac{h^*(X)}{h^*(X_\beta)}[\mu(r^E), E\in W]\right)\cdot a^C$$

where $\mathbf{C}_n$ denotes the set of all exponent sequences $C=(\varepsilon_0,\ldots,\varepsilon_{n-1})$ where $\varepsilon_i=0$ or 1 and $\{W\}$ denotes the directed set of all finite subsets of the set of all exponent sequences $\mathbf{E}$. Because $h^*(X)\cong\varprojlim h^*(X)/\overline{h^*(X_\beta)}$, one sees that

$$h^*(P(m))\hat{\otimes}_{h^*} h^*(X)\cong \prod_{C\in\mathbf{C}_n}\prod_{E\in\mathbf{E}} h^*(X)\cdot(\mu_h(r^E)a^C),$$

and this implies that $h^*(P(m))\hat{\otimes}_{h^*} h^*(-)$ is an additive cohomology theory over the category $\mathbf{W}$. From the universal property of the inverse limit one gets a canonical map

$$c: h^*(P(m))\otimes_{h^*} h^*(X)\rightarrow h^*(P(m))\hat{\otimes}_{h^*} h^*(X)$$

and it is not difficult to verify that the image of c is dense (with respect to the profinite topology) in $h^*(P(m))\hat{\otimes}_{h^*}h^*(X)$. Now c is monic and the map $t'_m \circ c^{-1}: im(c) \to h^*(P(m) \wedge X)$ is (uniformly) continuous (proof left to the reader), so $t'_m \circ c^{-1}$ extends uniquely to a transformation of (additive) cohomology theories

$$t_m: h^*(P(m)) \hat{\otimes}_{h^*} h^*(X) \to h^*((m) \wedge X)$$

which is an equivalence according to 3.5. That $t_0 = t$ follows from the remark 3.9.

4. Comodules over $h^*(BP \wedge BP)$

In this section, $h^*(-)$ is always understood to be a BP-module theory with locally finite coefficients, even if we do not mention it explicitly.

Suppose X, Y are module spectra over the ring spectrum BP and consider the map

4.1. $\psi_{X,Y}: BP \wedge BP \wedge X \wedge Y \to X \wedge Y$

defined by $\psi_{X,Y} = (\nu_X \wedge \nu_Y) \circ (id \wedge T \wedge id)$ where $T: BP \wedge X \to X \wedge BP$ is the switch map and ν_X, ν_Y are the module maps. Set $\psi = \psi_{BP,BP}$. Observe that $(BP \wedge BP, \psi)$ is a ring spectrum and that $\psi_{X,Y}$ makes $X \wedge Y$ a module spectrum over $BP \wedge BP$. Let $h^*(-)$ be a BP-module theory with locally finite coefficients. Repeated application of 3.3 shows that

4.2. $h^*(BP \wedge BP \wedge BP \wedge BP) \cong h^*(BP \wedge BP) \hat{\otimes}_{h^*} h^*(BP \wedge BP)$

and it is easily seen that ψ^* is given as follows:

4.3.
$$\begin{aligned}
\psi^*: h^*(BP \wedge BP) &\xrightarrow{t^{-1}} h^*(BP) \hat{\otimes}_{h^*} h^*(BP) \\
&\xrightarrow{m^* \hat{\otimes} m^*} h^*(BP \wedge BP) \hat{\otimes}_{h^*} h^*(BP \wedge BP) \\
&\xrightarrow{t^{-1} \hat{\otimes} t^{-1}} h^*(BP) \hat{\otimes}_{h^*} h^*(BP) \hat{\otimes}_{h^*} h^*(BP) \hat{\otimes}_{h^*} h^*(BP) \\
&\xrightarrow{(t \hat{\otimes} t)(id \hat{\otimes} T \hat{\otimes} id)} h^*(BP \wedge BP) \hat{\otimes}_{h^*} h^*(BP \wedge BP).
\end{aligned}$$

Using 4.3 one sees that $(h^*(BP \wedge BP), \psi^*)$ is a coalgebra in $\mathbf{Mod}_{h^*}^{\mathrm{prof}}$ with counit induced by the unit of $BP \wedge BP$. From 3.2 and 3.3 and the definition of the tensor product $\hat{\otimes}$ it follows that

4.4. $h^*(BP \wedge BP) \simeq h^*[[\mu(r^E), \mu(r^F)]]$

as a profinite h^*-module, so lemma 3.7 of [9] implies that $h^*(BP \wedge BP)$ is an exact coalgebra in $\mathbf{Mod}_{h^*}^{\mathrm{prof}}$.

If X and Y are (-1)-connected BP-module spectra, a consideration analogous to 4.3 shows that $\psi^*_{X,Y}$ may be written as

4.5. $\psi^*_{X,Y} = (t \hat{\otimes} id)(id \hat{\otimes} t^{-1}_{X \wedge Y}) t^{-1}_{BP \wedge X \wedge Y}(id \wedge T \wedge id)^*(\nu_X \wedge \nu_Y)^*.$

This and 4.3 show that $(h^*(X \wedge Y), \psi^*_{X,Y})$ is a $h^*(BP \wedge BP)$-comodule, provided X, Y are (-1)-connected.

4.6. EXAMPLE. For any non-negative integers m, n the $P(m)$, $P(n)$ are (-1)-connected BP-module spectra. So $h^*(P(m) \wedge P(n))$ becomes a $h^*(BP \wedge BP)$-comodule for any BP-module theory with locally finite coefficients. In this case, we denote the coaction map by $\Psi^*_{m,n}$. Notice that $\Psi^*_{0,0} = \Psi^*$.

Let $\theta : X \wedge Y \to h$ be a map of spectra, i.e. $\theta \in h^*(X \wedge Y)$, and consider the diagram

4.7.
$$\begin{tikzcd} BP \wedge BP \wedge X \wedge Y \arrow[r, "\psi_{X,Y}"] \arrow[d, "id \wedge id \wedge \theta"'] & X \wedge Y \arrow[d, "\theta"] \\ BP \wedge BP \wedge h \arrow[r, "\nu_h \circ (m \wedge id)"] & h. \end{tikzcd}$$

Suppose h is (-1)-connected. Then one sees from 3.3 that

4.8. $\nu_h^*(id_h) = \mu_h \hat{\otimes} id_h \in h^*(BP) \hat{\otimes}_{h^*} h^*(h).$

Because μ_h is a map of BP-module spectra the diagram

$$\begin{tikzcd} BP \wedge BP \arrow[r, "id \wedge \mu_h"] \arrow[d, "m"'] & BP \wedge h \arrow[d, "\nu_h"] \\ BP \arrow[r, "\mu_h"] & h \end{tikzcd}$$

commutes, so

4.9. $m^*(\mu_h) = \mu_h \hat{\otimes} \mu_h \in h^*(BP) \hat{\otimes}_{h^*} h^*(BP).$

4.10 LEMMA. *Suppose h is (-1)-connected. Then the diagram 4.7 commutes iff $\psi^*_{X,Y}(\theta) = (\mu_h m)\hat{\otimes}\theta$.*

Proof. This follows immediatly from 4.8 and 4.9.

An element a of the $h^*(BP \wedge BP)$-comodule $h^*(X \wedge Y)$ is called *primitive*, if $\psi^*_{X,Y}(a) = (\mu_h m)\hat{\otimes} a$. We denote by $Pr\{h^*(X \wedge Y)\}$ the graded h^*-module of all primitive elements of $h^*(X \wedge Y)$. The main purpose of this section is to calculate $Pr\{h^*(P(m) \wedge P(n))\}$ for a BP-module theory with the property that $\mu_h(I_m \cup I_n) = 0$.

4.11. LEMMA. *Suppose h is (-1)-connected and $\mu_h(I_k) = 0$ for some $k \geq \max\{m, n\}$. There is an isomorphism of $h^*(BP \wedge BP)$-comodules*

$$h^*(P(m) \wedge P(n)) \cong h^*(BP \wedge BP) \otimes_{h^*} E_{h^*}(a_0, \ldots, a_{m-1}, b_0, \ldots, b_{n-1})$$

where degree $a_i = 2p^i - 1$, degree $b_j = 2p^j - 1$. Moreover,

$$(\eta_{m-1} \wedge \eta_{n-1})^* : h^*(P(m) \wedge P(n)) \to h^*(P(m-1) \wedge P(n-1))$$

is split epic in the category of $h^*(BP \wedge BP)$-comodules.

The proof of 4.11 is similar to the proof of [9], prop. 4.12 and will be omitted.

4.12. PROPOSITION. *Suppose $h^*(-)$ is as in 4.11. Then there is an isomorphism of graded h^*-modules*

$$\alpha : Pr\{h^*(P(m) \wedge P(n))\} \cong E_{h^*}(a_0, \ldots, a_{m-1}, b_0, \ldots, b_{n-1}).$$

Proof. From the structure of $h^*(P(m) \wedge P(n))$ as $h^*(BP \wedge BP)$-comodule as given by Lemma 4.11 one sees immediately that there is an isomorphism of h^*-modules

$$Pr\{h^*(P(m) \wedge P(n))\} \cong Pr\{h^*(BP \wedge BP)\} \otimes_{h^*} E_{h^*}(a_0, \ldots, a_{m-1}, b_0, \ldots, b_{n-1}).$$

So it rests to show that there is an isomorphism

$$\beta : Pr\{h^*(BP \wedge BP)\} \cong h^*.$$

Let $a \in h^*(BP \wedge BP)$ be primitive, i.e. $\psi^*(a) = \mu_h m \hat{\otimes} a$. Now a may be written in the form

$$a=\sum_{E,F}\lambda_{E,F}(\mu(r^E)\hat{\otimes}\mu(r^F))$$

with uniquely determined coefficients $\lambda_{E,F}\in h^*$. From 4.3 one sees that

$$\begin{aligned}\psi^*(a)&=\sum_{E,F}\lambda_{E,F}\left\{\sum_{\substack{E_1+E_2=E\\F_1+F_2=F}}\mu r^{E_1}\hat{\otimes}\mu r^{F_1}\hat{\otimes}\mu r^{E_2}\hat{\otimes}\mu r^{F_2}\right\}\\&=\sum_{E,F}\lambda_{E,F}\{\mu m\hat{\otimes}\mu r^E\hat{\otimes}\mu r^F\},\end{aligned}$$

where the last equality holds because a is supposed to be primitive. This implies $\lambda_{E,F}=0$ for $E\neq 0$ or $F\neq 0$, so $a=\lambda_{0,0}(\mu\hat{\otimes}\mu)$ for some element $\lambda_{0,0}\in h^*$. If we define β by $\beta(a)=\lambda_{0,0}$ the proposition follows, because any element of h^* clearly occurs in this way.

4.13. COROLLARY. *Let p be an odd prime and suppose $k\geq\max\{m, n\}$. There is one and only one primitive element $a\in P(k)^0(P(m)\wedge P(n))$ such that $(\mu_m\wedge\mu_n)^*(a)=\mu_k m\in P(k)^0(BP\wedge BP)$.*

Proof. Observe first that $\mu_k m\in P(k)^0(BP\wedge BP)$ is primitive according to 4.7 and 4.10. From 4.11 we know that the morphism $(\mu_m\wedge\mu_n)^*$ is split epic in the category of $P(k)^*(BP\wedge BP)$-comodules, so there is at least one primitive element a which restricts to $\mu_k m$. One sees from 4.12 that

$$Pr\{P(k)^0(P(m)\wedge P(n))\}\cong(E_{P(k)^*}(a_0,\ldots,a_{m-1},b_0,\ldots,b_{n-1}))^0.$$

Let ε_i be 0 or 1 and denote by C, D exponent sequences of the form $C=(\varepsilon_0,\ldots,\varepsilon_{m-1})$, $D=(\varepsilon_0,\ldots,\varepsilon_{n-1})$. A dimension counting shows that

$$|a^Cb^D|\leq 2\left\{\frac{2(p^k-1)}{p-1}-k\right\}=r$$

for all exponent sequences C, D and that

$$|\lambda|\leq -2(p^k-1)$$

for all $\lambda\in P(k)^*$, $|\lambda|\neq 0$. Observe that for all $k\geq 1$,

$$r<2(p^k-1)\quad\text{iff}\quad[2/(p-1)-k/(p^k-1)]<1$$

and this is always the case if $p>2$. It follows that, for p odd, $0 \neq \lambda a^C b^D \in PrP(k)^0(P(m) \wedge P(n))$ iff $|\lambda| = 0$, so

$$Pr\{P(k)^0(P(m) \wedge P(n))\} \cong Pr\{P(k)^0(BP \wedge BP)\} \cong \mathbb{Z}_p.$$

The corollary follows.

Remark. Notice that the proof of 4.13 shows that 4.13 holds also for $p=2$, if one omits the uniqueness statement.

5. Construction of pairings $P(a) \wedge P(b) \to P(c)$

The purpose of this section is to furnish a proof of the following

5.1. THEOREM. *Let p be an odd prime. For all triples (a, b, c) of non-negative integers with the proparty that $c \geq \max\{a, b\}$ there is one and only one pairing*

$$m(a, b, c): P(a) \wedge P(b) \to P(c)$$

such that

(1) $m(a, b, c) \circ (\mu_a \wedge \mu_b) \simeq \mu_c \circ m$

and

(2) *the diagram*

$$\begin{array}{ccc} BP \wedge BP \wedge P(a) \wedge P(b) & \xrightarrow{(\nu_a \wedge \nu_b)(id \wedge T \wedge id)} & P(a) \wedge P(b) \\ \downarrow{\scriptstyle id \wedge id \wedge m(a,b,c)} & & \downarrow{\scriptstyle m(a,b,c)} \\ BP \wedge BP \wedge P(c) & \xrightarrow{\nu_c \circ (m \wedge id)} & P(c) \end{array}$$

is homotopy commutative.

These pairings have the following properties:

(3) *Commutativity. Let $T: P(a) \wedge P(b) \to P(b) \wedge P(a)$ be the switch map. Then*

$$m(a, b, c) \simeq m(b, a, c) \circ T.$$

(4) *Associativity. Suppose* $d \geq \max\{a, b\}$, $e \geq \max\{b, c\}$, $f \geq \max\{c, d\}$ *and* $f \geq \max\{a, e\}$. *Then*

$$m(a, e, f) \circ (id_{P(a)} \wedge m(b, c, e)) \simeq m(d, c, f) \circ (m(a, b, d) \wedge id_{P(c)}).$$

(5) *Unit. Let* $i: S^0 \to BP$ *be the unit of* BP. *Then* $i_a = \mu_a \circ i: S^0 \to P(a)$ *is a unit for the product* $m_a = m(a, a, a): P(a) \wedge P(a) \to P(a)$.

(6) $m(a+1, b+1, c+1) \circ (\eta_a \wedge \eta_b) \simeq \eta_c \circ m(a, b, c)$.

(7) *the diagram*

$$\begin{tikzcd} BP \wedge P(a) \arrow[r, "\nu_a"] \arrow[d, "\mu_a \wedge id"'] & P(a) \\ P(a) \wedge P(a) \arrow[ur, "m_a"'] & \end{tikzcd}$$

is homotopy commutative for all a.

5.2. *Remark.* Notice that Theorem 2.12 is an immediate consequence of 5.1.

Proof. We first prove existence and uniqueness of pairings $m(a, b, c): P(a) \wedge P(b) \to P(c)$ which satisfy conditions (1) and (2) of Theorem 5.1.

Suppose $c \geq \max\{a, b\}$. From 4.13 we know there is a unique element $m(a, b, c) \in Pr\{P(c)^0(P(a) \wedge P(b))\}$ such that $(\mu_a \wedge \mu_b)^*(m(a, b, c)) = \mu_c \circ m \in P(c)^0(BP \wedge BP)$, i.e. such that condition (1) of 5.1 holds. But from 4.10 we know that any pairing $m(a, b, c)$ which satisfies condition (2) must be a primitive element of $P(c)^0(P(a) \wedge P(b))$, so the assertion follows.

Now we go on to verify properties (3)–(7) of 5.1, beginning with the last one.

Proof of (7): $\mu_a \wedge id: BP \wedge P(a) \to P(a) \wedge P(a)$ is a morphism of $BP \wedge BP$-module spectra. Now $m_a \in Pr\{P(a)^0(P(a) \wedge P(a))\}$ by construction, so an easy calculation shows that $(\mu_a \wedge id)^*(m_a) \in Pr\{P(a)^0(BP \wedge P(a))\}$. Obviously, $\nu_a \in Pr\{P(a)^0(BP \wedge P(a)\}$. Because

$$(id \wedge \mu_a)^*(\mu_a \wedge id)^*(m_a) = (\mu_a \wedge \mu_a)^*(m_a) = \mu_a^*(m)$$

according to property (1) of 5.1 and $(id \wedge \mu_a)^*(\nu_a) = \mu_a^*(m)$ because μ_a is a map of BP-module spectra, it follows from 4.13 that $\nu_a = (\mu_a \wedge id)^*(m_a)$. This proves (7).

Before we proceed with the proof of properties (3)–(6) we remark that (7) implies that, for all $a \geq 0$, $P(a)^*(-)$ is a BP-module theory with a (canonical) product in the sense of Section 3. So it follows from Lemma 3.8 that if

$c \geq \max\{a, b\}$, m_c induces an isomorphism

5.3. $\quad P(c)^*(P(a)) \hat{\otimes}_{P(c)^*} P(c)^*(P(b)) \cong P(c)^*(P(a) \wedge P(b)).$

Using this isomorphism it is not difficult to see that the coaction map $\psi^*_{a,b}$ of $P(c)^*(P(a) \wedge P(b))$ may be described by the following formula:

5.4. $\quad \psi^*_{a,b}(x \hat{\otimes} y) = (id \hat{\otimes}\, T^* \hat{\otimes}\, id) \circ (\nu^*_a(x) \hat{\otimes}\, \nu^*_b(y)).$

Proof of (3): Observe that the rectangle in the following diagram commutes:

$$\begin{tikzcd}
BP \wedge BP \wedge P(a) \wedge P(b) \arrow[r, "\psi_{a,b}"] \arrow[d, "T \wedge T"'] & P(a) \wedge P(b) \arrow[d, "T"'] \arrow[dr, "m(a,b,c)"] & \\
BP \wedge BP \wedge P(b) \wedge P(a) \arrow[r, "\psi_{b,a}"] & P(b) \wedge P(a) \arrow[r, "m(b,a,c)"'] & P(c)
\end{tikzcd}$$

Recall that $m(b, a, c) \in Pr\{P(c)^0(P(b) \wedge P(a))\}$. Using 5.4 one sees that $T^*(m(b, a, c)) \in Pr\{P(c)^0(P(a) \wedge P(b))\}$. Because the product m is commutative, the diagram

$$\begin{tikzcd}
BP \wedge BP \arrow[d, "T"'] \arrow[dr, "\mu_c \circ m"] & \\
BP \wedge BP \arrow[r, "\mu_c \circ m"'] & P(c)
\end{tikzcd}$$

commutes. This fact and 4.13 imply that $m(a, b, c) = T^*(m(b, a, c))$.

To prove the associativity statement we need some preparation. Lemma 3.3 implies that $P(c)^*(BP)$ is a coalgebra in the category of profinite $P(c)^*$-modules with counit induced by $i: S^0 \to BP$. Moreover, if X is any (-1)-connected BP-module spectrum with module map $\nu_X: BP \wedge X \to X$, $P(c)^*(X)$ becomes a $P(c)^*(BP)$-comodule with coaction map ν^*_X. Denote by $Pr\{P(c)^*(X)\}$ the set of primitive elements of $P(c)^*(X)$, i.e. the set of all elements a such that $\nu^*_X(a) = \mu_c \hat{\otimes} a$. It is easily seen that the BP-versions of [9], Theorems 4.17 and 4.18 imply the following

5.5. LEMMA. *Suppose* $0 < b \leq c$. *There exists a unique primitive element* $x \in P(c)^0(P(b))$ *such that* $\mu^*_b(x) = \mu_c \in P(c)^0\ BP)$.

Suppose again that $c \geq \max\{a, b\}$. Because $m(a, b, c)$ satisfies condition (2) of

5.1, the following diagram commutes:

$$(*)\qquad \begin{tikzcd} P(c)^*(P(a)\wedge P(b)) \arrow[r, "\psi^*_{a,b}"] \arrow[d, "m(a,b,c)^*"'] & P(c)^*(BP\wedge BP)\hat{\otimes}_{P(c)_*} P(c)^*(P(a)\wedge P(b)) \arrow[d, "(id\wedge id)^*\hat{\otimes} m(a,b,c)^*"] \\ P(c)^*(P(c)) \arrow[r, "(m\otimes id)\circ\nu^*"] & P(c)^*(BP\wedge BP)\hat{\otimes}_{P(c)_*} P(c)^*(P(c)) \end{tikzcd}$$

Now let $x \in P(c)^*(P(c))$ be a primitive element. Using $(*)$ one sees that

$$\begin{aligned}\psi^*_{a,b}(m(a,b,c)^*(x)) &= (id\hat{\otimes} id\hat{\otimes} m(a,b,c)^*)\circ(m^*\hat{\otimes} id)\circ \nu^*_c(x)\\ &= \mu_c\hat{\otimes}\mu_c\hat{\otimes} m(a,b,c)^*(x)\end{aligned}$$

because x is primitive. This proves the first part of the following lemma, whose second part is an easy consequence of the description 5.4 of $\psi^*_{a,b}$ and the structure of $P(c)^*(BP\wedge BP\wedge P(a)\wedge P(b))$.

5.6. LEMMA. (1) *Suppose* $x\in Pr\{P(c)^*(P(c))\}$. *Then* $m(a,b,c)^*(x)\in Pr\{P(c)^*(P(a)\wedge P(b))\}$. (2) *Suppose* $z=z_a\hat{\otimes} z_b\in P(c)^*(P(a))\hat{\otimes}_{P(c)_*}P(c)^*(P(b))$. *Then* $z\in Pr\{P(c)^*(P(a)\wedge P(b))\}$ *iff* $z_a\in Pr\{P(c)^*(P(a))\}$ *and* $z_b\in Pr\{P(c)^*(P(b))\}$.

Proof of (4): Look at the diagram

$$\begin{tikzcd} P(a)\wedge P(b)\wedge P(c) \arrow[rrr, "m(a,b,d)\wedge id_c"] \arrow[ddd, "id_a\wedge m(b,c,e)"'] & & & P(d)\wedge P(c) \arrow[ddd, "m(d,c,f)"] \\ & BP\wedge BP\wedge BP \arrow[ul, "\mu_a\wedge\mu_b\wedge\mu_c"'] \arrow[r, "m\wedge id"] \arrow[d, "id\wedge m"] & BP\wedge BP \arrow[ur, "\mu_a\wedge\mu_c"] \arrow[d, "m"'] & \\ & BP\wedge BP \arrow[r, "m"] \arrow[dl, "\mu_a\wedge\mu_e"] & BP \arrow[dr, "\mu_f"'] & \\ P(a)\wedge P(e) \arrow[rrr, "m(a,e,f)"'] & & & P(f) \end{tikzcd}$$

Let $id_f: P(f)\to P(f)$ be the identity. id_f is obviously a primitive element of $P(f)^0(P(f))$. Now using 4.11, 4.13, the isomorphism 5.3 and Lemma 5.6 one sees that $(id\wedge m(b,c,e))^*\circ m(a,e,f)^*(id_f)$ may be written in the form $z_a\hat{\otimes} z_b\hat{\otimes} z_c$ where $z_a\in Pr\{P(f)^0(P(a))\}$, $z_b\in Pr\{P(f)^0(P(b))\}$ and $z_c\in Pr\{P(f)^0(P(c))\}$ and similarly for $(m(a,b,d)\wedge id)^*\circ m(d,e,f)^*(id_f)$. Because any proper subdiagram of the above diagram commutes, the assertion follows from 5.5.

Proof of (5). We have to show that the following diagram commutes:

$$\begin{array}{ccccc}
S^0\wedge P(a) & \xrightarrow{i\wedge id} & BP\wedge P(a) & \xrightarrow{\mu_a\wedge id} & P(a)\wedge P(a)\\
\wr\| & & & & \downarrow m_a\\
P(a) & \longleftarrow & id & \longrightarrow & P(a)\\
\wr\| & & & & \uparrow m_a\\
P(a)\wedge S^0 & \xrightarrow{id\wedge i} & P(a)\wedge BP & \xrightarrow{id\wedge\mu_a} & P(a)\wedge P(a).
\end{array}$$

This results from the relation (see 3.15)

$$m_a^*(id_a)=id_a\,\hat{\otimes}\,id_a$$

and the fact that i is a unit for BP.

Proof of (6). This is proved by the same sort of arguments used in the proof of (7).

5.7. *Remark.* For $p=2$, the map

$$Pr\{P(c)^0(P(a)\wedge P(b))\}\to Pr\{P(c)^0(BP\wedge BP)\}\cong\mathbb{Z}_2$$

is surjective but not injective. So in this case there are several different product maps $P(a)\wedge P(b)\to P(c)$ which satisfy conditions (1), (2) of Theorem 5.1. These products are in general not commutative (look at the case $P(1)\simeq BP\mathbb{Z}_2$). However, the same proof as for p odd shows that all of these products are associative and have a unit.

6. The structure of $P(n)_*(P(n))$

In this section we give a proof of Theorem 2.13. Let p be odd. Let $T:BP\to H\mathbb{Z}_p$ be the $\mathbb{Z}_p$ Thom map and denote by $T_n:P(n)\to H\mathbb{Z}_p (n>0)$ the map of spectra induced by the edge-homomorphism of the Atiyah–Hirzebruch spectral sequence of $P(n)^*(-)$. We get a diagram of ring spectra and maps of ring spectra

6.1.
$$\begin{tikzcd}
BP \arrow[rr, "\mu_n"] \arrow[dr, "T"'] & & P(n) \arrow[dl, "T_n"] \\
& H\mathbb{Z}_p &
\end{tikzcd}$$

which is easily seen to be commutative. Because $T_*:BP_*(BP)\to(H\mathbb{Z}_p)_*(BP)$ is epic the same holds for $(T_n)_*:P(n)_*(BP)\to(H\mathbb{Z}_p)_*(BP)$, so one sees by standard arguments that

6.2. $P(n)_*(BP) \cong BP_*(BP)/I_n \cdot BP_*(BP) \cong P(n)_* \otimes \mathbb{Z}_p[t_1, t_2, \ldots]$

as a $P(n)_*$-algebra and that the product of $P(n)$ induces an equivalence of homology theories

6.3. $P(n)_*(BP) \otimes_{P(n)_*} P(n)_*(X) \xrightarrow{\simeq} P(n)_*(BP \wedge X)$.

Now recall that there is an exact triangle

$$\begin{tikzcd} P(n) \arrow[r, "\theta_n"] & P(n) \arrow[r, "\eta_n"] & P(n+1) \arrow[ll, bend left, "\partial_n"] \end{tikzcd}$$

where θ_n is given by the composition $S^{|v_n|} \wedge P(n) \xrightarrow{\varphi \wedge id} BP \wedge P(n) \xrightarrow{\nu_n} P(n)$ and φ represents the element $v_n \in BP^*$. Because I_n is an invariant ideal the same argument as in the proof of [9], Lemma 3.15 shows that $P(m)^*(\varphi) = 0$ provided $n < m$. From this and 6.3 it follows easily that $P(m)_*(\theta_n) = 0$ and so we get for all $n < m$ a short and split exact sequence

$$0 \longrightarrow P(m)_*(P(n)) \xrightarrow{\eta_n} P(m)_*(P(n+1)) \xrightarrow{\partial_n} P(m)_*(P(n)) \longrightarrow 0.$$

From this it follows by induction that one has an isomorphism of $P(n)_*$-modules

6.4. $P(n)_*(P(n)) \cong P(n)_* \otimes \mathbb{Z}_p[s_1, s_2, \ldots] \otimes E(\alpha_0, \ldots, \alpha_{n-1})$.

Here α_i has degree $2p^i - 1$ and $s_i = (\mu_n)_*(t_i)$. In particular, $P(n)_*(P(n))$ is flat as a left $P(n)_*$-module and from [1] we know that this implies

6.5. PROPOSITION. *Let p be an odd prime. Then $P(n)_*(P(n))$ is for all n a Hopf algebra (in the sense of* [1]) *and $P(n)_*(-)$ takes values in the category of $P(n)_*(P(n))$-comodules.*

Because the maps $\eta_n : P(n) \rightarrow P(n+1)$ are (for all n) maps of ring spectra, we get (compare 6.1) a commutative diagram of Hopf algebras and homomorphisms of Hopf algebras

6.6.
$$\begin{tikzcd}
BP_*(BP) \arrow[rr, "(\mu_n\wedge\mu_n)_*"] \arrow[dr, "(T\wedge T)_*"'] & & P(n)_*(P(n)) \arrow[dl, "(T_n\wedge T_n)_*"] \\
& (H\mathbb{Z}_p)_*(H\mathbb{Z}_p) \arrow[d, equal] & \\
& \mathscr{A}(p)_* &
\end{tikzcd}$$

As is well known, there is an isomorphism of algebras

6.7. $\mathscr{A}(p)_* \cong \mathbb{Z}_p[\xi_1, \xi_2, \ldots] \otimes E(\tau_0, \tau_1, \ldots)$

where degree $\xi_i = 2(p^i - 1)$ and degree $\tau_j = 2p^j - 1$. We are now ready to give a

Proof of Theorem 2.13: From Zahler [12], Lemma 3.7 one knows that $(T \wedge T)_*(t_r) = \chi(\xi_r)$ where χ denotes the canonical anti-automorphism of the Hopf algebra $\mathscr{A}(p)_*$, and from work of Baas–Madsen [13] it follows that $(T_n)_*: H_*(P(n), \mathbb{Z}_p) \to H_*(H\mathbb{Z}_p, \mathbb{Z}_p)$ is an injective homomorphism of algebras with image isomorphic to

$$\mathbb{Z}_p[\chi(\xi_1), \chi(\xi_2), \ldots] \otimes E(\chi(\bar{\tau}_0), \ldots, \chi(\bar{\tau}_{n-1})),$$

where $\bar{\tau}_i \equiv \tau_i$ mod decomposable elements. Because the spectral sequence $H_*(P(n), P(n)_*) \cong H_*(P(n), \mathbb{Z}_p) \otimes P(n)_* \Rightarrow P(n)_*(P(n))$ collapses, these facts together with 6.4 and the commutativity of the product of $P(n)_*(P(n))$ imply by standard arguments part (1) of Theorem 2.13. Because $(\mu_n \wedge \mu_n)_*$ is a homomorphism of Hopf algebras, $P(n)_* = BP_*/I_n$ and the ideal I_n is invariant, part (2) follows from part (1). Let Δ denote the coaction map of $\mathscr{A}(p)_*$. We have

$$\begin{aligned}
\Delta \circ (T_n \wedge T_n)_*(a_k) &= \Delta \circ \chi(\tau_k) \\
&= (\chi \otimes \chi)\left(\sum_{i=0}^{k} \tau_i \otimes \xi_{k-1}^{p^i} + 1 \otimes \tau_k\right) \\
&= 1 \otimes \chi(\tau_k) + \sum_{i=0}^{k} \chi(\tau_i) \otimes \chi(\xi_{k-i}^{p^i}),
\end{aligned}$$

so it follows that

$$\psi_n(a_k) = 1 \otimes a_k + \sum_{i=0}^{k} a_i \otimes s_{k-i}^{p^i} + R_k$$

where $R_k \in \ker (T_n \wedge T_n)_*$. One easily sees that $\ker (T_n \wedge T_n)_*$ contains no non-zero elements of dimension $2p^k - 1$ $(k = 0, 1, \dots, n-1)$, so $R_k = 0$ and the first half of 2.13, (3) is proved. The same sort of argument shows that $c_n(a_k) = -a_k$. The last statement of 2.13 follows from 2.13 (2) because η_L^n factors obviously through $C(n)_*$ and $\eta_R^n = c_n \cdot \eta_L^n$.

Let $u^{BP} \in BP^2(P_\infty\mathbb{C})$ denote the usual $BP^*(-)$-Euler class of the canonical complex line bundle over $P_\infty\mathbb{C}$. We always consider $P(n)^*(-)$ as complex-oriented by $u^{P(n)} = \mu_n(u^{BP})$. The formal group law $F_{P(n)}(X, Y)$ associated to $u^{P(n)}$ is clearly p-typical and it is of height n because the ideal $I_n \subset BP^*$ agrees with the ideal of BP^* generated by the coefficients of $x, x^p, \dots, x^{p^{n-1}}$ in the power series $[p]_{F_{BP}}(X)$, this may be seen for example by the methods of [3], §6. As a first application of 2.13 we state the following

6.8. PROPOSITION. *Let $h^*(-)$ be a complex-oriented multiplicative cohomology theory with coefficient ring of characteristic $p > 2$ and p-typical formal group law $F_h(X, Y)$ of finite height n. Then there exists a multiplicative transformation $\rho_n : P(n)^*(-) \to h^*(-)$ such that $\rho_n(u^{P(n)}) = u^h$. If $h^{1-2p^k}(pt) = 0$ for $k = 0$, $1, \dots, n-1$, ρ_n is unique.*

Proof. Using the first part of Theorem 2.13 and the fact that $BP^*(-)$ is universal for multiplicative cohomology theories with coefficients a $\mathbb{Z}_{(p)}$-algebra and p-typical formal group ([3], Theorem 7.2), this may be proved by arguments similar to those used in the proof of [2], Lemma 4.6. We leave the details to the reader.

One sees by standard arguments that there is an isomorphism

6.9. $\quad P(n)^*P(n) \cong \mathrm{Hom}^*_{P(n)_*}(P(n)_*(P(n)), P(n)^*)$.

In particular this means that the algebra of stable $P(n)^*(-)$-operations is – at least theoretically – determined by Theorem 2.13. For each exponent sequence $E = (e_1, e_2, \dots)$ of non-negative integers we denote by r_n^E the stable operation dual to $s_1^{e_1} s_2^{e_2} \cdots \in P(n)_*(P(n))$. The r_n^E have degree $\|E\| = \sum_{i \geq 1} 2(p^i - 1)e_i$, they satisfy a Cartan-formula and it is not difficult to show that they generate (topologically) a $P(n)^*$-subalgebra of $P(n)^*(P(n))$ isomorphic to $BP^*(BP)/I_n$.

The operations r_n^E give rise to homology operations $r_E^n : P(n)_*X \to P(n)_*X$ of degree $-\|E\|$. Using these we define for all X a map

6.10. $\quad \varphi_X : P(n)_*(X) \to BP_*(BP)/I_n \otimes_{P(n)^*} P(n)_*(X)$

by $\varphi_X(x) = \sum_E c_n(t^E) \otimes r_E^n(x)$. It is easy to see that φ_X makes $P(n)_*(X)$ a comodule over $BP_*(BP)/I_n$. This remark is useful because it permits us to apply directly Landweber's filtration theorem and exact functor theorem [6], [14] on $P(n)_*(X)$ (see [10], [15] for a different discussion of these topics). In particular one gets the following lemma which will be useful in the next section:

6.11. LEMMA. *Let G be a $P(n)_*$-module. The functor $-\otimes_{P(n)_*} G$ is exact on the category of $BP_*(BP)/I_n$-comodules which are finitely presented $P(n)_*$-modules if and only if multiplication by v_k is monic on $G/(v_n, v_{n+1}, \ldots, v_{n+k-1})$ for all $k \geq n$.*

6.11 has been proved in [10] and [15] by different methods.

We close this section by a remark concerning the case $p=2$. For any prime p and any positive integer n define $A(p, n) \subset \mathbb{Z}$ by

$$A(p, n) = \left\{ q \in \mathbb{Z} - \{0\} \,\middle|\, q = -\sum_{i=0}^{n-1} \varepsilon_i (2p^i - 1),\ \varepsilon_i = 0 \text{ or } 1 \right\}.$$

For any graded ring R set

$$J(R, p, n) = \prod_{q \in A(p,n)} R^q$$

The following proposition is an immediate consequence of the BP-analogs of Theorems 4.17 and 4.18 of [9].

6.12. PROPOSITION. *Let $h^*(-)$ be a BP-module theory with locally finite coefficients, $n > 0$ an integer and suppose $\mu_h(I_h) = 0$. Then there exists a transformation of BP-module theories $\rho_n : P(n)^*(-) \to h^*(-)$ over the category* **W**. *Moreover, ρ_n is unique iff $J(h^*, p, n) = 0$.*

As the case $n = \infty$ indicates, for $p = 2$ we cannot expect in general to find a subalgebra of $P(n)^*(P(n))$ isomorphic to $BP^*(BP)/I_n$, but 6.12 may be used (following [9], §5) to construct stable operations θ_n^E for all exponent sequences E of the form $(0, \ldots, 0, e_d, e_{d+1}, \ldots)$ even in the case $p = 2$, here $d = 1$ if $n < 2(p-1)$ and $d = n$ otherwise. The structure of the subalgebra of $P(n)^*(P(n))$ generated by the θ_n^E may be determined, it turns out that – for $n < 2(p-1)$ – this algebra is again isomorphic to $BP^*(BP)/I_n$.

6.13. *Remark.* 6.8 contains obviously a uniqueness statement for ring theories $h^*(-)$ with formal group of finite height n such that h^* satisfies the conditions of 6.11. In particular, $K(n)^*(-)$ is uniquely determined by $K(n)^*$ and its formal group.

7. Proof of Theorem 2.14

Again, let p be an *odd* prime. We first note that it follows from 6.11 that $P(n)^*(-) \otimes_{P(n)^*} K(n)^*$ is a cohomology theory over the category $\mathbf{W}^f$. Now any cohomology theory $h^*(-)$ with locally finite coefficients defined over $\mathbf{W}^f$ admits a unique additive extension $Eh^*(-)$ over the category $\mathbf{W}$ of all complexes, namely

7.1. $Eh^*(X) = \overset{\lim}{\longleftarrow} h^*(\text{finite subcomplexes of } X).$

From 6.12 we know that there is a unique transformation of BP-module theories $\lambda_n : P(n)^*(-) \to K(n)^*(-)$. From this and the comparison theorem for cohomology theories we now see that there is a unique equivalence of BP-module theories

7.2. $\tilde{\lambda}_n : E(P(n)^*(-) \otimes_{P(n)^*} K(n)^*) \xrightarrow{\cong} K(n)^*(-).$

Because $P(n)^*(-)$ is multiplicative the same holds for the left side of 7.2, hence $K(n)^*(-)$ becomes a multiplicative theory via the isomorphism $\tilde{\lambda}_n$. Let $i_n : P(n)^*(-) \to E(P(n)^*(-) \otimes_{P(n)^*} K(n)^*)$ be the natural map. We have $\lambda_n = \tilde{\lambda}_n \circ i_n$ according to 6.12, so λ_n is multiplicative by the definition of the product of $K(n)^*(-)$. This proves Theorem 2.14.

7.3. *Remarks.* (1) Our argument shows that – in the case $p = 2$ – whatever product structure exists on $P(n)^*(-)$, the same kind of structure exists also on $K(n)^*(-)$ (compare 5.7).

(2) Notice that $k(n)$ is equivalent to the (-1)-connected cover of $K(n)$. The universal property of the (-1)-connected cover construction implies that the $k(n)$ are ring spectra, too $(p > 2)$.

8. Remarks

8.1. Because $K(n)^* = \mathbb{F}_p[v_n, v_n^{-1}]$ is a "graded field", all graded $K(n)^*$-modules are free. The fact that $K(n)$ is a ring spectrum then easily implies the Künneth formulas

8.1.1.
$$K(n)_*(X \wedge Y) \cong K(n)_*(X) \otimes_{K(n)^*} K(n)_*(Y)$$
$$K(n)^*(X \wedge Y) \cong K(n)^*(X) \hat{\otimes}_{K(n)^*} K(n)^*(Y)$$

for X, Y arbitrary CW-complexes. If $p=2$, the isomorphisms 8.1.1 hold for any choice of the product map (compare 5.7). Another easy consequence is that the Kronecker product induced by $K(n)\wedge K(n)\to K(n)$ defines an isomorphism

8.1.2. $K(n)^*(X)\cong \mathrm{Hom}^*_{K(n)_*}(K(n)_*(X), K(n)_*)$

over the category **W**.

8.2. Because $P(n)$ is a ring spectrum and from Theorem 2.13 it follows that for p odd, one may set up an Adams spectral sequence based on $P(n)_*(-)$,

$$E_2^{**}(X)\cong \mathrm{Ext}^{**}_{P(n)_*(P(n))}(P(n)_*, P(n)_*(X))\Rightarrow \pi_*^S(X)\otimes \mathbb{Z}_p$$

which interpolates between the Adams spectral sequence based on BP and the classical one. We hope to come back to this somewhere else.

8.3. Notice that because $P(n)_*(P(n))$ is a free $P(n)_*$-module it follows from [2], Lemma 11.1 that the spectrum $P(n)\wedge P(n)$ is equivalent to a wedge of suspensions of $P(n)$. A similar statement holds for the spectrum $K(n)\wedge K(n)$.

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