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The extremum problem for analytic functions with finite area integral

by J. M. ANDERSON

§1. Introduction

Let $D=\{z:|z|<1\}$ denote the open unit disc and let I denote the Banach space of functions $f(z)$, analytic in $|z|<1$ for which the norm

$$\|f\|=\frac{1}{\pi}\iint_D |f(z)|\,dx\,dy$$

is finite. If $\kappa(z)$ is a function in $L^\infty(D)$, the space of complex-valued bounded measurable functions in D with

$$\|\kappa\|_\infty=\text{ess sup}\,|\kappa(z)|,\qquad z\in D,$$

then we may associate with κ a linear functional L_κ on I defined by

$$L_\kappa(f)=\frac{1}{\pi}\iint_D \kappa(z)f(z)\,dx\,dy.$$

Clearly

$$\|L_\kappa\|\leq\|\kappa\|_\infty \tag{1}$$

and some importance attaches to the question of when equality holds in (1). This is shown by the following theorem:

THEOREM A. *A function* $\kappa(z)\in L^\infty(D)$ *with* $\|\kappa\|_\infty<1$ *is the complex dilatation of an extremal quasi-conformal mapping of* D *onto itself if and only if equality holds in* (1.1).

The necessity part of this theorem was established in [3] and the sufficiency in [10]. If ϕ is a quasi-conformal mapping of D onto itself, then ϕ induces a homeomorphism of the boundary ∂D onto itself. A quasi-conformal mapping θ of D onto itself is called *extremal* if it has the smallest maximal dilatation in the class $Q(\phi)$ of all quasi-conformal mappings of D onto D which coincide with ϕ on ∂D i.e.,

$$K(\theta) = \min K(\psi), \qquad \psi \in Q(\phi),$$

where $K(\theta)$ denotes the maximal dilatation of θ. An argument involving normal families (see e.g. [6] p. 75) shows that the minimum is always attained, though the extremal mapping θ need not be unique.

It is frequently difficult to determine, for a given function $\kappa(z)$ whether or not equality holds in (1.1). Although the above considerations permit of a geometric approach to this problem, there have also been several attempts [4], [5], [7], [9] by analytic methods to gain some insight into the problem, and the present note is also in this spirit. We define

$$\mathfrak{K} = \{\kappa : \kappa \in L^\infty(D), \qquad \|L_\kappa\| = \|\kappa\|_\infty\},$$

$$\mathfrak{K}(T) = \left\{\kappa : \kappa = \alpha \frac{\bar{f}(z)}{|f(z)|}, \qquad f \in I, \qquad \alpha \in \mathbf{C}\right\}.$$

Obviously $\mathfrak{K}(T) c \mathfrak{K}$. The extremal quasi-conformal mappings associated with dilatations in $\mathfrak{K}(T)$ are called Teichmüller extremals. It is only for functions κ in $\mathfrak{K}(T)$ that there exists an f_0 in I with $L_\kappa(f_0) = \|L_\kappa\|$ (see [9] Lemma 0.3). However it was first shown by Strebel [11], see also [9] example 0.1, that there are functions $\kappa \in \mathfrak{K}$ which do not belong to $\mathfrak{K}(T)$.

Given $\kappa \in L^\infty(D)$ we shall say that a sequence $\{\phi_n\}$ in I is an *extremal sequence* for κ if $\|\phi_n\| = 1$, $n = 1, 2, 3, \ldots$ and

$$L_\kappa(\phi_n) \to \|L_\kappa\| \qquad (n \to \infty).$$

The condition that $\|\phi_n\| = 1$, $n = 1, 2, 3, \ldots$ shows that the family of functions $\{\phi_n(z)\}$ is uniformly bounded on compact subsets of D and so, by the usual argument involving normal families we may assume, by passing to a subsequence if necessary, that

$$\phi_n(z) \to \phi_0(z) \qquad (n \to \infty),$$

locally uniformly in D. If the limit function $\phi_0(z)$ vanishes identically the extremal sequence $\{\phi_n\}$ is said to degenerate. This terminology is due to Strebel. We require the following theorem of Reich ([9]), p. 433; see also [4] Proposition 1.2).

THEOREM B. *Suppose that* $\kappa \in L^\infty(D)$. *Then either every extremal sequence* $\{\phi_n(z)\}$ for κ degenerates, or

$$L_\kappa(f) = \|L_\kappa\| \iint_D \frac{\bar{\phi}_0(z)}{|\phi_0(z)|} f(z)\,dx\,dy$$

for all $f \in I$, *where* $\phi_0(z) = \lim_{n\to\infty} \phi_n(z)$. *In particular, if* $\kappa \in \mathfrak{K}$ *then every extremal sequence for* κ *degenerates or* $\kappa \in \mathfrak{K}(T)$ *(or possibly both)*

§2. An analytic approach

Leaving the applications aside for the moment, the question of when equality holds in (1) is an interesting extremal problem for analytic functions. One of the difficulties is that in attempting to identify the dual space of I as a quotient space the annihilator subspace

$$I^\perp = \{\kappa : L_\kappa(f) = 0 \quad \text{for all} \quad f \in I\}$$

plays a role. Yet it is known that I^*, the dual space of I, can be identified, unfortunately not isometrically, with another space of analytic functions, and this quite often yields some information. We now describe this in detail, though it is similar to [1] p. 17.

A function $f(z)$, analytic in D and with $f(0) = 0$ is called a *Bloch function* if the norm

$$\|f\|_B = \sup (1 - |z|^2)\,|f'(z)|, \qquad z \in D,$$

is finite. The set of all such functions, denoted by B, is a Banach space with the norm indicated. For information on B see [1] or [2].

We now fix $\kappa(z) \in L^\infty(D)$ and, for $n \geqslant 0$ we set

$$b_n = (n+2) L_\kappa(z^n),$$

i.e., we take moments of the function $\kappa(z)$. Note that b_n depends on κ, but for notational convenience we suppress this. We set

$$g(\zeta)=\sum_{n=0}^{\infty} b_n\zeta^n$$

and note that $g(\zeta)$ is analytic for $|\zeta|<1$, in fact $b_n=0(n)$, $(n\to\infty)$. Suppose now that $f(z)=\sum_{n=0}^{\infty} a_n z^n$ belongs to I and fix ρ, $0<\rho<1$. Then

$$L_\kappa(f(\rho z))=\sum_{n=0}^{\infty} a_n\rho^n L_\kappa(z^n)=\sum_{n=0}^{\infty}\frac{a_n b_n}{n+2}\rho^n.$$

Since $\|f(\rho z)-f(z)\|_I\to 0$ as $\rho\to 1-$ we obtain that, for each $f\in I$,

$$L_\kappa(f)=\lim_{\rho\to 1-}\sum_{n=0}^{\infty}\frac{a_n b_n}{n+2}\rho^n.$$

But also

$$\frac{d}{d\zeta}(\zeta g(\zeta))=g(\zeta)+\zeta g'(\zeta)=\sum_{n=0}^{\infty}(n+1)b_n\zeta^n$$

$$=\sum_{n=0}^{\infty}\frac{b_n}{n+2}(n+1)(n+2)\zeta^n,$$

and so, for fixed ζ, $|\zeta|<1$,

$$\frac{d}{d\zeta}(\zeta g(\zeta))=L_\kappa\left(\sum_{n=0}^{\infty}(n+1)(n+2)\zeta^n z^n\right)$$

$$=L_\kappa\left(\frac{2}{(1-\zeta z)^3}\right).$$

We now let $G(\zeta)=\zeta g(\zeta)$ and note that $G(0)=0$. Then

$$|G'(\zeta)|\le 2\|L_\kappa\|\cdot\|(1-\zeta z)^{-3}\|_I$$

$$=\frac{2}{\pi}\|L_\kappa\|\int_0^1\int_0^{2\pi}\frac{r\,dr\,d\theta}{|1-\zeta re^{i\theta}|^3}$$

$$\le 4\|L_\kappa\|\int_0^1\frac{r}{(1-|\zeta|r)}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-\zeta re^{i\theta}|^2}\right)dr$$

$$=4\|L_\kappa\|\int_0^1\frac{r\,dr}{(1-|\zeta|r)(1-|\zeta|^2r^2)},$$

since the inner integral above is just the Poisson kernel. Now, for $0<r<1$, $0<|\zeta|<1$, we have $r/1+|\zeta|r<1/1+|\zeta|$ and so

$$|G'(\zeta)|\leq\frac{4\|L_\kappa\|}{1+|\zeta|}\int_0^1\frac{dr}{(1-|\zeta|r)^2}=\frac{4\|L_\kappa\|}{1-|\zeta|^2}.$$

Hence $G(\zeta)$ is a Bloch function with Bloch norm at most $4\|L_\kappa\|$.

More accurately, we have

$$\int_0^1\int_0^{2\pi}\frac{r\,dr\,d\theta}{|1-\zeta re^{i\theta}|^3}=\pi\sum_{n=0}^{\infty}\frac{|\alpha_n|^2|\zeta|^{2n}}{n+1},\tag{2}$$

by Parseval's formula, where the $\alpha_n=(2n+1)!/(4^n(n!)^2)$ are the Taylor coefficients of the function $\phi(x)=(1-x)^{-3/2}$. An application of Stirling's formula to (2) now yields

$$\lim_{|\zeta|\to 1-}(1-|\zeta|^2)\int_0^1\int_0^{2\pi}\frac{r\,dr\,d\theta}{|1-\zeta re^{i\theta}|^3}=4$$

so that

$$\limsup_{|\zeta|\to 1-}(1-|\zeta|^2)\,|G'(\zeta)|\leq\frac{8}{\pi}\|L_\kappa\|.$$

The inequality in the other direction is more relevant for our purposes. If $|\zeta|<1$ we obtain, from Parseval's formula,

$$\frac{1}{2\pi}\int_0^{2\pi}f(re^{i\theta})G'(\zeta re^{-i\theta})\,d\theta$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n=0}^{\infty}a_nr^ne^{in\theta}\right)\left(\sum_{n=0}^{\infty}(n+1)b_n\zeta^nr^ne^{-in\theta}\right)d\theta=\sum_{n=0}^{\infty}a_nb_n\zeta^n(n+1)r^{2n}.$$

Thus we obtain

$$\frac{1}{2\pi}\int_0^1\int_0^{2\pi}f(re^{i\theta})G'(\zeta re^{-i\theta})(1-r^2)r\,dr\,d\theta$$
$$=\sum_{n=0}^{\infty}(n+1)\,\zeta^na_nb_n\int_0^1r^{2n+1}(1-r^2)\,dr=\frac{1}{2}\sum_{n=0}^{\infty}\frac{a_nb_n}{n+2}\zeta^n.\tag{3}$$

Since this latter expression is just $\frac{1}{2}L_\kappa(f)$ as $\zeta \to 1$, it follows that

$$|L_\kappa(f)| \le \left\{ \sup_{|z|<1} (1-|z|^2)\,|G'(z)| \right\} \cdot \|f\|_I,$$

and so $\|L_\kappa\| \le \|G\|_B$

Summing up, we have proved

THEOREM 1. *For a given* $\kappa(z) \in L^\infty(D)$ *we have*

$$\|L_\kappa\| \le \|G\|_B \le 4\|L_\kappa\|,$$

where

$$G'(\zeta) = \frac{2}{\pi} \iint_D \frac{\kappa(z)\,dx\,dy}{(1-\zeta z)^3}.$$

Theorem 1 asserts that the Banach space B and the dual space I^* are *isomorphic* as Banach spaces, i.e., that norms are preserved up to certain multiplicative constants which are bounded away from zero and infinity. In order to apply Theorem A directly, however, we require to identify I^* *isometrically* as a Banach space, i.e., without change of norm. I do not know of any space of analytic functions which achieves this.

§3. Further estimates

A closer examination of (3) shows that somewhat more precise estimates can be obtained. If $\kappa(z)$ is such that some extremal sequence $\{\phi_n\}$ for κ degenerates then we obtain that there is a sequence $\{r_n\}$, $0 < r_n < 1$, $r_n \to 1-$ as $n \to \infty$ with the following property: given $\varepsilon > 0$ there is an n_0 such that, for all $n > n_0$,

$$|L_\kappa(\phi_n)| \le \varepsilon \|G\|_B + \left[\sup_{r_n \le |z| < 1} (1-|z|^2)\,|G'(z)| \right] \|\phi_n\|_I.$$

THEOREM 2. *If* $\kappa \in \mathfrak{K}$ *admits a degenerate extremal sequence then*

$$\limsup_{|z| \to 1} (1-|z|^2)\,|G'(z)| \ge \|L_\kappa\|,$$

where $G(z)$ is defined in §2. *In particular, if*

$$\iint_D \frac{\kappa(z)\,dx\,dy}{(1-\zeta z)^3} = o(1-|\zeta|^2)^{-1} \qquad (|\zeta|\to 1-), \tag{4}$$

then κ cannot belong to $\mathfrak{K}$ unless it belongs to $\mathfrak{K}(T)$.

Condition (4) is to be thought of as a *smoothness* condition on $\kappa(z)$. For any $\kappa \in L^\infty(D)$ we always have that the above integral is $0(1-|\zeta|^2)$ as $|\zeta|\to 1-$. Various other smoothness conditions on $\kappa(z)$, yielding the same conclusion have been discussed in [7] and [4], but they are different from Theorem 2.

However, a more far-reaching deduction can be made from (3) by "completing the kernel," the method one usually adopts in dealing with problems involving annihilator subspaces. We obtain from (3) that

$$\sum_{n=0}^{\infty} \frac{a_n b_n}{n+2}\zeta^n = \frac{1}{\pi}\int_0^1\int_0^{2\pi} f(re^{i\theta})[G'(\zeta re^{-i\theta}) + H(\zeta re^{i\theta})](1-r^2)r\,dr\,d\theta, \tag{5}$$

where $H(z)$ is any function analytic in $|z|<1$ with $H(0)=0$. This may substantially reduce the Bloch norm, as we shall see below and so give a better upper bound for $\|L_\kappa\|$. In particular, if $G'(0)=0$ we may take $H(z)=G'(z)$ or $H(z)=iG'(z)$ to obtain a variant of Theorem 2 involving only the real or imaginary part of $G'(z)$.

§4. The Reich example

The following example has been suggested by Reich as a prototype:

$$\begin{aligned} \alpha(z) &= +1, \qquad \operatorname{Im} z \geqslant 0, \qquad |z|<1, \\ &= -1, \qquad \operatorname{Im} z < 0, \qquad |z|<1. \end{aligned}$$

There now exist several proofs, both of a geometric and analytic nature, [8], [9], [7], of the fact that $\|L_\alpha\| < 1 = \|\alpha\|_\infty$. It is instructive for us to consider this example also. In this case.

$$\begin{aligned} b_n &= \frac{4i}{n\pi}, \quad n \text{ odd} \\ &= 0, \quad n \text{ even,} \end{aligned}$$

so that

$$g(\zeta)=\frac{4i}{\pi}\sum_{m=0}^{\infty}\frac{\zeta^{2m+1}}{2m+1}=\frac{2i}{\pi}\log\left(\frac{1+\zeta}{1-\zeta}\right).$$

Hence

$$G'(\zeta)=\frac{2i}{\pi}\left[\log\left(\frac{1+\zeta}{1-\zeta}\right)+\frac{2\zeta}{1-\zeta^2}\right]$$

and thus

$$\lim_{|\zeta|\to 1-}(1-|\zeta|^2)\,|G'(\zeta)|=\frac{4}{\pi}.$$

(Note that, in fact, $|G\|_B>(4/\pi)$). This does not yield a contradiction to Theorem 2, but a suitable use of (5) will yield some information.

Since $L_\alpha(z)=4i/3\pi$ and $\|z\|_I=2/3$ we see that

$$\|L_\alpha\|\geq\frac{|L_\alpha(z)|}{\|z\|}=\frac{2}{\pi}.$$

In fact it has been shown that $\|L_\alpha\|>0.779$ ([5] *p.* 167).

The estimates required for the upper bound are somewhat more delicate. We suppose, first of all, that an extremal sequence, $\{\phi_n\}$ for α degenerates. In (5) we take $H(z)=-uG'(z)$ where $0\leq u\leq 1$. We then see that

$$|L_\alpha(f)|\leq\|f\|_I\cdot\lim_{r\to 1-}(1-r^2)\sup_{0\leq\theta\leq 2\pi}|G'(re^{-i\theta})-uG'(re^{i\theta})|.$$

In this particular case the term involving $\log(1+re^{i\theta}/1-re^{i\theta})$ yields zero on passing to the limit and we obtain

$$|L_\alpha(f)|\leq\frac{4}{\pi}\|f\|_I S(u)$$

where

$$S(u)=\limsup_{r\to 1-}\left\{(1-r^2)\left|\frac{re^{i\theta}}{1-r^2e^{2i\theta}}-u\frac{re^{-i\theta}}{1-r^2e^{-2i\theta}}\right|\right\}$$

$$=\lim_{r\to 1-}(1-r^2)\max_{\theta}\left\{\frac{[(1-u)^2(1-r^2)^2+4(\sin^2\theta)(1+ur^2)(u+r^2)]^{1/2}}{(1-r^2)+4r^2(\sin^2\theta)}\right\}.$$

Some elementary but tiresome calculations yield the following values for $S(u)$

$$\begin{aligned} S(u) &= 1-u, & 0 &\le u \le \frac{\sqrt{2}-1}{\sqrt{2}+1}, \\ &= \frac{(1+u)^2}{4u^{1/2}}, & \frac{\sqrt{2}-1}{\sqrt{2}+1} &\le u \le 1. \end{aligned}$$

Hence, with the assumption that L_α has an extremal sequence $\{\phi_n\}$ which degenerates, we obtain

$$\|L_\alpha\| \le \frac{4}{\pi} \min_{0\le u\le 1} S(u).$$

It is interesting, and somewhat unexpected, to note that this minimum does not occur for $u=0$ or $u=1$, where $S(u)$ takes the value 1, but for $u=\frac{1}{3}$. We obtain

$$\frac{4}{\pi} \min_{0\le u\le 1} S(u) = \frac{16}{3\pi\sqrt{3}} = 0.98014\ldots < 1.$$

This is not as good a result as Theroem 4.1 of [5] where it is shown, under the assumption that L_α has a degenerate extremal sequence, that $\|L_\alpha\| \le 2/\pi$. Summing up, therefore, we obtain for the Reich example that either $\|L_\alpha\|$ is attained, in which case $\|L_\alpha\| < 1$, or $\|L_\alpha\|$ is approached through a degenerate extremal sequence, in which case $\|L_\alpha\| \le 16/3\pi\sqrt{3} < 1$. It seems quite likely that this latter estimate is an upper bound for $\|L_\alpha\|$, but I am unable to verify this. The difficulty arises in the presence of the logarithmic term in the expression for $G'(\zeta)$.

§5. Concluding remarks

For applications it is the inequality $\|L_\kappa\| \le \|G\|_B$ of Theorem 1 that is important. For that reason it is important in (5) to choose the function $H(z)$ so as to minimize the Bloch norm of $G(z)+H(z)$. Unfortunately it is a feature of the method of "completing the kernel" that no indication is given of how to choose the function $H(z)$ in (5) in the general case. Although in the Reich example it was reasonable to choose $H(z) = -uG'(z)$ for some $u>0$ since $G'(z)$ was real and large for z real, there was no reason to expect that $u=\frac{1}{3}$ was the correct value to choose. Even after the fact it is difficult to know what importance to attach to this.

The question of which other spaces are isomorphic to the dual space I^* is considered in [2]. We mention only, for $n \geq 1$, the spaces B_n of functions

$$g(z) = \sum_{k=n}^{\infty} a_k z^k,$$

analytic for $|z|<1$ and such that the norm

$$\|g\|_n = \sup_{|z|<1} (1-|z|^2)^n \, |g^n(z)|$$

is finite. With suitable adjustments for the values of the functions at the origin all of these spaces are isomorphic, but again not isometric, to I^*.

It seems reasonable, in conclusion, to enquire whether smoothness conditions like (4) can have a geometrical meaning in this problem. The Bloch condition (4) gives us information about the image of $|z|<1$ under the mapping $w = G(z)$, but this is already several stages removed from information about $\kappa(x, y)$.

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