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A remark on Abel's Theorem and the mapping of linear series

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Let X and Y be closed Riemann surfaces of positive genera, and let $\phi: X \to Y$ be a holomorphic map onto.

Remark. If $D = \sum m_i Q_i$ is a positive divisor of degree n and (projective) dimension r on X, then $\phi(D) = \sum m_i \phi(Q_i)$ is a positive divisor of degree n and (projective) dimension $\geq r$ on Y. If the complete linear series determined by D is without fixed points, then so is that determined by $\phi(D)$.

Proof. An arbitrary positive divisor of degree r on Y may be written as $\phi(D')$ where D' is a positive divisor of degree r on X. If D is of dimension r, there is a positive divisor D'' on X such that $D \sim D' + D''$ (linear equivalence). Connecting the points of D and $D' + D''$ by curves and applying Abel's theorem to the pullbacks to X of the holomorphic differentials on Y, we immediately see that $\phi(D) \sim \phi(D') + \phi(D'')$ on Y. Since $\phi(D')$ was an arbitrary positive divisor of degree r on Y, we conclude that $\phi(D)$ is of dimension $\geq r$.

The preimage under ϕ of the set of points occuring in $\phi(D)$ is a finite set of points on X. If D determines a complete linear series without fixed points, then we can find a positive divisor D' on X such that $D' \sim D$, and no point in the preimage occurs in D'. Then, since $\phi(D) \sim \phi(D')$ on Y, the complete linear series determined by $\phi(D)$ must be without fixed points.

COROLLARY 1. *If X can be displayed as an n-sheeted covering of the Riemann sphere, then so can Y. In particular, if X is hyperelliptic then Y is hyperelliptic or elliptic.*

Proof. X can be displayed as an n-sheeted covering of the Riemann Sphere if and only if it admits a positive divisor of degree n and dimension ≥ 1 that determines a complete linear series without fixed points.

COROLLARY 2. *If D and $\phi(D)$ both are divisors of dimension* 1, *then the branch points of the coverings of the Riemann sphere determined by D are mapped on the branch points of the coverings determined by $\phi(D')$ without reduction of branching order.*

Proof. Under the stated assumption, the branch points of any cover are precisely those that appear with multiplicity ≥ 2 in the divisors of the linear series.

APPLICATION. We consider the case when X is hyperelliptic. Then Y is hyperelliptic or elliptic, and in any case a display of Y as a 2-sheeted cover of the Riemann sphere must have $2h+2$ branch points, where h is the genus of Y. If g is the genus of X, the $2g+2$ branch points of X must map on the branch points of Y, whence $2g+2 \leq m(2h+2)$ or

$$g \leq m(h+1)-1$$

where m is the degree of ϕ. Thus, for instance, as noted by Accola and Farkas, a hyperelliptic surface of genus >3 cannot be a 2-sheeted covering of an elliptic surface. Combining the above formula with the Riemann–Hurwitz relation, we get

$$m(h-1)+1+\tfrac{1}{2}b_\phi = g \leq m(h+1)-1$$

or

$$b_\phi \leq 4(m-1)$$

where b_ϕ is the total branch order of ϕ.

(If only Y is known to be hyperelliptic, certain restrictions are imposed on X. Thus, if Y is of genus ≥ 3, no function of order 3 can exist on Y and hence not on X. The above formulas can be generalized to non-hyperelliptic cases).

Each divisor of degree 2 of the hyperelliptic series on X is mapped on a divisor of degree 2 in the image series on Y. Hence a sheet interchange on X followed by ϕ is equivalent to ϕ followed by a sheet interchange on Y. Thus, the function field on Y pulls back to a subfield on X invariant under the hyperelliptic involution.

(The above argument can be carried out for complete, irreducible algebraic curves, over a general groundfield, whose canonical images in the jacobian variety are invariant (modulo translation) under the involution $u \rightarrow (-u)$. This yields a geometric interpretation of a result of Tamme [*Ein Satz über Hyperelliptische Funktionenkörper*, J. Reine Angew. Math. *257* (1972) 217–220]. Of course, our main remark can also be proved in the more general context).

The result for hyperelliptic surfaces given in Corollary was mentioned by R. D. M. Accola, *Advances in the Theory of Riemann Surfaces*, Ann. of Math. Studies, No. 66, Princeton Univ. Press, Princeton N.J. 1971, pp. 7–18. A special case of our main remark was proven by M. Newman, Math. Ann. *196* (1972) 198–217.

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