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Strange inner product spaces

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Introduction

It has been a puzzling question for some time whether there exist or not quadratic spaces of an infinite dimension κ such that the orthogonals of all infinite dimensional subspaces are of dimension less than κ. In [3] it had been shown how to construct such spaces over uncountable fields. For countably infinite or finite fields the problem remained open. We shall give here a solution for this remaining case under the additional proviso that the continuum hypothesis holds in the underlying set theory.

Another question asks if each vector space supplied with a non degenerate symmetric form contains some subspace U spanned by an orthogonal basis and whose orthogonal $U^{\perp}$ is the null space. (Clearly if E itself admits an orthogonal basis no problem exists; likewise, if E has no non-zero isotropic vectors then an application of Zorn's lemma will provide us with a subspace U of the required sort.) In [1] (Theorem 6) it has been shown that the answer is positive for a certain interesting class of $\aleph_1$-dimensional spaces. Here we shall show how to construct spaces which fail to have subspaces U of the required shape. The construction rests on the additional assumption that all sets of the underlying set theory are constructible (in the sense of Gödel) which, intuitively, means that the set theory is the smallest universe which satisfies the usual set theory axioms and which is transitive and which contains all ordinals. More precisely, we use Jensen's combinatorial principle $\diamond$ (see e.g. [2]) which has already been used successfully by Shelah [5] in order to solve the Whitehead problem.

The solution of the first problem bears on the existence of locally algebraic elements in the orthogonal group of a sesquilinear space (cf. [3]); the second question arises quite often in the proof of theorems; good examples for the point in case are Theorem 1 in [4] and Theorems 6–8 in [1].

Terminology. Throughout the paper $\bar{k}$ is an arbitrary finite or countably infinite commutative field and k a fixed subfield. ω_0 is the first infinite ordinal, and ω_1 is the first uncountable ordinal. As usual, an ordinal α is identified with the set of its predecessors.

Dedicated to Professor Dr. Beno Eckmann to his sixtieth Birthday.

1. Extending forms

All forms Φ considered here are symmetric and bilinear. If Φ is defined on the k-vectorspace E then $\bar{\Phi}$ is the usual $\bar{k}$-ification of Φ to the space $\bar{E}=\bar{k}\otimes_k E$.

LEMMA 1. *Let E be a hyperplane in the $\aleph_0$-dimensional k-vectorspace E_1 and $(U_i)_{i<\omega_0}$ a family of subspaces of $\bar{E}$ with* $\dim U_i=\aleph_0$. *Assume that Φ is a non degenerate form on E. Then there exists a non degenerate extension Φ_1 of Φ to E_1 such that for all $i<\omega_0$ the orthogonal complement of U_i in $\bar{E}_1$ is contained in $\bar{E}$.*

Proof. Let e span a supplement of E in E_1 and let $(\langle x(j), i(j)\rangle)_{j<\omega_0}$ be an enumeration of $\bar{E}\times\omega_0$. The definition of Φ_1 proceeds in stages. Assume that we have already defined elements $\Phi_1(w, e)\in k$ for all w in some finite dimensional subspace $W_{j-1}\subseteq E$ such that $\Phi_1(w, e)$ is linear in the first argument. The j-th step is done as follows. Pick $u\in U_{i(j)}-\bar{W}_{j-1}$. There are finitely many $u_1,\ldots, u_m\in E$ and $\bar{\lambda}_1,\ldots,\bar{\lambda}_m\in\bar{k}$ such that $u=\sum_\mu\bar{\lambda}_\mu\otimes u_\mu$. Assume that the first r among the u_μ are a basis of a supplement of W_{j-1} in $W_j:=W_{j-1}+k(u_1,\ldots,u_m)$. There are furthermore finitely many $x_\nu\in E$ such that $x(j)=\sum_\nu\bar{\xi}_\nu\otimes x_\nu$ for certain $\bar{\xi}_\nu\in\bar{k}$.

We can certainly assign elements of k as values to $\Phi_1(u_\mu, e)$ $(\mu=1,\ldots,r)$ such that

$$\sum_{\mu,\nu}\bar{\lambda}_\mu\bar{\xi}_\nu\Phi(u_\mu, x_\nu)+\sum_\mu\bar{\lambda}_\mu\Phi_1(u_\mu, e)\neq 0. \tag{1}$$

This procedure gives us a sequence $(W_n)_{n<\omega_0}$ in E and defines the form Φ_1 on part of E_1, namely $W\oplus k(e)$ where $W:=\bigcup\{W_n\mid n<\omega_0\}$. (We define $\Phi_1(e, e)$ any way we like, and $\Phi_1|_W=\Phi|_W$). We complete the definition of Φ_1 by extending it to all of $E\oplus k(e)$ in some arbitrary fashion, but such that $\Phi_1|_E=\Phi$.

Assume now that for some non-zero $\bar{\lambda}\in\bar{k}$ we had a vector $z=\bar{x}+\bar{\lambda}\otimes e\in\bar{E}\oplus\bar{k}(e)=\bar{k}\otimes E_1$ orthogonal to U_i. Let $j<\omega_0$ be such that $\langle x(j), i(j)\rangle=\langle(1/\bar{\lambda})\bar{x}, i\rangle$. By (1) we have a vector $u\in U_i$ and $\bar{\Phi}_1(u, z)\neq 0$, contradiction.

THEOREM 1. $(2^{\aleph_0}=\aleph_1)$ *There exists a non degenerate form on a k-vectorspace E of dimension $\aleph_1$ such that for all infinite dimensional subspaces X of $\bar{E}=\bar{k}\otimes E$ we have* $\dim_{\bar{k}} X^\perp\leq\aleph_0$.

Proof. Pick some k-space $E=\bigoplus_{\alpha<\omega_1}k(e_\alpha)$ and let $\mathcal{U}=(U_\alpha)_{\omega_0\leq\alpha<\omega_1}$ be an enumeration of all $\aleph_0$-dimensional subspaces of $\bar{E}$ (there are $(\aleph_1\cdot\operatorname{card}\bar{k})^{\aleph_0}=\aleph_1^{\aleph_0}=\aleph_1$ of them). For all $\alpha<\omega_1$ we set $E_\alpha=\bigoplus_{\beta<\alpha}k(e_\beta)$. We can always renumber $\mathcal{U}$

such that $U_\alpha \subseteq \bar{E}_\alpha$ for all $\alpha \geq \omega_0$. We define Φ on E_α by transfinite induction. We start on E_{ω_0} with some arbitrary non degenerate form. Assume that Φ is defined on E_α. We extend Φ to $E_{\alpha+1} = E_\alpha \oplus k(e_\alpha)$ by means of the lemma where now $(U_\beta)_{\omega_0 \leq \beta < \alpha}$ plays the role of $(U_i)_{i \in \omega_0}$. By forming unions we extend Φ to E_γ with γ limit numbers. In this way we get a form Φ on E with the desired properties.

2. Some remarkable properties of the space of Theorem 1

A vectorspace automorphism T of the k-space E is called locally algebraic if for each $x \in E$ there is some polynomial $f_x \in k[X]$ such that $f_x(T)x = 0$; if f_x does not depend on x we call T algebraic.

In Theorem 1 we may let $\bar{k}$ be the algebraic closure of some countable field k. This will permit us to reproduce the proof of Theorem 2 in [3] and we obtain

THEOREM 2. *If (E, Φ) is as in Theorem 1 then the set $\mathscr{A}$ of all locally algebraic isometries is a group. $\mathscr{A}$ coincides with the set of all algebraic isometries on E; furthermore $\mathscr{A}$ is generated by $-\mathbf{1}$ and the symmetries about non degenerate hyperplanes of E. In particular, $\mathscr{A}$ is a normal subgroup of the orthogonal group of E.*

We may choose k to be a finite field; all maximal totally isotropic subspaces of the space E of Theorem 1 are then of dimension $\aleph_0$. One may arrange for some such subspaces X to have $X = X^\perp$ and for certain others to satisfy $\dim X^\perp/X = 1$ so that the orthogonal group will not be transitive on the set of maximal totally isotropic subspaces even though they are all of dimension $\aleph_0$. Incidentally, none of them will admit a Witt decomposition.

We also remark that if k is an ordered field and the form in Lemma 1 is positive definite, then it is easy to arrange for Φ_1 in the lemma to be positive definite as well. Hence we obtain

THEOREM 3. *If in Theorem 1 we let k be ordered then Φ may be required to be positive definite. For such a space the set $L_s = \{X \subseteq E \mid X^\perp + X = E\}$ of splitting subspaces, ordered by inclusion, is a lattice (in fact, a modular lattice).*

It is not easy to find examples different from real complex and quaternion Hilbertspace where the set L_s is a lattice.

3. A space which admits no dense orthogonal family

A subset C of ω_1 is called closed iff the supremum of any countable subset of C is again in C.

LEMMA 2. *Let (E, Φ) be of dimension $\aleph_1$, and let $(E_\alpha)_{\alpha<\omega_1}$ be an increasing sequence of countable subspaces of E such that $E = \bigcup_{\alpha<\omega_1} E_\alpha$ and $E_\alpha = \bigcup_{\beta<\alpha} E_\beta$ if α is limit. If U is a subspace spanned by a basis $(u_\gamma)_{\gamma<\omega_1}$ and with $U^\perp = (0)$ then the set $A(U) := \{\alpha \mid \alpha$ a limit ordinal and $U \cap E_\alpha = \bigoplus_{u_\gamma \in E_\alpha} k(u_\gamma)$ and $(U \cap E_\alpha)^\perp \cap E_\alpha = (0)\}$ is both closed and cofinal in ω_1.*

Indeed, closedness is obvious, and to show cofinality let $\beta < \omega_1$ be given. Define a sequence $(\beta_i)_{i<\omega_0}$ by $\beta_0 = \beta$ and $\beta_{i+1} = \min\{\beta \mid \beta > \beta_i$ & $U \cap E_{\beta_i} \subseteq \bigoplus_{u_\gamma \in E_\beta} k(u_\gamma)$ & $(U \cap E_\beta)^\perp \cap E_{\beta_i} = (0)\}$. Then $\alpha = \sup\{\beta_1 \mid i < \omega_0\} \in A(U)$.

A subset of ω_1 is stationary iff it intersects every closed cofinal subset A of ω_1. If our set theory satisfies the axiom of constructibility ("$V = L$") then the following combinatorial principle of Jensen holds (see p. 48 in [2]).

($\diamond$) There is a family $(X_\alpha)_{\alpha<\omega_1}$ with the following properties: (i) $X_\alpha \subseteq \alpha$, (ii) for all $X \subseteq \omega_1$ the set $\{\alpha \mid X \cap \alpha = X_\alpha\}$ is stationary in ω_1.

THEOREM 4. ($V = L$). *There exists a non degenerate k-space (E, Φ) of dimension $\aleph_1$ such that every subspace U spanned by an orthogonal basis has $U^\perp \neq (0)$.*

Proof. I. Construction of (E, Φ). Let $E = \bigoplus_{\alpha<\omega_1} k(e_\alpha) \oplus k(e'_\alpha)$ and for each $\alpha < \omega_1$ let E_α be the sum of the planes $k(e_\beta) \oplus k(e'_\beta)$ with $\beta < \alpha$. Since card $E = \aleph_1$ we may and often shall identify E with the set ω_1; we may do this in such a fashion that the subsets E_α satisfy

$$\alpha \text{ is a limit number} \rightarrow E_\alpha = \alpha.$$

When identifying E and ω_1 then some of the sets X_α in the family of ($\diamond$) may happen to correspond to linear subspaces of E!

Assume now that on E_α a non degenerate symmetric form Φ has already been defined. The extension to $E_{\alpha+1}$ is done as follows. First we extend Φ to the space $E_\alpha \oplus k(e_\alpha)$ according to Lemma 1 with the set $\{X_\beta \mid \beta \leq \alpha$ & X_β is a linear subspace of E_α of dimension $\aleph_0\}$ playing the role of the family $(U_i)_{i<\omega_0}$. Secondly, we extend this new form to all of $E_{\alpha+1}$ by setting $\Phi(E_\alpha, e'_\alpha) = \{0\}$, $\Phi(e'_\alpha, e_\alpha) = 1$, $\Phi(e'_\alpha, e'_\alpha) = 0$. Since we can take care of limit ordinals simply by taking unions starting out with some arbitrary non degenerate form on E_{ω_0} we get a non degenerate form Φ on all of E.

II. Consider then a subspace U spanned by an orthogonal basis and with $U^{\perp}=(0)$. If dim $U=\aleph_0$ then $U\subseteq E_\alpha$ for some α and $e'_\alpha \perp E_\alpha$, hence $U^{\perp}\neq(0)$. If dim $U=\aleph_1$ then let $(u_\alpha)_{\alpha<\omega_1}$ be an orthogonal basis. Choose α in the intersection of the set $A(U)$ of Lemma 2 and the stationary set of $(\diamond)$ with $X=U$. This means $X_\alpha=U\cap\alpha=U\cap E_\alpha=\bigoplus_{u_\gamma\in E_\alpha} k(u_\gamma)$ and $(U\cap E_\alpha)^{\perp}\cap E_\alpha=(0)$. Since for every $\beta\geqslant\alpha$ X_α is one of the subspaces taken into consideration when extending the form at step $\beta+1$ we see that $X_\alpha^{\perp}\cap E_\beta$ is totally isotropic and increasing for $\beta>\alpha$. Hence $X_\alpha^{\perp}$ is totally isotropic, and, since X_α is an orthogonal summand of U, we conclude $U^{\perp}=X_\alpha^{\perp}\neq(0)$.

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