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Slowly growing subharmonic function II

MATTS ESSÉN

1. Introduction

Let u be subharmonic and not constant in the open z-plane and let

$$B(r)=B(r,u)=\sup_{|z|=r} u(z). \tag{1.1}$$

If $B(r)$ grows sufficiently slowly, it is known that for "most" values of $z=re^{i\theta}$, $u(z)$ is not much smaller than $B(r)$. In other words, for each $\varepsilon>0$, the set

$$E=E(\varepsilon)=\{z:u(z)<(1-\varepsilon)B(r)\}$$

must be small. In [3], Essén, Hayman and Huber proved that if

$$B(r)=O((\log r)^2), \qquad r\to\infty, \tag{1.2}$$

then a generalized Wiener condition will hold at infinity for E. In particular, E can only be a small subset of each annulus $\omega_n=\{z:2^n\leq|z|<2^{n+1}\}$ when n is large. The purpose of the present paper is to study the case when condition (1.2) is replaced by

$$B(r)=O(\psi(r)), \qquad r\to\infty, \tag{1.3}$$

where ψ is an increasing function such that

$$\limsup_{r\to\infty} \psi(r)/(\log r)^2=\infty \tag{1.4}$$

$$\log\psi(r)/\log r\to 0, \qquad r\to\infty. \tag{1.5}$$

Here, the situation is different: E can now also contain almost all points in a sequence of annuli $\{\omega_{n_k}\}_1^\infty$, although the sequence $\{n_k\}$ must be rather sparse.

(References to earlier work on small subharmonic functions satisfying (1.3) and (1.4) can also be found in [3].)

The emphasis is here on the study of E for subharmonic functions of order zero, i.e., we assume that (1.5) holds. If this is not the case and we can consider functions of positive order, the set E can be almost the whole plane, as simple examples show.

The main results are stated in Section 2. In the proofs, we need certain results from the theory of series which are given in Section 3. After the proof of the main results in Sections 4 and 5, we give examples in Section 6. In Section 7, we consider a question of J. M. Anderson: does there exist a path going out to infinity which does not meet $E(\varepsilon)$? We also sharpen a related result of M. N. M. Talpur.

This work started as a joint effort by W. K. Hayman, A. Huber and myself to study growth problems for small subharmonic functions. The first part of this project deals with the case when (1.2) holds, and our results are given in [3]. It turned out, however, that I was responsible for the research on the remaining part of the project, and it was therefore decided that this work will appear as a paper by one author only. I am grateful to my co-authors from [3] for their generosity. Also, I want to thank W. K. Hayman for interesting discussions and J. M. Anderson for suggesting the problem mentioned above.

2. The main results

Let u be a subharmonic function of order zero. Without loss of generality, we can assume that u is harmonic in $\{|z|<1\}$ and that $u(0)=0$. Thus u has a representation of the form

$$u(z)=\int \log|1-z/\zeta|\, d\mu(\zeta), \tag{2.1}$$

where μ is the Riesz mass of u and the integral is taken over the open plane (cf. [3, Section 4]). If $n(t)$ is the mass in $\{|z|<t\}$, we have $n(1)=0$. We define

$$N(r)=\int_0^r n(t)/t\, dt \tag{2.2}$$

$$u^*(z)=\int \log|1+z/|\zeta||\, d\mu(\zeta)=\int_0^\infty \log|1+z/t|\, dn(t). \tag{2.3}$$

$$B^*(r)=\sup_{|z|=r} u^*(z)=u^*(r). \tag{2.4}$$

As examples of possible growth rates, we mention, if α is a positive constant

$$\psi(r)=(\log r)^{\alpha}, \qquad r\geq e, \qquad \alpha>2,$$

$$\psi(r)=\exp\{(\log\log r)^{\alpha}\}, \qquad r\geq e, \qquad \alpha>1.$$

More generally, we assume that $\psi:[0,\infty)\to(0,\infty)$ is a positive, increasing and continuous function for which (1.4) and (1.5) hold and which is either such that

$$\psi(2r)/\psi(r)\to 1, \qquad r\to\infty, \tag{2.5a}$$

or which is such that for some constant $C\geq 1$, we have

$$\int_r^{\infty}\psi(t)/t^2\,dt\leq(C+o(1))\psi(r)/r, \qquad r\to\infty. \tag{2.5b}$$

The two examples mentioned above satisfy both assumptions (2.5a) and (2.5b) with $C=1$. If we are interested essentially in functions ψ which grow in this very regular way, it is sufficient to use (2.5a). Our results hold, however, also for a function ψ satisfying the weaker assumption (2.5b). We note that when $C=1$, (2.5a) and (2.5b) are equivalent. As a consequence of (1.3) and (2.5b), we obtain

$$B^*(r)=O(\psi(r)), \qquad r\to\infty. \tag{2.6}$$

Proof of (2.6). From Lemma 3.5.1 in Boas [2], we see that

$$B^*(r)\leq N(r)+\int_r^{\infty}N(t)/t^2\,dt\leq B(r)+r\int_r^{\infty}B(t)/t^2\,dt\leq\text{Const. }\psi(r)$$

and (2.6) is proved. We have used that $N(r)\leq B(r)$.

If $\varepsilon>0$ is given, we consider the set

$$E^*=E^*(\varepsilon)=\{z:u(z)<(1-\varepsilon)B^*(r)\}.$$

We clearly have $E\subset E^*$. We can now state our main results which describe how small the set E^* is.

THEOREM 1. *Let ψ satisfy* (1.4), (2.5*a*) *or* (2.5*b*) *and let u be a non-constant subharmonic function of order* 0 *such that* (1.3) *ψ non-constant holds. For a fixed* *positive* ε, *let* $E_k^* = E^* \cap \omega_k$. *We then have*

$$\sum_1^n \{\log(2^{k+3}/\text{cap } E_k^*)\}^{-1} \leq \text{Const. } \varepsilon^{-1}\{1+\log \psi(2^{n+3})\} \tag{2.7}$$

If $\psi(r) = (\log r)^\alpha$, $r > e$, *where* $\alpha > 2$ *is given,* (2.7) *is replaced by*

$$\sum_1^n \{\log(2^{k+3}/\text{cap } E_k^*)\}^{-1} \leq \text{Const. } \varepsilon^{-1}(1+\alpha \log(n+3)). \tag{2.8}$$

The constant does not depend on α *or* ε.

COROLLARY. *Let* $\delta > 0$ *be given. If* (2.7) *holds, there exists a covering of* E^* *by disks* $\{z : |z - R_k e^{i\theta_k}| < r_k\}$ *which is such that*

$$\sum (\log(2R_k/r_k))^{-1-\delta} = O(\varepsilon^{-1} \log \psi(2^{n+3})), \qquad n \to \infty. \tag{2.9}$$

We have summed over those indices k *which are such that the union of the associated disks covers* $E^* \cap \{|z| < 2^{n+1}\}$.

Remark. In Section 6, examples are given which illustrate the precision of (2.8) and (2.9). There are additional results which explain why $\alpha = 2$ is a critical value and which will appear elsewhere. Let me mention one estimate of this type. Let the sequences $\{c_n\}$ and $\{\delta_n\}$ be defined as in Section 3. Let $\alpha > 2$ be given and assume that

$$c_n = O(n^\alpha), \qquad n \to \infty.$$

Then we have

$$\limsup_{n \to \infty} (\log n)^{-1} \sum_1^n \delta_k \leq c_0(\alpha - 2),$$

where $c_0 = \sup_{0<x<1} (1-x)^2(\log(1/x))^{-1} \approx 0.407$. There exists an admissible sequence $\{c_n\}$ which gives equality.

Next, we consider the following result of P. D. Barry ([1, Corollary to Theorem 5, p. 475; also cf. Section 7.4]).

THEOREM A. *Let u be a subharmonic function such that* (1.3) *holds with* $\psi(r)=(\log r)^{\alpha}$, *where* $\alpha<3$. *Let* $\varepsilon>0$. *Then*

$$\inf_{\theta} u(re^{i\theta})>(1-\varepsilon)B(r),$$

outside a set F for which

$$\int_F (t\log t)^{-1}\,dt<\infty.$$

This result can be strengthened in the following way.

THEOREM 2. *Let u and ψ be as in Theorem* 1 *and assume furthermore that ψ is strictly increasing and continuously differentiable on* $[1,\infty)$. *Let f be a decreasing, nonnegative and continuously differentiable function which is such that*

$$\int_1^{\infty} f(s)\,ds/s<\infty.$$

If $\varepsilon>0$ is given, then we have

$$\inf_{\theta} u(re^{i\theta})>(1-\varepsilon)B^*(r),\qquad r\geq 1 \tag{2.10}$$

outside a set F such that

$$\int_F f(\psi(t))\,dt/t<\infty. \tag{2.11}$$

COROLLARY. *Let* $\psi(r)=(\log r)^{\alpha}$, $\alpha>2$. *Then for each* $h>0$, *we have*

$$\int_F (\log\log t)^{-1-h}\,dt/t<\infty. \tag{2.12}$$

Remark. An example given in Section 6 will show that we cannot take $h=0$ in the Corollary.

3. A lemma on series.

As in [3], our problem will be reduced to a problem in the theory of series. In this section, we give a simple lemma

Let ψ be as in Section 2. Let $\{a_n\}_1^\infty$ be a given non-negative sequence and let

$$b_n = \sum_{k=1}^{n} a_k, \qquad c_n = \sum_{k=1}^{n} b_k.$$

We also define

$$\delta_n = \begin{cases} a_n/c_n, & c_n > 0, \\ 0, & c_n = 0. \end{cases}$$

If $c_n = O(\psi(2^n))$, $n \to \infty$, $\{\delta_n\}$ is almost a ψ-sequence as defined in Section 2 in [3]. We say almost because we do not assume that $\psi(r) = O((\log r)^2)$, $r \to \infty$ (which is part of the definition of ψ-sequence in [3]); we assume in fact that (1.4) holds. In [3], we gave results on the series $\sum_1^\infty \delta_\nu^\lambda$, where $\lambda \geq 1/2$; here we study $\sum_1^\infty \delta_\nu$. Since we assume less on ψ, the situation is more complicated than that discussed in [3].

LEMMA 1. *Let ψ be as in Theorem 1. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{\delta_n\}$ be as above. If*

$$c_n \leq \psi(2^n), \qquad n = 1, 2, \ldots \tag{3.1}$$

then we have $\qquad n = 1, 2, \ldots.$

$$\sum_1^n \delta_k \leq \text{Const.} + \min\{\log \psi(2^n), \log(\psi(2^{2n})/n)\}. \tag{3.2}$$

If furthermore we assume that $a_n = o(c_n)$, $n \to \infty$, then we have

$$\sum_1^n \delta_k = o(\log \psi(2^n)), \qquad n \to \infty. \tag{3.3}$$

Remark. In the case $\psi(r)=(\log r)^{\alpha}$, where $\alpha>2$ is given, we deduce from (3.2) that

$$\sum_{1}^{n}\delta_k\le \text{Const.}+(\alpha-1)\log n,\qquad n=1,2,\dots. \tag{3.2$'$}$$

The case $\alpha=2$ is treated in [3, Theorem 4]: then we know that $\sum_1^\infty \delta_n<\infty$.

Proof of Lemma 1. We know that $b_n\le\min\{c_n, c_{2n}/n\}$. Using [3, (2.19)], we deduce that

$$\sum_{n_0}^{n} a_k/c_k\le\sum_{n_0}^{n} a_k/b_k\le\log b_n-\log b_{n_0},$$

where n_0 is the smallest integer such that b_n does not vanish. It is now clear that (3.2) follows from (3.1). To prove (3.3), we note that if $\varepsilon>0$ is given, there exists an integer n_1 such that $a_n/c_n\le\varepsilon^2$, $n\ge n_1$. If $n\ge n_1$ and $b_n/c_n>\varepsilon$, we see that $a_n/b_n<\varepsilon$. It follows that

$$\begin{aligned}\sum_{n_1}^{n} a_k/c_k&=\Bigg(\sum_{b_k/c_k>\varepsilon}{}' + \sum_{b_k/c_k\le\varepsilon}{}''\Bigg)(a_k/b_k)(b_k/c_k)\\ &\le\varepsilon\left(\sum_{n_1}^{n} b_k/c_k+\sum_{n_1}^{n} a_k/b_k\right)\le 2\varepsilon(\text{Const.}+\log\psi(2^n)).\end{aligned}$$

In the last step, we used once more [3, (2.19)]. The lemma is proved.

Remark. If $a_n=o(c_n)$, $n\to\infty$, we claim that we also have

$$b_n=o(c_n),\qquad n\to\infty. \tag{3.4}$$

To prove this, let ε be given, $0<\varepsilon<1/2$, and choose n_1 as in the proof of (3.3). If $p=[1/\varepsilon]$, we see that if $n>n_1$, we have

$$b_n\le b_{n-p}+\varepsilon^2\sum_{n-p+1}^{n} c_k\le b_{n-p}+p\varepsilon^2 c_n,$$

$$c_n\ge\sum_{1}^{n-p}(n+1-k)a_k\ge p b_{n-p}.$$

It follows that

$$b_n/c_n \le (b_{n-p} + p\varepsilon^2 c_n)/c_n \le p^{-1} + \varepsilon \le 3\varepsilon.$$

Thus (3.4) holds.

Lemma 1 is clearly a very simple result, but it is all we need in the proofs of Theorems 1 and 2, assuming that ψ satisfies (1.4), (2.5a) or (2.5b). Lemma 1 is unsatisfactory from another point of view: it does not explain why the value $\alpha = 2$ is crucial when $\psi(r) = (\log r)^\alpha$. This question was discussed in the Remark after the Corollary to Theorem 1.

4. Proof of Theorem 1

We start with

LEMMA 2. *Let u and ψ be as in Theorem 1 and let ε_0 be given, $0 < \varepsilon_0 < 1/6$. Let G be the set of positive integers k which are such that*

$$2^k \int_{2^k}^{\infty} n(t)/t^2\, dt > \varepsilon_0 \sum_{j=1}^{k} 2^j \int_{2^j}^{\infty} n(t)/t^2\, dt.$$

Let $\pi(p)$ be the number of elements in G in the set $\{k\}_{k=1}^{p}$. Then

$$\pi(p) \le \varepsilon_0^{-1} \log \psi(2^p) + O(1), \qquad p \to \infty. \tag{4.1}$$

In the complement of G, we have

$$2^k \int_{2^k}^{\infty} n(t)/t^2\, dt \le 3\varepsilon_0 N(2^k). \tag{4.2}$$

Proof. We define $q_k = 2^k \int_{2^k}^{\infty} n(t)/t^2\, dt$ and $Q_k = \sum_{j=1}^{k} q_j$. Then we have

$$\varepsilon_0 \pi(p) \le \sum_{k=1}^{p} q_k/Q_k \le 1 + \sum_{k=2}^{p} \log(Q_k/Q_{k-1})$$

$$= 1 + \log(Q_p/Q_1) = \log\left(\sum_{j=1}^{p} 2^j \int_{2^j}^{\infty} n(t)/t^2\, dt\right) - \log\left(2 \int_{2}^{\infty} n(t)/t^2\, dt\right) + 1. \tag{4.3}$$

To estimate the right hand side of (4.3), we note that

$$\sum_{j=1}^{p} 2^j\left\{\int_{2^j}^{2^p}+\int_{2^p}^{\infty}\right\}n(t)/t^2\,dt \le \int_2^{2^p} n(t)/t^2\Big(\sum_{1\le j,2^j\le t} 2^j\Big)\,dt$$

$$+2^{p+1}\int_{2^p}^{\infty} n(t)/t^2\,dt \le 2\left\{\int_2^{2^p} n(t)/t\,dt+2^p\int_{2^p}^{\infty} n(t)/t^2\,dt\right\}$$

$$\le 2^{p+1}\int_{2^p}^{\infty} N(t)/t^2\,dt \le O(\psi(2^p)), \qquad p\to\infty. \quad (4.4)$$

In the last step, we used three estimates: the fact that $N(r)\le B(r)$, (1.3) and (2.5b). Combining (4.3) and (4.4), we obtain (4.1).

To prove (4.2), we note that if $k\notin G$, it follows from (4.4) that

$$2^k\int_{2^k}^{\infty} n(t)/t^2\,dt \le \varepsilon_0\sum_{j=1}^{k} 2^j\int_{2^j}^{\infty} n(t)/t^2\,dt \le 2\varepsilon_0\Big(N(2^k)+2^k\int_{2^k}^{\infty} n(t)/t^2\,dt\Big).$$

Since $0<\varepsilon_0<1/6$, we obtain (4.2). This completes the proof of Lemma 2.

Let ω_k be as in the introduction. We also define $\Omega_k=\{z: 2^{k-1}\le|z|\le 2^{k+2}\}$, $D_k=\{z:|z|<2^{k-1}\}$, $F_k=\{z:|z|>2^{k+2}\}$.

The exceptional set $E^*(\varepsilon)$ is contained in the union of two sets E_I and E_{II}. We first define $E_I=\bigcup\omega_k$, $k+2\in G$, where G is the set of integers defined in Lemma 2; we shall see that we can take $\varepsilon_0=\varepsilon/35$.

To define E_{II}, we have to study the potentials which are associated with the subharmonic functions u and u^* (cf. (2.1) and (2.3)). Let us choose k such that $k+2\notin G$ and assume that $z=re^{i\theta}\in\omega_k$. We want to estimate

$$u(z)-B^*(r)=\left\{\int_{D_k}+\int_{\Omega_k}+\int_{F_k}\right\}(\log|1-z/\zeta|\,d\mu(\zeta)$$

$$-\log(1+r/t)\,dn(t))=I_1+I_2+I_3.$$

First, we estimate I_1 and I_3 from below:

$$I_1\ge\int_{D_k}\log((|z|-t)/(|z|+t))\,dn(t)\ge -\log 3n(2^{k-1}),$$

$$I_3\ge\int_{F_k}\log((t-|z|)/(t+|z|))\,dn(t)\ge -2\log 3|z|\int_{F_k} t^{-1}\,dn(t)$$

$$\ge -2^{k+2}\log 3\Big(\int_{F_k} n(t)/t^2\,dt-n(2^{k+2})/2^{k+2}\Big)$$

Adding up, we obtain

$$I_1+I_3\geq -2^{k+2}\log 3\int_{F_k} n(t)/t^2\,dt\geq -3\varepsilon_0\log 3N(2^{k+2}),\qquad k+2\notin G.$$

It follows that

$$u(z)-B^*(r)\geq\int_{\Omega_k}\log(|\zeta-z|2^{-k-3})\,d\mu(\zeta)-3\varepsilon_0\log 3N(2^{k+2}),\quad k+2\notin G. \tag{4.5}$$

Let E_k be the set in ω_k where

$$\int_{\Omega_k}\log(|\zeta-z|2^{-k-3})\,d\mu(\zeta)\leq -\varepsilon_0 N(2^{k+3}). \tag{4.6}$$

We define $E_{II}=\bigcup E_k$, $k+2\notin G$. From (4.5), it follows that

$$u(z)-B^*(r)\geq -\varepsilon_0(1+3\log 3)N(2^{k+3})\geq -35\varepsilon_0 B^*(r),\quad z\notin E_I\cup E_{II}. \tag{4.7}$$

In the last step, we used the inequality

$$N(2^{k+3})\leq B^*(2^{k+3})\leq 8B^*(2^k)\leq 8B^*(r).$$

Choosing $\varepsilon_0=\varepsilon/35$ we see that $E^*(\varepsilon)\subset E_I\cup E_{II}$.

It remains to prove that the set $E_I\cup E_{II}$ is small in the sense described in Theorem 1. We first claim that

$$\sum_{k=1}^{n}\{\log(2^{k+3}/\mathrm{cap}(E_I\cap\omega_k))\}^{-1}\leq \text{Const. }\varepsilon^{-1}(1+\log\psi(2^{n+2})). \tag{4.8}$$

For each k, there are two possibilities: either $E_I\cap\omega_k=\omega_k$ or $E_I\cap\omega_k=\varnothing$. In the first case, we have $(E_I\cap\omega_k)2^{-k-3}=\{z:\frac{1}{8}\leq|z|\leq\frac{1}{4}\}$, and the sum in (4.8) is majorized by the number of indices in $G\cap[1,n+2]$ times the constant $(\log(1/\mathrm{cap}\,\omega_{-3}))^{-1}$. Our conclusion follows from Lemma 2.

To discuss E_{II}, let a_k be the Riesz mass in Ω_k, i.e., $a_k=\mu(\Omega_k)$. Applying Lemma 4 in [3], we see from (4.6) that

$$\mathrm{cap}\,E_k\leq\exp\{-\varepsilon_0 N(2^{k+3})/a_k\}2^{k+3},$$

$$\sum_1^n(\log\{2^{k+3}/\mathrm{cap}\,E_k\})^{-1}\leq\varepsilon_0^{-1}\sum_1^n a_k/N(2^{k+3}). \tag{4.9}$$

Starting from $\{a_k\}_1^\infty$, we define $\{b_k\}$ and $\{c_k\}$ as in Section 3. We note that

$$3N(2^{k+3}) \geq 3 \sum_{\nu=1}^{k+2} n(2^\nu) \log 2 \geq (\log 2) \sum_{\nu=1}^{k} b_\nu = c_k \log 2.$$

Hence the right hand member of (4.9) is majorized by Const. $\varepsilon^{-1} \sum_1^n a_k/c_k$. Since $c_n = O(\psi(2^{n+3}))$, $n \to \infty$, we can use Lemma 1. Combining (4.8) and (4.9), we obtain (2.7) in Theorem 1.

To prove the Corollary, we argue in exactly the same way as in the proof of Theorem 7 in [3]. We omit the details.

5. Proof of Theorem 2

We start from the Corollary to Theorem 1. Let $\delta > 0$ be given and consider a covering of $E^*(\varepsilon)$ by disks $\{\Delta_k\} = \{\{z : |z - R_k e^{i\theta_k}| < r_k\}\}$ which is such that (2.9) holds. Furthermore, we assume that $r_k \leq R_k$, $k = 1, 2, \ldots$. It is known that

$$Dx \geq (\log 2x)^{1+\delta}, \qquad x \geq 1,$$

where D is a positive constant which depends on δ. It follows from (2.9) that

$$\sum r_k/2R_k \leq D \sum (\log (2R_k/r_k))^{-1-\delta} = O(\varepsilon^{-1} \log \psi(2^{n+3})), \qquad n \to \infty; \tag{5.1}$$

we sum over those indices k which are such that the corresponding disks intersect $\bigcup_1^n \omega_\nu$. We now define $F = \{r \geq 1 : r = |z| \text{ for some } z \in \bigcup_1^\infty \Delta_k\}$. It is clear that (2.10) holds outside F. It follows from (5.1) that

$$\int_{F \cap [1,r]} dt/t \leq \text{Const. } \varepsilon^{-1} \log \psi(16r). \tag{5.2}$$

Applying a standard argument, we see from (5.2) that

$$\int_{F \cap [1,r]} f(\psi(16t))\, dt/t \leq \text{Const. } \varepsilon^{-1} \int_1^r f(\psi(16t))\, d \log \psi(16t)$$

$$= \text{Const. } \varepsilon^{-1} \int_{\psi(16)}^{\psi(16r)} f(s)\, ds/s. \tag{5.3}$$

Letting $r\to\infty$, we see that the integral in the left hand member in (5.3) converges, and we have proved (2.11).

Proof of the Corollary. We choose $f(s)=(\log s)^{-1-h}$, $s\geq e$.

6. Examples

We shall give examples of subharmonic functions of order zero where exceptional sets of type E_I and E_{II} occur. We start with E_{II}.

Let ψ and the nonnegative sequences $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$ be as in Section 3. We assume that $c_n=O(\psi(2^n))$, $n\to\infty$ and that $a_n=o(c_n)$, $n\to\infty$. From (3.4), we see that we also have

$$b_n=o(c_n), \qquad n\to\infty. \tag{6.1}$$

If $D_0=\{z:1/8\leq|z|\leq 1/4,\ \operatorname{Re} z\leq 0\}$, we put $L=\log(1/\operatorname{cap} D_0)$. If $D\subset\{\operatorname{Re} z\leq 0\}\cap\omega_n$, we know that

$$0\leq\{\log(2^{n+3}/\operatorname{cap} D)\}^{-1}\leq L^{-1}.$$

Suppose that the positive number ε is given. Since $a_n=o(c_n)$, $n\to\infty$, there exists n_0 such that $a_n/(2\varepsilon c_n)\leq L^{-1}$, $n\geq n_0$. Thus for $n\geq n_0$, there exists a closed subset $E_n\subset\omega_n\cap\{\operatorname{Re} z\leq 0\}$ which is symmetric around the real axis and such that

$$\{\log(2^{n+3}/\operatorname{cap} E_n\}^{-1}=a_n/(2\varepsilon c_n).$$

If $n<n_0$, we put $E_n=\varnothing$.

Let μ_n be the equilibrium distribution on E_n. In particular, the total mass of μ_n is 1 and we have

$$\int\log|z-\zeta|\,d\mu_n(\zeta)=\log(\operatorname{cap} E_n)\ \text{p.p. on } E_n.$$

We now define the measure $\mu=\Sigma\,\mu_n a_n$ and the subharmonic function

$$u(z)=\int\log|1-z/\zeta|\,d\mu(\zeta). \tag{6.2}$$

We claim that for this function u, we have

$$B(r)=B(r,u)=(1+o(1))N(r)=(\log 2+o(1))c_n, \qquad 2^n\leq r\leq 2^{n+1}. \tag{6.3}$$

To prove (6.2), we first note that (cf. (4.21) in [3])

$$c_{n-1}\log 2\le N(r)\le c_n\log 2,\qquad 2^n\le r<2^{n+1}, \tag{6.4}$$

$$N(r)=O(\psi(2r)),\qquad r\to\infty. \tag{6.5}$$

It is known that

$$B(r)\le\int_0^\infty r(t+r)^{-1}n(t)/t\,dt=\int_0^\infty r(t+r)^{-2}N(t)\,dt.$$

Since (6.1) holds, we have $n(r)=o(N(r))$, $r\to\infty$, and thus

$$N(r)\le B(r)\le N(r)+\int_r^\infty r(t+r)^{-1}n(t)/t\,dt\le$$

$$\le N(r)+o(1)r\int_r^\infty N(t)/t^2\,dt\le N(r)+o(1)4\int_0^\infty r(r+t)^{-2}N(t)\,dt$$

$$\le N(r)+o(B(r)). \tag{6.6}$$

Formula (6.3) now follows from (6.1), (6.4) and (6.6). From (6.5), we see that $B(r)=O(\psi(2r))$, $r\to\infty$.

Since supp $\mu_n\subset\{\mathrm{Re}\,z\le 0\}$ and E_n is symmetric around the real axis, we have $B(r)=u(r)$. As in the proof of Theorem 1, we write, if $z\in\omega_n$,

$$u(z)-B(r)=\left\{\int_{D_n}+\int_{\Omega_n}+\int_{F_n}\right\}(\log|1-z/\zeta|-\log|1-r/\zeta|)\,d\mu(\zeta)=J_1+J_2+J_3.$$

Estimates similar to those in the proof of Theorem 1 show that if $z\in\omega_n$, we have $|J_1|\le(\log 3)n(2^{n-1})=O(b_{n-1})=o(c_n)$, $|J_3|=o(c_{n+2})=o(c_n)$.

We also have

$$\log|r-\zeta|=n\log 2+O(1),\quad r\in\omega_n,\quad \zeta\in(\mathrm{supp}\,\mu)\cap\Omega_n.$$

We deduce the following representation formula which can be compared to (4.11) in [3]. Uniformly as $n\to\infty$, for $z\in\omega_n$, we have

$$u(z)-B(r)=\int_{\Omega_n}\log|z-\zeta|\,d\mu(\zeta)-(n\log 2+O(1))(a_{n-1}+a_n+a_{n+1})+o(c_n). \tag{6.7}$$

In particular, we have

$$u(z)-B(r)\leq -2\,\varepsilon\, c_n+na_n \log 2+(a_{n-1}+a_{n+1})(n \log 2+O(1))$$

$$-(n \log 2+O(1))(a_{n-1}+a_n+a_{n+1})+o(c_n)=-2\,\varepsilon\, c_n+o(c_n) \text{ p.p. on supp } \mu_n.$$

From (6.3), we finally obtain

$$u(z)\leq(1-\varepsilon)B(r), \qquad z\in \operatorname{supp} \mu_n, \qquad n\geq n_1. \tag{6.8}$$

The exceptional set $E=\bigcup_{n_1}^{\infty} E_k$ is such that

$$\sum_{n_1}^{n} \{\log (2^{k+3}/\operatorname{cap}(E\cap \omega_k))\}^{-1}=(2\varepsilon)^{-1}\sum_{n_1}^{n} a_k/c_k.$$

Thus the upper bound in (4.9) is of the right order of magnitude for sequences which satisfy (6.1): furthermore, we have from (3.3) that

$$\sum_{n_1}^{n} a_k/c_k=\sum_{n_1}^{n} \delta_k=o(\log n), \qquad n\to\infty. \tag{6.9}$$

It is easy to construct examples of sequences such that $\sum_{n_1}^{n} \delta_k$ is close to the upper bound in (6.9), when $n\to\infty$.

The subharmonic function u constructed above has the property that for all n, the exceptional set E is such that $E\cap\omega_n$ is a small subset of ω_n in the sense that

$$\{\log (2^{n+3}/\operatorname{cap}(E\cap\omega_n))\}^{-1}\to 0.$$

In our next example, we shall consider an exceptional set E of type E_I which is such that $E\cap\omega_n$ is for certain indices a large part of the annulus ω_n.

Let $\alpha>2$, $\delta\in(0,\alpha-2)$ and $\lambda\in(0,1)$ be given. If we want to construct a subharmonic function with the property $B(r)=O((\log r)^{\alpha})$, $r\to\infty$, we must clearly have $n(r)=O((\log r)^{\alpha-1})$, $r\to\infty$. We shall construct $n(r)$. We choose sequences $\{A_n\}$, $\{B_n\}$ and $\{K_n\}$ tending to infinity which are such that

$$(\log A_{\nu-1})^{\alpha-1}\leq(\log A_\nu)^{\alpha-2-\delta}, \qquad B_\nu=(\log A_\nu)^{\delta}, \qquad K_\nu=(\log B_\nu)^{2/\lambda}. \tag{6.10}$$

In the sequel, we consider the example given by $A_\nu = \exp\{\exp \gamma^\nu\}$, where $\gamma > (\alpha-1)(\alpha-2-\delta)^{-1}$. The increasing, continuous function $n(r)$ is defined by

$$n(r) = (\log A_{\nu-1})^{\alpha-1}, \qquad A_{\nu-1} \le r \le A_\nu/B_\nu, \qquad \nu = 1, 2, \ldots.$$

$$dn(r) = \alpha_\nu \, dr^\lambda, \qquad A_\nu/B_\nu < r < A_\nu, \qquad \nu = 1, 2, \ldots.$$

where $\alpha_\nu = ((\log A_\nu)^{\alpha-1} - (\log A_{\nu-1})^{\alpha-1})A_\nu^{-\lambda}(1-B_\nu^{-\lambda})^{-1} \approx A_\nu^{-\lambda}\,(\log A_\nu)^{\alpha-1}$. In the interval $(0, e^e)$, we let $n(r)$ be 0 in $(0, 1]$ and then increase to the value $n(e^e) = e$ in $[1, e^e]$. Our example is the subharmonic function

$$u(z) = \int_0^\infty \log|1 + z/t| \, dn(t). \tag{6.11}$$

It is easy to prove that $B(r) = u(r) = O((\log r)^\alpha)$, $r \to \infty$. We note that $u(A_\nu^2) \approx (\log A_\nu)^\alpha$. The sequence $\{A_\nu^2\}$ is well outside the support of the measure $dn(r)$.

We note for later reference that all constants in the O- and o-relations in the inequalities occurring in the deduction of the estimate of $C_\nu(r)$ below can be chosen to be independent of λ.

Let $z = re^{i\theta}$ and assume that $K_\nu A_\nu/B_\nu \le r \le A_\nu/K_\nu$. We have from (6.10) that

$$\int_0^{A_{\nu-1}} \log(1 + r/t) \, dn(t) \le (\log A_\nu)(\log A_{\nu-1})^{\alpha-1} \le (\log A_\nu)^{\alpha-1-\delta}.$$

$$\int_{A_\nu}^\infty \log(1 + r/t) \, dn(t) \le \int_{A_\nu}^\infty (r/t) \, dn(t) = O((r/A_\nu)(\log A_\nu)^{\alpha-1}).$$

It follows that

$$\begin{aligned} u(z) &= \int_{A_\nu/B_\nu}^{A_\nu} \log|1 + z/t| \, dn(t) + O((r/A_\nu)(\log A_\nu)^{\alpha-1}) \\ &= (\log A_\nu)^{\alpha-1}(1+o(1))\left\{\left\{\int_0^\infty - \int_0^{A_\nu/B_\nu} - \int_{A_\nu}^\infty\right\} \log|1+z/t| \, d(t^\lambda/A_\nu^\lambda) + O(r/A_\nu)\right\}. \end{aligned}$$

We need the following estimate:

$$\begin{aligned} \int_0^{A_\nu/B_\nu} \log(1 + (A_\nu/K_\nu t)) \, d(t/A_\nu)^\lambda &= K_\nu^{-\lambda} \int_0^{K_\nu/B_\nu} \log(1 + s^{-1}) \, ds^\lambda \\ &\le B_\nu^{-\lambda} \log(1 + B_\nu) + K_\nu^{-\lambda} \int_0^{K_\nu/B_\nu} s^{\lambda-1} \, ds = B_\nu^{-\lambda}(\log(1+B_\nu) + \lambda^{-1}). \end{aligned}$$

Simple computations now show that

$$|u(z)-(\log A_\nu)^{\alpha-1}(1+o(1))\pi(\sin \pi\lambda)^{-1}\mathrm{Re}(z^\lambda/A_\nu^\lambda)|\leq C_\nu(r).$$

where

$$C_\nu(r)\leq(\log A_\nu)^{\alpha-1}(1+o(1))(B_\nu^{-\lambda}\log(1+B_\nu)+\lambda^{-1}B_\nu^{-\lambda}+O((r/A_\nu)(1-\lambda)^{-1}))$$
$$=O(r/A_\nu)(1-\lambda)^{-1}(\log A_\nu)^{\alpha-1},\qquad K_\nu A_\nu/B_\nu\leq r\leq A_\nu/K_\nu.$$

We finally obtain

$$u(z)=(\log A_\nu)^{\alpha-1}(r^\lambda/A_\nu^\lambda)\{\pi\cos\lambda\theta(\sin\pi\lambda)^{-1}+o(1)\},\tag{6.12}$$

$$K_\nu A_\nu/B_\nu\leq r\leq A_\nu/K_\nu;$$

the error estimates are uniform in z as $\nu\to\infty$.

We note that if $\varepsilon>0$ is small, the set $E_k^*(\varepsilon)$ is almost the whole annulus ω_k for indices k which are large and such that $2^k\in\bigcup_\nu[K_\nu A_\nu/B_\nu, A_\nu/K_\nu]$. Since cap $E_k^*\approx 2^k$, it follows that

$$\limsup_{n\to\infty}(\log n)^{-1}\sum_1^n\{\log(2^{k+3}/\text{cap } E_k^*)\}^{-1}$$
$$\geq \text{Const.}\lim_{n\to\infty}(\log n)^{-1}(\log B_n-2\log K_n)=\text{Const. }\delta.$$

From (2.8), we see that

$$\limsup_{n\to\infty}(\log n)^{-1}\sum_1^n\{\log(2^{k+3}/\text{cap } E_k^*)\}^{-1}\leq \text{Const. }\alpha/\varepsilon.$$

Since δ can be chosen close to $\alpha-2$, the discrepancy between the orders of magnitude in these two estimates is essentially the factor ε^{-1}. The upper bound given by (2.8) is not far from the lower bound which holds for the function u defined by (6.11).

This function u also gives an example which shows that the Corollary of Theorem 2 is sharp. If $F=\{r\geq 1:\ re^{i\theta}\in E_k^*(\varepsilon)$ for some $\theta\}$ we have

$$\int_{E_I} dt/(t\log\log t)\approx\sum_1^\infty(\log B_\nu)/(\log\log A_\nu)\approx\sum_1^\infty 1=\infty.$$

7. The growth of u on paths going out to infinity

Let u be a subharmonic function in the plane such that $B(r) = O((\log r)^\alpha)$, $r \to \infty$. In the case $\alpha = 2$, Hayman proved in [4] that for almost all θ in $[0, 2\pi]$, the ray $\{\arg z = \theta\}$ meets the set $E(\varepsilon)$ in a bounded set. An improvement of this result is given in Theorem 6 in [3]: the exceptional set of θ:s is in fact of capacity zero. Hayman's result [4] led Dr. Milne Anderson to ask the following question in the case $\alpha > 2$ (private communication): if $\varepsilon \in (0, 1]$ is given, does there exist a path Γ going out to infinity such that $u(z) \geq (1-\varepsilon)B(|z|)$, $z \in \Gamma$?

M. Talpur proved in [5] that if u is subharmonic and non-constant in the plane, there exists a path Γ going out to infinity on which $u(z) \to \infty$. Clearly, we have $u(z) > 0$, $z \in \Gamma$, z large. The answer to Anderson's question is thus yes when $\varepsilon = 1$. In the remaining case, we have the following result which answers Anderson's question in the negative.

THEOREM 3. *Let $\alpha > 2$ be given. There exists a subharmonic function u such that $B(r, u) = O((\log r)^\alpha)$, $r \to \infty$, and such that for any path Γ going out to infinity, we have*

$$\liminf u(z)/B(|z|) = 0, \qquad z \to \infty, \qquad z \in \Gamma.$$

Remark. For convenience, we have restricted ourselves to the case $\psi(r) = (\log r)^\alpha$. The method used in constructing the example also works for a general growth rate ψ satisfying the conditions of Theorem 1.

In [6], Talpur proved that if u is subharmonic in the plane and of order zero, there exists a path going out to infinity on which $u(z) > (1-\varepsilon)B(|z|^{1-\varepsilon})$. Assuming that a little more is known about the growth of $B(r, u)$, we can prove the following result.

THEOREM 4. *Let ψ be as in Theorem 2 and assume furthermore that*

$$\limsup_{r\to\infty} \log\log \psi(r)/\log\log r < 1, \tag{7.1}$$

$$\log \psi(r^2) \leq \text{Const.} \log \psi(r), \qquad r > 1. \tag{7.2}$$

Let u be a subharmonic function in the plane of order zero which satisfies (1.3) *and let $\varepsilon \in (0, 1)$ be given. Then for all positive h, there exists a path Γ going out to infinity and a constant $C(h)$ such that*

$$u(z) > (1-\varepsilon)B(|z| \exp(-C(h) \log \psi(|z|))^{1+h})), \qquad z \to \infty, \qquad z \in \Gamma.$$

If in particular $\psi(r) = (\log r)^\alpha$, *where* $\alpha > 2$, *we have*

$$u(z) > (1-\varepsilon)B(|z| \exp(-C(h) \operatorname{loglog}|z|)^{1+h})), \qquad z \to \infty, \qquad z \in \Gamma.$$

Proof of Theorem 3. Let $\delta \in (0, \alpha - 2)$ be given and let $\{\lambda_n\}$ be an increasing sequence of positive numbers tending to 1 which is such that $\lambda_1 \geq 1/2$ and

$$n(1-\lambda_n)/\log n \to \infty, \qquad n \to \infty. \tag{7.3}$$

The sequences $\{A_n\}$ and $\{B_n\}$ are defined as in Section 6. Also here, we can take $A_n = \exp\{\exp \gamma^n\}$, where $\gamma > (\alpha-1)/(\alpha-2-\delta)$. We also introduce $K_n = (\log B_n)^{2/\lambda_n}$. Next, we define three nondecreasing, continuous functions $n_1(r)$, $n_2(r)$ and $n_3(r)$ via the relations

$$dn_1(r) = \begin{cases} (\log A_\nu)^{\alpha-1}\, d(r/A_\nu)^{\lambda_\nu}, & A_\nu/B_\nu < r < A_\nu/B_\nu^{1/3},\ \nu = 1, 2, \ldots. \\ 0, & r \notin \cup (A_\nu/B_\nu, A_\nu/B_\nu^{1/3}), \end{cases}$$

$$dn_2(r) = \begin{cases} (\log A_\nu)^{\alpha-1}\, d(r/A_\nu)^{\lambda_\nu}, & A_\nu/B_\nu^{2/3} < r < A_\nu,\ \nu = 1, 2, \ldots. \\ 0, & r \notin \cup (A_\nu/B_\nu^{2/3}, A_\nu), \end{cases}$$

$$dn_3(r) = \begin{cases} (\log A_\nu)^{\alpha-1}\, d(r/A_\nu)^{\lambda_\nu}, & A_\nu/B_\nu < r < A_\nu,\ \nu = 1, 2, \ldots. \\ 0, & r \notin \cup (A_\nu/B_\nu, A_\nu). \end{cases}$$

We consider the subharmonic functions

$$u_1(z) = \int_0^\infty \log|1 + z/t|\, dn_1(t),$$

$$u_2(z) = \int_0^\infty \log|1 - z/t|\, dn_2(t),$$

$$v(z) = \int_0^\infty \log|1 + z/t|\, dn_3(t).$$

It is easy to see that $n_k(r) = O((\log r)^{\alpha-1})$, $r \to \infty$, $k = 1, 2, 3$. Consequently, the maximum moduli of these three functions are all $O((\log r)^\alpha)$, $r \to \infty$.

Our example is the subharmonic function $U = \max(u_1, u_2)$. We clearly have the following property of the auxiliary function v:

$$u_k(iy) \leq v(iy), \qquad y \in R, \qquad k = 1, 2. \tag{7.4}$$

To find the properties of these subharmonic functions, we use the same method as in the proof of (6.12). The only difference is that in each interval $[A_\nu/B_\nu, A_\nu]$, we replace in all formulas the constant λ by λ_ν. If this change is made in the function u defined in (6.11), we obtain the following estimate of $C_\nu(r)$ which is the crucial quantity in the proof of (6.12):

$$\begin{aligned}C_\nu(r) &= (\log A_\nu)^{\alpha-1}(1+o(1))(B_\nu^{-\lambda_\nu}\log(1+B_\nu)+(r/A_\nu)(1-\lambda_\nu)^{-1})\\ &= o((r/A_\nu)^{\lambda_\nu}(\log A_\nu)^{\alpha-1})(1-\lambda_\nu)), \quad K_\nu A_\nu/B_\nu \le r \le A_\nu/K_\nu.\end{aligned}$$

In the last step, we used the fact that $(1-\lambda_\nu)\log K_\nu + 2\log(1-\lambda_\nu) \to \infty$ $\nu \to \infty$ which is a consequence of (7.3) and our special choice of A_ν. Arguing in this way, we obtain

$$\begin{aligned}u_1(z) &= (\log A_\nu)^{\alpha-1}(r/A_\nu)^{\lambda_\nu}\{\pi\cos\lambda_\nu\theta(\sin\pi\lambda_\nu)^{-1}+o(1-\lambda_\nu)\},\\ &\qquad\qquad K_\nu A_\nu/B_\nu \le r \le A_\nu/(K_\nu B_\nu^{1/3}),\\ u_2(z) &= (\log A_\nu)^{\alpha-1}(r/A_\nu)^{\lambda_\nu}\{\pi\cos\lambda_\nu(\pi-\theta)(\sin\pi\lambda_\nu)^{-1}+o(1-\lambda_\nu)\}\\ &\qquad\qquad K_\nu A_\nu/B_\nu^{2/3} \le r \le A_\nu/K_\nu,\\ U(iy) \le v(iy) &= (\log A_\nu)^{\alpha-1}(r/A_\nu)^{\lambda_\nu}(\pi\cos(\pi\lambda_\nu/2)(\sin\pi\lambda_\nu)^{-1}+o(1-\lambda_\nu)\},\\ &\qquad\qquad K_\nu A_\nu/B_\nu \le r \le A_\nu/K_\nu.\end{aligned}$$

In the last estimate, we used (7.4). The error estimates are uniform in z as $\nu \to \infty$.

Let us now consider u_2 when $K_\nu A_\nu/B_\nu \le r \le A_\nu/(K_\nu B_\nu^{2/3})$. From (6.10) and (7.3), we see that

$$\begin{aligned}u_2(z) &\le \left\{\int_0^{A_{\nu-1}} + \int_{A_\nu/B_\nu^{2/3}}^{A_\nu} + \int_{A_\nu}^{\infty}\right\}\log(1+r/t)\,dn_2(t)\\ &\le (\log A_\nu)^{\alpha-1-\delta} + (\log A_\nu)^{\alpha-1}(r/A_\nu)(1-\lambda_\nu)^{-1}B_\nu^{2(1-\lambda_\nu)/3}\\ &\quad + (r/A_{\nu+1})(\log A_{\nu+1})^{\alpha-1}B_\nu^{2(1-\lambda_{\nu+1})/3}(1-\lambda_{\nu+1})^{-1} = o((\log A_\nu)^{\alpha-1}(r/A_\nu)^{\lambda_\nu}(1-\lambda_\nu)).\end{aligned}$$

For all large ν, we thus have

$$\begin{cases} U(z) = u_1(z), & \operatorname{Re} z \ge 0,\\ U(z) \le U(i|z|), & \operatorname{Re} z < 0,\end{cases} \qquad K_\nu A_\nu/B_\nu \le r \le A_\nu/(K_\nu B_\nu^{2/3}).$$

The next step is to consider u_1 when $K_\nu A_\nu/B_\nu^{1/3} \le r \le A_\nu/K_\nu$. We have

$$u_1(z) \le \left\{ \int_0^{A_\nu/B_\nu^{1/3}} + \int_{A_{\nu+1}/B_{\nu+1}}^{\infty} \right\} \log(1+r/t)\, dn_1(t) = I_1 + I_2.$$

To estimate I_1, we write

$$I_1 = \sum_{k=1}^{\nu} \int_{A_k/B_k}^{A_k/B_k^{1/3}} \log(1+r/t)\, dn_1(t) = \sum_{k=1}^{\nu} J_k.$$

Since $n_1(A_k/B_k^{1/3}) - n_1(A_k/B_k) = (\log A_k)^{\alpha-1} B_k^{-\lambda_k/3}$, an integration by parts shows that for $k = 1, 2, \dots,$ we have $J_k \le (\log A_k)^{\alpha-1} B_k^{-\lambda_k/3}(\log(1 + rB_k^{1/3}/A_k) + (2/3)\log B_k)$. Since $r \le A_\nu/K_\nu$, we obtain

$$I_1 \le 3(\log B_\nu)(\log A_\nu)^{\alpha-1} B_\nu^{-\lambda_\nu/3}.$$

In the second term I_2, we use the crude estimate $n_1(r) \le (\log r)^{\alpha-1}$. If $r \le A_\nu/K_\nu$, we have

$$I_2 \le \int_{A_{\nu+1}/B_{\nu+1}}^{\infty} r/t\, dn_1(t) \le (rB_{\nu+1}A_{\nu+1}) n_1(A_\nu) + r\int_{A_{\nu+1}/B_{\nu+1}}^{\infty} (\log t)^{\alpha-1}\, dt/t^2$$
$$\le 2(A_\nu B_{\nu+1}/A_{\nu+1})(\log A_{\nu+1})^{\alpha-1}.$$

Since $r \ge K_\nu A_\nu/B_\nu^{1/3}$, we finally obtain the estimate

$$u_1(z) \le I_1 + I_2 = o((\log A_\nu)^{\alpha-1})(r/A_\nu)^{\lambda_\nu}(1-\lambda_\nu)), \qquad \nu \to \infty,$$

where $K_\nu A_\nu/B_\nu^{1/3} \le |z| \le A_\nu/K_\nu$.

It follows that for all large ν, we have also

$$\begin{cases} U(z) \le U(i|z|), & \operatorname{Re} z \ge 0, \\ U(z) = u_2(z), & \operatorname{Re} z < 0, \end{cases} \qquad K_\nu A_\nu/B_\nu^{1/3} \le r \le A_\nu/K_\nu.$$

Thus, if $|z| = r$, we have

$$U(z) \le (1 + o(1)) \cos(\pi\lambda_\nu/2) B(r, U), \qquad z \in \Gamma_\nu,$$

where Γ_ν is a closed curve which consists of the following arcs:

$$\begin{cases} |z| = K_\nu A_\nu / B_\nu, & |\arg z - \pi| \leq \pi/2, \\ |z| = A_\nu / K_\nu, & |\arg z| \leq \pi/2, \\ K_\nu A_\nu / B_\nu \leq |\mathrm{Im}\, z| \leq A_\nu / K_\nu, & \mathrm{Re}\, z = 0. \end{cases}$$

Clearly, each curve Γ_ν separates the origin from infinity. Since $\cos(\pi\lambda_\nu/2) \to 0$, $\nu \to \infty$, we have our example and Theorem 3 is proved.

Proof of Theorem 4. In the discussion of (4.11) in [3], we use Lemma 2 in [3] which says that there exist $K > 1$ and r_0 such that if $r_\nu = r_0 K^\nu$, we have

$$u(z) > (1 - o(1)) B(r_\nu), \qquad |z| = r_\nu, \qquad \nu \to \infty. \tag{7.5}$$

This result is based on a covering result of Hayman (cf. (1.4) and (1.5) in [3]) which is true only if $B(r) = O(\log r)^2$, $r \to \infty$. If we only assume that $B(r) = O((\log r)^\alpha)$, $r \to \infty$, where $\alpha > 2$, we can use the Corollary of Theorem 2 to deduce a similar, but weaker result which is the starting point of this proof.

We claim that there exists a positive number h and an increasing sequence $\{r_\nu\}$ tending to infinity such that (7.5) holds and which is such that

$$\log(r_{\nu+1}/r_\nu) = O((\log \psi(r_\nu))^{1+h}), \quad \to \infty$$

To see this, we first note that if the interval $[r, R]$ is contained in F (where F is defined as in Theorem 2), it follows from (2.11) that if $h > 0$, there is a constant C such that

$$\int_r^R (\log \psi(t))^{-1-h}\, dt/t < C. \tag{7.6}$$

It follows from (7.6) that $\log R < 2 \log r$. In fact, if $\log R > 2 \log r$, we deduce from (7.6) and (7.2) that

$$\log r < \text{Const.}\, (\log \psi(r))^{1+h}.$$

Taking logarithms dividing by loglog r and letting $r \to \infty$, we obtain

$$1 < (1 + h) \limsup_{r \to \infty} \operatorname{loglog} \psi(r) / \operatorname{loglog} r.$$

It follows from (7.1) that there exists $h>0$ such that this is a contradiction. From now on, we work with such an h; then we know that $R<r^2$, and our claim follows from (7.6) and (7.2).

Consider now the set $G_\nu=\{z:u(z)>(1-\varepsilon)B(r_\nu),\ r_\nu\le|z|\le r_{\nu+1}\}$. The work of Talpur (cf. [6, Lemma 1]) shows that there exists a path going from $\{|z|=r_\nu\}$ to $\{|z|=r_{\nu+1}\}$ which is contained in G_ν. The curve Γ mentioned in Theorem 4 is the union of these paths and circular arcs with radii $\{r_\nu\}$ which are chosen in such a way that Γ will be connected. If $z\in G_\nu\cap\Gamma$, we have

$$u(z)>(1-\varepsilon)B(r_\nu)\ge(1-\varepsilon)B(r_{\nu+1}\exp(-C(h)(\log\psi(r_\nu))^{1+h}))$$
$$\ge(1-\varepsilon)B(|z|\exp(-C(h)(\log\psi(|z|))^{1+h}).$$

This completes the proof of Theorem 4.

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