Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 55 (1980)

Artikel: Lower central series, augmentation quotients and homology of groups.
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DOI: https://doi.org/10.5169/seals-42368

Comment. Math. Helvetici **55** (1980) 159–177 Birkhäuser Verlag, Basel

Lower central series, augmentation quotients and homology of groups

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1. Introduction

For a group G and a subring R of the rationals $\mathbf{Q}$ the following three graded structures are somehow related refinements of the "R-abelianization" $H_1(G, R) = R \otimes G_{ab}$ and they partly reflect the structure of G or its R-localization G^R:
—The homology $H_*(G, R)$ of G with coefficients in R.
—The graded Lie algebra $L^R G = \bigoplus_{n \geq 1} R \otimes G_n/G_{n+1}$, where

$$\cdots \leq G_{n+1} \leq G_n \leq \cdots \leq G_2 \leq G_1 = G$$

is the lower central series of G.

—The graded algebra $\operatorname{gr} RG = \bigoplus_{n \geq 0} JG^n/JG^{n+1}$, where JG is the augmentation ideal of the group algebra RG.

It is the purpose of this paper to study some of the relationships between these structures. There are of course natural surjections

$$\alpha : \mathfrak{L}^R H_1(G, R) \to L^R G, \qquad \beta : T^R H_1(G, R) \to \operatorname{gr} RG, \qquad \gamma : UL^R G \to \operatorname{gr} RG,$$

where $\mathfrak{L}^R$ is the free Lie algebra functor, T^R the tensoralgebra functor and U the universal envelope functor. By results of Witt and Magnus all these maps are isomorphisms if G is free. The map γ has been studied in several papers, so in [9] for $R = \mathbf{Q}$ and in [1] for $R = \mathbf{Z}$. In general neither of the three maps in bijective. In order to be able to use more homological information, the maps α and β will be replaced by two spectral sequences $E(R, G)$, $\tilde{E}(R, G)$ and a natural map $\kappa : E(R, G) \to \tilde{E}(R, G)$, which converge, in a sense to be made precise, to the Lie algebra $L^R G$, the graded algebra $\operatorname{gr} RG$ and the natural map $\kappa : L^R G \to \operatorname{gr} RG$, respectively. Their initial terms are homology invariants and the relevant differentials can in principle be computed in terms of certain Fox derivatives. For a free group G the spectral sequences collapse and reduce to the isomorphisms of Witt and Magnus. $E(R, G)$ is essentially the lower central series spectral sequence of a free simplicial resolution X of G, $\tilde{E}(R, G)$ is obtained by filtering the simplicial algebra RX by the powers of the augmentation ideal and κ is induced by the map $\kappa: X \to RX$, $\kappa(x) = x - 1$. The indices are chosen such that the differentials d^r are of degree $(r, -1)$ and $E^r_{n,m} = 0 = \tilde{E}^r_{n,m}$ unless $n > 0$ and $m > 0$ or $n \geq 0$ and $m = 0$. Our approach is influenced by Stallings' work [12], where the Cobar construction is used to relate $H_*(G)$ and gr ZG.

In section 2 we recall and slightly generalize some results by A. Dold [3] on the homotopy and homology of simplicial modules. These will be used in section 3 to prove the following theorem concerning the spectral sequences $E(R, G)$ and $\tilde{E}(R, G)$.

THEOREM A. *For every group G and every principal ideal domain R* (char $R = 0$) *there are spectral sequences $E(R, G)$, $\tilde{E}(R, G)$ and a natural map $\kappa: E(R, G) \to \tilde{E}(R, G)$ with the following properties:*

(i) *$E^1_{*,m}(R, G)$ and $\tilde{E}^1_{*,m}(R, G)$ are homology invariants of G depending on $H_{i+1}(G, R)$ for $i \leq m$ only. In particular, $E^1_{*,0} = \mathfrak{L}H_1(G, R)$, $\tilde{E}^1_{*,0} = TH_1(G, R)$ and $E^1_{1,m} = H_{m+1}(G, R) = \tilde{E}^1_{1,m}$.*
(ii) *$\kappa^1_{*,0}: E^1_{*,0} \to \tilde{E}^1_{*,o}$ is injective.*
(iii) *$E^\infty_{n,0} = E^n_{n,0} = L^R_n G$ and $\tilde{E}^\infty_{n,0} = \tilde{E}^n_{n,0} = \mathrm{gr}_n\, RG$.*
(iv) *Every sequence of homomorphisms $\{h_i : H_{i+1}(G, R) \to H_{i+1}(K, R) \mid i \leq m\}$ induces homomorphisms $h_{*,i}: E^1_{*,i}(R, G) \to E^1_{*,i}(R, K)$ and $\tilde{h}_{*,i}: \tilde{E}^1_{*,i}(R, G) \to \tilde{E}^1_{*,i}(R, K)$ for $i \leq m$. If h_i is bijective for $i < m$ and surjective for $i = m$ then the same is true for the induced sequences.*
(v) *If n is invertible in R then $\kappa^1_{n,*}: E^1_{n,*}(R, G) \to \tilde{E}^1_{n,*}(R, G)$ is a split monomorphism. In particular, $\kappa^1: E^1(\mathbf{Q}, G) \to \tilde{E}(\mathbf{Q}, G)$ is a monomorphism, $E^1(\mathbf{Q}, G) \cong \mathfrak{L}H^{\mathbf{G}}(G, \mathbf{Q})$ and $\tilde{E}^1(\mathbf{Q}, G) \cong TH^{\mathbf{G}}(G, \mathbf{Q})$ where $H^{\mathbf{G}}_*(G, \mathbf{Q}) = H_{*+1}(G, \mathbf{Q})$ is the (free group) cotriple homology of G.*

These spectral sequences can usefully be applied to get information about the Lie algebra $L^R G$ and the graded ring gr RG or to transform homological information about G into group theoretic information. Applications of this sort are the subject of section 4. Relationships between the homological behaviour of group homomorphisms and its group theoretic properties in the sense of Stallings and Stammbach are discussed. Some homological conditions on G under which the maps α and β are isomorphism fall out naturally in this framework. We also generalize some results of J. B. S. Passi concerning the natural map $\chi: \mathrm{gr}\, RG \otimes_R \mathrm{gr}\, RK \to \mathrm{gr}\, R(G \times K)$.

This paper was completed at the Forschungsinstitut für Mathematik at the ETH in Zürich. I am very grateful for the hospitality of this institute and to its director B. Eckmann for his support.

2. Preliminaries on simplicial modules and algebras

2.1. To fix notation let us recall the definition of a simplicial object. A simplicial object $X.$ in a category $\mathfrak{C}$ is a family $\{X_q \mid q \geq 0\}$ of objects of $\mathfrak{C}$ together with two families of morphisms of $\mathfrak{C}$ for each $q \geq 0$, the faces $\{\varepsilon_i : X_{q+1} \to X_q \mid 0 \leq i \leq q+1\}$ and the degeneracies $\{\sigma_i : X_q \to X_{q+1} \mid 0 \leq i \leq q\}$, which satisfy the identities

$$\begin{matrix} \varepsilon_i \varepsilon_{j+1} = \varepsilon_j \varepsilon_i & (i \leq j) \\ \sigma_{j+1} \sigma_i = \sigma_i \sigma_j & (i \leq j) \end{matrix}, \qquad \varepsilon_i \sigma_j = \begin{cases} \sigma_{j-1} \varepsilon_i & (i < j) \\ 1 & (i = j, j+1) \\ \sigma_j \varepsilon_{i-1} & (i > j+1) \end{cases}$$

A simplicial map $f. : X. \to Y.$ is a family $\{f_q : X_q \to Y_q \mid q \geq 0\}$ of morphisms of $\mathfrak{C}$ compatible with the faces and degeneracies. For the notion of homotopy of simplicial maps see for instance [7].

2.2. We shall make use of the equivalence between the category sm of simplicial modules over a commutative ring R and the category cm of chain modules [3], [7]. The equivalence is realized by the normal chain complex functor

$$N : sm \to cm,$$

where $N_q(X) = \bigcap_{i<q} \ker(\varepsilon_i : X_q \to X_{q-1})$ and where the differential is given by the last face ε_q. The reciprocal equivalence

$$K : cm \to sm$$

is given by $K_q(C) = cm(N(q), C)$, where $N(q) = NX(q)$ and $X(q)$ is the free simplicial module of the standard q-simplex Δ_q. We shall not need the explicit description of K but only that N is an equivalence and that N and K "preserve" homotopy. $N(X)$ is naturally isomorphic to the normalized chain module X_N of X, which in turn is homotopy equivalent to the chain module $X = (X, \partial_q = \sum_{i=0}^{q} (-1)^i \varepsilon_i)$.

$$X_N = X/DX$$

where $D_q X = \sum_{i=0}^{q-1} \sigma_i(X_{q-1})$ is the submodule of X_q generated by the degenerate elements of X_q. By a theorem of J. C. Moore [3], [7], the homotopy $\pi_*(X)$ is the same as the homology $H_*(NX)$. Thus, there are natural isomorphisms

$$\pi(X) = H(NX) \cong H_*(X) \cong H(X_N).$$

We shall usually identify all these groups and just write $H(X)$, if there is no danger of confusion.

2.3. Define the q-skeleton $C^{(q+1)}$ of a chain module C by

$$C_n^{(q+1)} = \begin{cases} C_n & (n \leq q) \\ C_n/Z_n & (n = q+1) \\ 0 & (n > q+1) \end{cases}$$

with the obvious induced differentials Z_n is the submodule of n-cycles. The projection $\varphi_q : C \twoheadrightarrow C^{(q+1)}$ induces isomorphisms

$$H_n(\varphi_q) : H_n(C) \xrightarrow{\sim} H_n(C^{(q+1)})$$

for $n \leq q$, whereas $H_n(C^{(q+1)}) = 0$ for $n > q$. If $C = N(X)$ for some simplicial R-module X then

$$X^{(q+1)} = K(C^{(q+1)})$$

is the q-skeleton of X and

$$H_n(X^{(q+1)}) = \begin{cases} H_n(X) & (n \leq q) \\ 0 & (n > q) \end{cases}$$

2.4. PROPOSITION [3]: *Let X and X′ be simplicial modules over a heriditary ring R.*

(i) *If X is projective, then any sequence of homomorphisms $\{h_q : H_q(X) \to H_q(X') \mid q \geq 0\}$ is induced by a simplicial map $f : X \to X'$,* i.e. $h_q = H_q(f)$.
(ii) *If X and X' are both projective then they are homotopy equivelent if and only if $H(X) \cong H(X')$.*

Proof. (i) It suffices to find a chain map $g : NX \to NX'$ such that $H_q(g) = h_q$ for all $q \geq 0$. This is done in the standard way. Since R is heriditary and X is projective NX is projective. Furthermore, $N_qX = Z_q \oplus B_q$, where Z_q is the submodule of q-cycles and $B_q \cong N_q/Z_q$ is a complementary submodule. Z_q and B_q are projective for every $q \geqslant 0$. It is therefore possible to find maps g'_q and g''_{g+1} that make the diagram with exact rows

$$\begin{array}{ccccc} B_{q+1} & \rightarrowtail & Z_q & \twoheadrightarrow & H_q(X) \\ \vdots\, g''_{q+1} & & \vdots\, g'_q & & \downarrow h_q \\ N_{q+1}(X') & \rightarrowtail & Z'_q & \twoheadrightarrow & H_q(X') \end{array}$$

commutative. These maps then give the required chain map g.
(ii) Since the functors N and K preserve homotopy X and X' are homotopy equivalent if and only if NX and NX' are homotopy equivalent, in which case $H(X) \cong H(X')$. Conversely, if X is projective and $H(X) \cong H(X')$ then by (i) there is a simplicial map $f: X \to X'$ such that $N(f)_*: H_*(X) \xrightarrow{\sim} H_*(X')$. If X' is also projective then $N(f)$ and thus f itself is a homotopy equivalence. □

The next proposition is a simple generalization of Theorem 5.11 in [3].

2.5. PROPOSITION: *Let $F: \mathfrak{m}_R \to \mathfrak{m}_S$ be any functor from R-modules to S-modules, where R is a heriditary ring, and let X and X' be projective simplicial R-modules. Then*

(i) *Any sequence of homomorphisms $\{h_i: H_i(X) \to H_i(X') \mid i \leqslant q\}$ induces homomorphisms $h_{F,i}: H_i(FX) \to H_i(FX')$ for $i \leqslant q$.*
(ii) *If h_i is bijective for $i < q$ and surjective for $i = q$ the same is true for the sequence $\{h_{F,i} \mid i \leqslant q\}$.*

Proof. (i) By 2.2 and 2.3 we can assume that $\{h_i \mid i \leqslant q\}$ is induced by a simplicial map $h: X^{(q+1)} \to X'^{(q+1)}$, i.e.

$$h_i: H_i(X) \cong H_i(X^{(q+1)}) \xrightarrow{H_i(h)} H_i(X'^{(q+1)}) \cong H_i(X')$$

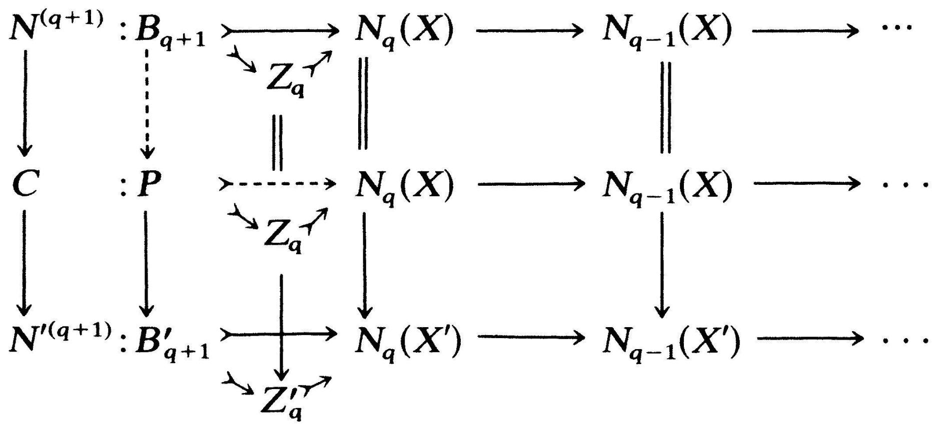
for $i \leqslant q$. In addition $X^{(q+1)}$ is a direct summand of X such that $X_i^{(q+1)} = X_i$ for $i \leqslant q$. It follows that $F(X^{(q+1)})$ is a direct summand of $F(X)$ with $F(X_i^{(q+1)}) = FX_i$ for $i \leqslant q$ and thus

$$H_i(FX) = H_i(FX^{(q+1)})$$

for $i \leqslant q$. The sequence $\{h_{F,i} \mid i \leqslant q\}$ is now given by

$$h_{F,i} : H_i(FX) \cong H_i(FX^{(q+1)}) \xrightarrow{H_i(Fh)} H_i(FX'^{(q+1)}) \cong H_i(FX').$$

(ii) To prove the second assertion let us decompose the chain map $Nh : N^{(q+1)} \to N'^{(q+1)}$ as follows:



where the lower left hand square is a pullback. Now $H_i(\beta) : H_i(C) \to H_i(N'^{(q+1)})$ is an isomorphism for all $i \geqslant 0$. Since C is also projective β is a homotopy equivalence. Thus, $K\beta : KC \to X'^{(q+1)}$ is a homotopy equivalence and since F obviously preserves homotopy $FK\beta : FKC \to FX'^{(q+1)}$ is a homotopy equivalence. $H_i(\alpha) : H_i(N^{(q+1)}) \to H_i(C)$ is the identity for $i < q$ and surjective for $i = q$. Moreover, $X^{(q+1)} = KC$ and thus $FX^{(q+1)} = FKC$ in dimension $\leqslant q$. We conclude that $H_i(FK\alpha) : H_i(FX^{(q+1)}) \to H_i(FKD)$ is the identity for $i < q$ and surjective for $i = q$. The composition

$$h_{F,i} : H_i(FX) \cong H_i(FX^{(q+1)}) \xrightarrow{H_i(FK\alpha)} H_i(FKC) \xrightarrow{H_i(FK\beta)} H_i(FX'^{(q+1)}) \cong H_i(X')$$

is therefore bijective for $i < q$ and surjective for $i = q$. □

2.6. By the Eilenberg–Zilber theorem the shuffle map g and the Alexander–Whitney map f define natural homotopy-inverse homotopy equivalences of chain modules

$$N(X) \otimes N(Y) \underset{f}{\overset{g}{\rightleftarrows}} N(X \otimes Y)$$

with $fg = id$ [6]. The Künneth-Formula can therefore be used to compute the homotopy groups of a tensorproduct of simplicial modules over a principal ideal domain. The homotopy groups of a simplicial algebra or Lie algebra form a graded algebra or a graded Lie algebra, respectively. These facts will be used in the sequel without special mention.

3. The spectral sequences associated with a simplicial group

3.1. Henceforth, if not mentioned otherwise, R shall always be a sub-ring of $\mathbf{Q}$ containing $\mathbf{Z}$. The notion of R-localization can of course be extended dimensionwise to nilpotent simplicial groups. If X is a simplicial group then RX is a simplicial augmented R-algebra. The homotopy exact couple associated with the extensions

$$L_n^R X \rightarrowtail \Gamma_{n+1}^R X \twoheadrightarrow \Gamma_n^R X$$

gives rise to a spectral sequence $E(R, X)$ of graded Lie algebras. The superscript R means R-localization and $\Gamma_n G = G/G_n$. On the other hand the augmentation filtration of RX leads to a spectral sequence $\tilde{E}(R, X)$ of graded R-algebras. The initial terms are given by

$$E^1_{n,m}(R, X) = \pi_m(L_n^R X) = H_m(L_n^R X), \quad \tilde{E}^1_{n,m}(R, X) = \pi_m(\mathrm{gr}_n RX) = H_m(\mathrm{gr}_n RX)$$

and the differentials d^r are of degree $(r, -1)$. These spectral sequences have some nice and useful properties although, except for weak convergence on the n-axis, there is no convergence in general.

3.2. PROPOSITION. *Let X be a simplicial group. Then:*

(i) *There are natural isomorphisms of spectral sequences* $R\otimes E(\mathbf{Z}, X) \xrightarrow{\sim} E(R, X)$, $R\otimes \tilde{E}(\mathbf{Z}, X) \xrightarrow{\sim} \tilde{E}(R, X)$.
(ii) *The canonical injection* $\kappa : X \to RX$, $\kappa(x) = x - 1$, *induces a natural homomorphism* $\kappa : E(R, X) \to \tilde{E}(R, X)$.
(iii) $E^\infty_{n,0}(R, X) = E^n_{n,0}(R, X) = L_n^R \pi_0(X)$, $\tilde{E}^\infty_{n,0}(R, X) = \tilde{E}^n_{n,0}(R, X) = \mathrm{gr}_n R\pi_0(X)$

Proof. (i) If X is a simplicial group then $\pi(X) = H(NX)$ [7], where NX is the (non-abelian) group complex defined as in 2.2, i.e. $N_q X = \bigcap_{i<q} \ker(\varepsilon_i : X_q \to X_{q-1})$ and the differential is given by the last face operator ε_q. The computation of the homotopy groups of X therefore only involves taking kernels, finite intersections and quotients. Since R-localization of nilpotent groups

preserves subgroups, extensions and intersections [13], [4] there is a natural isomorphism $R\otimes\pi(Y)\to\pi(R\otimes Y)$ for simplicial nilpotent groups Y. It now follows easily from the appropriate exact couples in the definition of $E(\mathbf{Z}, X)$ and of $E(R, X)$ that $R\otimes E(\mathbf{Z}, X)$ and $E(R, X)$ are naturally isomorphic. Since R is a principal ideal domain of characteristic zero the natural isomorphism $R\otimes \mathbf{Z}G\xrightarrow{\sim} RG$ induces isomorphisms $R\otimes J^nG\xrightarrow{\sim} J_R^nG$, where J_RG is the augmentation ideal of RG. This together with the universal coefficient theorem gives the second isomorphism. In view of (i) it suffices to prove (ii) for $R=\mathbf{Z}$. The canonical map $\kappa: X\to\mathbf{Z}X$ induces natural maps

$$\begin{array}{ccccc} L_nX & \rightarrowtail & \Gamma_{n+1}X & \twoheadrightarrow & \Gamma_nX \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{gr}_n\,\mathbf{Z}X & \rightarrowtail & \mathbf{Z}X/J^{n+1}X & \twoheadrightarrow & \mathbf{Z}X/J^nX. \end{array}$$

All the maps in these diagrams are group homomorphisms if JX/J^kX is considered a group under the "Jacobson operation" $u\circ v=u+v+uv$ and if $\mathbf{Z}X/J^kX$ is identified with $\mathbf{Z}\times JX/J^kX$. Moreover, for $i>0$ the group structure of the homotopy groups $\pi_i(Y)$ of the underlying simplicial set of a simplicial group Y coincides with the group structure induced by that of Y, if the identity of Y is chosen as base point [7]. This implies that the exact couple defining $\tilde{E}(\mathbf{Z}, X)$ and the exact couple obtained by replacing the ordinary addition on JX/J^kX by the "Jacobson operation" coincide everywhere, except for their tails

$$\to\pi_0(\mathrm{gr}_n\,\mathbf{Z}X)\xrightarrow{\alpha_n}\pi_0(\mathbf{Z}X/J^{n+1}X)\twoheadrightarrow\pi_0(\mathbf{Z}X/J^nX).$$

But all that is needed from the tails of the exact couples in the construction of the associated spectral sequences are the images of the maps α_n and the subgroups thereof. The image of α_n however inherits its group structure from that of $\pi_0(\mathrm{gr}_n\,\mathbf{Z}X)$, which is the same in both cases. We conclude that the two spectral sequences coincide. Thus, κ induces a natural homomorphism $\kappa: E(\mathbf{Z}, X)\to\tilde{E}(\mathbf{Z}, X)$ of spectral sequences.

(iii) Since $E_{n,m}(R, X)=0=\tilde{E}_{n,m}(R, X)$ unless $n>0$ and $m>0$ or $n\geq 0$ and $m=0$ and since $\deg d^r=(r,-1)$ it follows immediately that $E^\infty_{n,0}=E^n_{n,0}$ and $\tilde{E}^\infty_{n,0}=\tilde{E}^n_{n,0}$. An easy calculation shows that $E^n_{n,0}=\ker(\pi_0(\Gamma^R_{n+1}X)\to\pi_0(\Gamma^R_nX))$. Since the functors Γ^R_n preserve coequalizers we conclude that $E^\infty_{n,o}=E^n_{n,0}=L^R_n\pi_0(X)$. On the other hand $\tilde{E}^\infty_{n,0}=\mathrm{gr}_n\,\pi_0(RX)$ is the graded algebra associated with the augmentation filtration of $\pi_0(RX)$. But the group algebra functor preserves coequalizers and hence $\tilde{E}^\infty_{n,0}=\mathrm{gr}_n\,R\pi_0(X)$.

3.3. THEOREM: *If X and Y are free simplicial groups, then:*

(i) $E^1_{n,m}=\pi_m(\mathfrak{L}^R_n(R\otimes X_{ab}))$, $\tilde{E}^1_{n,m}=\pi_m(T^R_n(R\otimes X_{ab}))$, *where $\mathfrak{L}^R$ is the free Lie algebra functor and T^R is the tensor algebra functor. In particular* $E^1_{1,m}=H_m(R\otimes X_{ab})=\tilde{E}^1_{1,m}$, $E^1_{n,0}=\mathfrak{L}^R_n H_0(R\otimes X_{ab})$, $\tilde{E}^1_{n,0}=T^R_n H_0(R\otimes X_{ab})$.

(ii) $\kappa^1_{*,0}:E^1_{*,0}\to\tilde{E}^1_{*,0}$ *is injective.*

(iii) $E^\infty_{n,0}=E^n_{n,0}=L^R_n\pi_0(X)$, $\tilde{E}^\infty_{n,0}=\tilde{E}^n_{n,0}=\mathrm{gr}_n\, R\pi_0(X)$.

(iv) *Any sequence of homomorphisms* $\{h_i:H_i(R\otimes X_{ab})\to H_i(R\otimes Y_{ab})\mid i\leq m\}$ *induces homomorphisms* $h_{*,i}:E^1_{*,i}(R,X)\to E^1_{*,i}(R,Y)$ *and* $\tilde{h}_i:\tilde{E}_{*,i}(R,X)\to\tilde{E}_{*,i}(R,Y)$ *for $i\leq m$. If h_i is bijective for $i<m$ and surjective for $i=m$ then the same holds for the induced sequences* $\{h_{*,i}\mid i\leq m\}$ *and* $\{\tilde{h}_{*,i}\mid i\leq m\}$. *In particular,* $E^1_{*,m}(R,X)$ *and* $\tilde{E}^1_{*,m}(R,X)$ *depend only on* $\{H_i(R\otimes X_{ab})\mid i\leq m\}$.

Proof. (i) The freeness of X implies that $L^RX=\mathfrak{L}^R(R\otimes X_{ab})$ and $\mathrm{gr}\,RX=T^R(R\otimes X_{ab})$. Hence, $E^1_{n,m}=\pi_m(\mathfrak{L}^R_n(R\otimes X_{ab}))$ and $\tilde{E}^1_{n,m}=\pi_m(T^R_n(R\otimes X_{ab}))$. If

$$A \underset{\xrightarrow{\varepsilon_1}}{\overset{\xrightarrow{\varepsilon_0}}{\xleftarrow{\sigma_0}}} B$$

is a diagram of R-algebras or R-Lie algebras satisfying $\varepsilon_0\sigma_0=1=\varepsilon_1\sigma_0$ then the underlying R-module of the coequalizer of the pair $(\varepsilon_0,\varepsilon_1)$ is the same as the coequalizer of $(\varepsilon_0,\varepsilon_1)$ in the category of R-modules. Since in addition the functors $\mathfrak{L}^R$ and T^R preserve coequalizers we conclude that $H_0(\mathfrak{L}^R_n(R\otimes X_{ab}))=\mathfrak{L}^R_n H_0(R\otimes X_{ab}))$ and $H_0(T^R_n(R\otimes X_{ab}))=T^R_n H_0(R\otimes X_{ab})$. Since the universal envelope functor U^R also preserves coequalizers (ii) follows directly. (iii) is the same as 3.2 (iii). To prove (iv) use Proposition 2.5 with $F=\mathfrak{L}^R_n:\mathfrak{m}_R\to\mathfrak{m}_R$ and $F=T^R_n:\mathfrak{m}_R\to\mathfrak{m}_R$, respectively. The proposition applies since R is a PID and $R\otimes X_{ab}$ is R-free. □

Note: a) By proposition 3.2 it suffices to consider the spectral sequences $E(\mathbf{Z},X)$ and $\tilde{E}(\mathbf{Z},X)$ as far as functorial properties of the spectral sequences are concerned. This is of course not the case in 3.3 (iv) since the maps h_i need not be induced by a map from X to Y.

b) Similar spectral sequences can be obtained over the field $\mathbf{F}_p$ if the lower central series is replaced by the lower central p-series

$$\cdots\leq G^{(p)}_{n+1}\leq G^{(p)}_n\leq\cdots\leq G^{(p)}_2\leq G^{(p)}_1=G,$$

where $G_n^{(p)}$ is the subgroup of G generated by $\{[x_1, \ldots, x_s]^{p^\nu} \mid sp^\nu \geq n\}$. $L^{(p)}G = \otimes_{i\geq 1} G_i^{(p)}/G_{i+1}^{(p)}$ has the structure of a restricted Lie algebra over $\mathbf{F}_p$. If G is free then $L^{(p)}G$ is isomorphic to the free restricted Lie algebra $\mathfrak{L}^{(p)}(G/G_2^{(p)})$ and $\operatorname{gr} \mathbf{F}_p G \cong T(G/G_2^{(p)})$ [15]. With the obvious modifications theorem 3.3 still holds in this case.

c) In the situation of Theorem 3.3 $\tilde{E}^1_{n,m}(R, G) \cong H_m(T_n^R N(R\otimes X_{ab}))$ by the Eilenberg–Zilber theorem. The Künneth formula can then be used to compute $\tilde{E}^1(R, X)$ in terms of $H_*(R\otimes X_{ab})$

$$\tilde{E}^1_{n,m}(R, X) \cong \sum_{i+j=m} \tilde{E}^1_{n-1,i} \underset{R}{\otimes} \tilde{E}_{1,j} \oplus \sum_{i+j=m-1} \operatorname{Tor}^R(\tilde{E}^1_{n-1,i}, \tilde{E}^1_{1,j}).$$

For some values of n the Künneth theorem and the Eilenberg–Zilber map can also be used to get more information about $E^1_{n,m}(R, X)$.

3.4. PROPOSITION: *If X is a free simplicial group and n is invertible in R then $\kappa^1_{n,*}: E^1_{n,*}(R, X) \to \tilde{E}^1_{n,*}(R, X)$ is a split monomorphism and $E^1_{n,*}(R, X) \cong H(\mathfrak{L}_n N(R\otimes X_{ab}))$. In particular, $\kappa^1: E^1(\mathbf{Q}, X) \to \tilde{E}^1(\mathbf{Q}, X)$ is a (split) monomorphism, $E^1(\mathbf{Q}, X) \cong \mathfrak{L}H(\mathbf{Q}\otimes X_{ab})$ and $\tilde{E}^1(\mathbf{Q}, X) \cong TH(\mathbf{Q}\otimes X_{ab})$*

Proof: If M is an R-module and n is invertible in R then the map $\rho_n: T_n M \to \mathfrak{L}_n M$ given by

$$\rho(m_1\otimes m_2 \otimes \cdots \otimes m_n) = \begin{cases} 0, & \text{if } n=0 \\ \frac{1}{n}[m_1[m_2, \ldots [m_{n-1}, m_n] \cdots]], & \text{if } n>0, \end{cases}$$

is a left inverse for the canonical inclusion $\mu_n: \mathfrak{L}_n M \to T_n M$. The assertions now readily follow from the commutative diagram [9], Part I, 4.5

$$\begin{array}{ccccc}
\mathfrak{L}_n H(N(R\otimes X_{ab})) & \to & H(\mathfrak{L}_n N(R\otimes X_{ab})) & \to & \pi(\mathfrak{L}_n(R\otimes X_{ab})) \\
\mu_n \downarrow \uparrow \rho_n & & H(\mu_n) \downarrow \uparrow H(\rho_n) & & \pi(\mu_n) \downarrow \uparrow \pi(\rho_n) \\
T_n H(N(R\otimes X_{ab})) & \xrightarrow{f} & H(T_n N(R\otimes X_{ab})) & \xrightarrow[\cong]{g} & \pi(T_n(R\otimes X_{ab}))
\end{array}$$

where f is given by the Künneth theorem and g is the Eilenberg–Zilber isomorphism.

3.5. The proof of Theorem A is now an application of 3.3 and 3.4 to free simplicial resolutions of a group G.

A short exposition of free simplicial resolutions with all the necessary proofs can be found in [5]. Here is a brief description. Let X be a simplicial group with $\pi_0(X) = G$ and let $\varepsilon : X \twoheadrightarrow G$ be the augmentation. For $n \geq 0$ the subgroup

$$Z_nX = \{(x_0, x_1, \ldots, x_{n+1}) \in X_n^{n+2} \mid \varepsilon_i(x_j) = \varepsilon_{j-1}(x_i) \text{ for } i<j\}$$

is called the n^{th} *simplicial kernel* of the augmented simplicial group $\varepsilon : X \twoheadrightarrow G$. It comes equipped with faces $\{p_i : Z_nX \to X_n \mid 0 \leq i \leq n+1\}$ and degeneracies $\{q_j : X_n \to Z_nX \mid 0 \leq j \leq n\}$, where $p_i(x_0, x_1, \ldots, x_{n+1}) = x_i$ and $q_j(x) = (\sigma_{j-1}\varepsilon_0 x, \ldots, \sigma_{j-1}\varepsilon_{j-1}x, x, x, \sigma_j\varepsilon_{j+1}x, \ldots, \sigma_j\varepsilon_n x)$, respectively. The homomorphism

$$\varepsilon^{(n)} : X_{n+1} \to Z_nX$$

defined by $\varepsilon^{(n)}x = (\varepsilon_0 x, \varepsilon_1 x, \ldots, \varepsilon_{n+1}x)$ is the unique map that must exist by the universal property of simplicial kernels. The augmented simplicial group $\varepsilon : X \twoheadrightarrow G$ is called a *resolution of* G if $\varepsilon^{(n)} : X_{n+1} \to Z_nX$ is surjective for all $n \geq 0$, i.e. if $\pi_n(X) = 0$ for $n > 0$ and $\pi_0(X) = G$. It is called a *free simplicial resolution* of G if in addition X_n is free for each $n \geq 0$ and if sets $B_n \subset X_n$ of free generators can be chosen in such a way that $\sigma_i B_n \subset B_{n+1}$ for $o \leq i \leq n$. This definition is constructive in the sense that it allows for a *step by step construction* of free simplicial resolutions. There is a *comparison theorem*, which says: "If $\varepsilon : X \twoheadrightarrow G$ is a free simplicial group over $G = \pi_0(X)$ and $\varepsilon : Y \twoheadrightarrow H$ is a resolution of H then any homomorphism $f : G \to H$ can be extended to a simplicial map $\varphi : X \to Y$ such that $\pi_0(\varphi) = f$ and any two such extensions are homotopic. In particular, any two free simplicial resolutions of G are homotopy equivalent". For a free simplicial resolution X of G write $E(R, X) = E(R, G)$ and $\tilde{E}(R, X) = \tilde{E}(R, G)$. By the comparison theorem these spectral sequences do not depend on the particular choice of X and are therefore functorial in G. If X is any free simplicial resolution of G then $H_*(R \otimes X_{ab}) = H_*^{\mathbb{G}}(G, R) = H_{*+1}(G, R)$ is the cotriple homology of G.

3.6. Remark. a) Let $\{x; r\}$ be a free presentation of the group G. If F and R are the free groups on x and r, respectively, then the diagram

$$R * F \;\underset{\varepsilon_0}{\overset{\varepsilon_0}{\underset{\longleftarrow\sigma_0\longrightarrow}{\longrightarrow}}}\; F \xrightarrow{\varepsilon} G$$

is an initial segment of a free simplicial resolution of G which can be extended step by step. The maps in the diagram are given by $\varepsilon_0(r) = 1$, $\varepsilon_1(r) = r$, $\varepsilon_0(x) = \varepsilon_1(x) = x = \sigma_0(x)$. Using such a free resolution the differentials $d^s : \tilde{E}^s_{n,1} \to \tilde{E}^s_{n+s,0}$ can be computed in terms of Fox derivatives. For instance if $\tilde{r}$ is a representative of an element r in $\tilde{E}^s_{1,1}$ then

$$d^s(\bar{r}) = \sum \eta(D_{i_1}D_{1_2}\cdots D_{i_s}r)\overline{x_{i_1}\otimes\cdots\otimes x_{i_s}},$$

where $\eta:\mathbf{Z}F\to\mathbf{Z}$ is the augmentation map. The diagram above can also be extended to a free simplicial group X with $X_n = \underbrace{R\times\cdots\times R}_{n\text{-times}}\times F$. X is not a resolution of G anymore but $\pi_0(X)=G$ and Theorem 3.3 still holds. The resulting spectral sequences are similar to those used by Sjogren [11].
b) There are also functorial procedures to get free simplicial resolutions. The standard simplicial **G**-resolution, where **G** is the free group cotriple, is functorial. Another is obtained by applying Kan's Loop group construction to the classifying simplicial set of the group G [7].

4. Applications

In this section we shall apply Theorem A as described in the introduction. Again, R shall be a subring of $\mathbf{Q}$ containing $\mathbf{Z}$. Many of the results, especially those that depend on $\tilde{E}$ only, also hold if R is replaced by any principal ideal domain of characteristic zero. If G is a group then $\hat{G}^R = \varprojlim \Gamma^R_n G$ is the R-completion of G and G^R is our symbol for the Bousfield R-localization.

4.1. THEOREM: *If a grouphomomorphism $f:G\to K$ induces an isomorphism $f_*:H_1(G,R)\xrightarrow{\sim}H_1(K,R)$ and an epimorphism $f_*:H_2(G,R)\twoheadrightarrow H_2(K,R)$ then it induces isomorphisms $L^R(f):L^RG\xrightarrow{\sim}L^RK$ and $\operatorname{gr}R(f):\operatorname{gr}RG\xrightarrow{\sim}\operatorname{gr}RH$. In particular the R-completions $\hat{f}:\hat{G}^R\to\hat{K}^R$ and $\hat{R}(f):\hat{R}G\to\hat{R}K$ are isomorphisms.*

Proof: In view of our hypotheses and Theorem A (iv) the homomorphism $f:G\to K$ induces isomorphisms $E^1_{*,0}(R,G)\xrightarrow{\sim}E^1_{*,0}(R,K)$, $\tilde{E}^1_{*,0}(R,G)\xrightarrow{\sim}\tilde{E}^1_{*,0}(R,K)$ and epimorphisms $E^1_{*,1}(R,G)\twoheadrightarrow E^1_{*,1}(R,K)$, $\tilde{E}^1_{*,1}(R,G)\twoheadrightarrow\tilde{E}^1_{*,1}(R,K)$ which are of course compatible with the differentials. The assertions now follow from the next Lemma and an obvious induction argument.

LEMMA. *Let $g:E\to\bar{E}$ be a morphism of spectral sequences with differentials d^r of degree $(r,-1)$ and let $E_{n,m}=0=\bar{E}_{n,m}$ whenever $n<0$ or $m<0$. If for some $r\geq 1$ g^r has the property that $g^r_{*,0}$ is bijective and $g^r_{*,1}$ is surjective then g^s has the same property for all $s\geq r$ and $g^\infty_{*,0}$ is an isomorphism.*

Proof. This follows by induction on $s \geqslant r$. If g^s has the property then the commutative diagram with exact rows

$$\begin{array}{ccccc} E^s_{n,1} & \xrightarrow{d^s} & E^s_{n+s,0} & \twoheadrightarrow & E^{s+1}_{n+s,0} \\ \downarrow & & \downarrow & & \downarrow \\ \bar{E}^s_{n,1} & \xrightarrow{d^s} & \bar{E}^s_{n+s,0} & \twoheadrightarrow & \bar{E}^{s+1}_{n+s,0} \end{array}$$

implies that im $d^s \to$ im $\bar{d}^s$ is bijective, ker $d^s \to$ ker $\bar{d}^s$ is surjective and thus that g^{s+1} has the property. The assumptions on the form of the spectral sequences then imply that $E^\infty_{n,0} = E^{n+1}_{n,0} \xrightarrow{\sim} \bar{E}^{n+1}_{n,0} = \bar{E}^\infty_{n,0}$ for all $n \geqslant 0$. □

Note that the R-localization map $l: G \to G^R$ by definition satisfies the hypotheses of the theorem.

COROLLARY. *The R-localization map $l: G \to G^R$ induces isomorphisms $L^R G \xrightarrow{\sim} L^R G^R$ and* gr $RG \xrightarrow{\sim}$ gr RG^R. *If G and K are finitely generated groups with isomorphic p-localizations for every prime p, then $LG \cong LK$ and* gr $\mathbf{Z}G \cong$ gr $\mathbf{Z}K$ *as graded abelian groups.*

While the first isomorphism of the theorem and of the corollary is well known the second is not. However, in the special case where $L^R K$ is free as an R-module one shows as in [1] for $R = \mathbf{Z}$ that the universal envelope of $L^R K$ is isomorphic to gr RK and then the second isomorphism follows from the first. Since by the Künneth theorem $\bar{E}^1(R, G)$ can be computed explicitly in terms $H_*(G, R)$ the spectral sequence $\bar{E}(R, G)$ is usually easier to handle than $E(R, G)$. Much information about $E(R, G)$ comes from the natural map κ. The proof of the following theorem is a case in point.

4.2. THEOREM: *Let G be a group with $H_1(G, R)$ torsionfree and $H_2(G, R)$ torsion. Then:*

(i) $T^R(R \otimes G_{ab}) \xrightarrow{\sim}$ gr RG *and* $\mathfrak{L}^R(R \otimes G_{ab}) \xrightarrow{\sim} L^R G$.

(ii) *If a homomorphism $f: G \to K$ induces a monomorphism $f_*: H_1(G, R) \rightarrowtail H_1(K, R)$ and if $H_2(K, R)$ is torsion then $L^R(f): L^R G \to L^R K$ and* gr $R(f)$: gr $RG \to$ gr RK *are injective.*

(iii) *If a homomorphism $g: K \to G$ induces a monomorphism $g_*: H_1(K, R) \to H_1(Q, R)$ then $L^R(g): L^R K \to L^R G$ and* gr $R(g)$: gr $RK \to$ gr RG *are injective.*

Proof. If X is a free simplicial resolution of G then $\tilde{E}^1_{n,m}(R,G)=H_m(T^R_n(R\otimes X_{ab}))$ and by the Künneth formula

$$\tilde{E}^1_{n,m}\cong\sum_{i+j=m}\tilde{E}^1_{n-1,i}\underset{R}{\otimes}\tilde{E}^1_{1,j}\oplus\sum_{i+j=m-1}\mathrm{Tor}^R(\tilde{E}^1_{n-1,i},\tilde{E}^1_{1,j}).$$

In particular $\tilde{E}^1_{n,1}\cong\tilde{E}^1_{n-1,1}\otimes_R\tilde{E}^1_{1,0}\oplus\tilde{E}^1_{n-1,0}\otimes\tilde{E}^1_{1,1}\oplus\mathrm{Tor}^R(\tilde{E}^1_{n-1,0},\tilde{E}^1_{1,0})$ and $\tilde{E}^1_{n,0}=\tilde{E}^1_{n-1,0}\otimes_R\tilde{E}^1_{1,0}$. Induction on $n\geq 1$ shows that $\tilde{E}^1_{n,1}$ and thus $\tilde{E}^r_{n,1}$ is torsion for all $n\geq 1$ and all $r\geq 1$ if $H_2(G,R)=\tilde{E}^1_{1,1}$ is. Consider the commutative squares

$$\begin{array}{ccc} E^r_{n,1} & \xrightarrow{d^r} & E^r_{n+r,0} \\ \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\kappa} \\ \tilde{E}^r_{n,1} & \xrightarrow{\tilde{d}^r} & \tilde{E}^r_{n+r,0}. \end{array}$$

By induction on $r\geq 1$ we first conclude that $\tilde{d}^r=0$ and then since by Theorem A $\kappa^1_{*,0}:E^1_{*,0}\to\tilde{E}^1_{*,0}$ is injective also that $d^r=0$ for all $r\geq 1$. This establishes claim (i) in view of Theorem A. To prove (ii) note first that by the hypotheses on G and K the map $\tilde{E}^1_{*,0}(R,G)\to\tilde{E}^1_{*,0}(R,K)$ induced by f is injective and $\tilde{E}^r_{*,1}(R,K)$ is torsion for all $r\geq 1$. Induction on $r\geq 1$ shows that $\tilde{E}^r_{*,0}(R,G)\to\tilde{E}^r_{*,0}(R,K)$ is injective for all $r\geq 1$. The rest follows from the commutative squares

$$\begin{array}{ccc} E^r_{*,0}(R,G) & \longrightarrow & E^r_{*,0}(R,K) \\ \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\kappa} \\ \tilde{E}^r_{*,0}(R,G) & \rightarrowtail & \tilde{E}^r_{*,0}(R,K). \end{array}$$

The proof of (iii) is similar. □

For $R=\mathbf{Z}$ these results have been obtained with different methods in a recent paper by R. Strebel [14]. If $R=\mathbf{Q}$ the condition on $H_1(G,R)$ is of course redundant and assertion (i) always holds if $H_2(G)$ is torsion. Moreover, if $H_2(G)$ is torsion then there exists a minimal subring R of $\mathbf{Q}$ for which the conditions of 4.3 are satisfied, namely $R=\mathbf{Z}[\Sigma^{-1}]$, where $\Sigma=\{p\in\mathbf{Z}\mid p$ prime and $H_1(G)$ has non-trivial p-torsion$\}$. If $H_1(G)$ has no non-trivial p-torsion for some prime p then $R\neq\mathbf{Q}$. This happens in particular if $H_1(G)$ is finitely generated. If $H_1(G,R)$ is free as an R-module then the isomorphisms of 4.3 (i) are induced by a free subgroup $F\subseteq G$.

4.3. COROLLARY: *If $H_1(G, R)$ is a free R-module and $H_2(G)$ is torsion then there exists a free subgroup $F \subseteq G$ such that* $\operatorname{gr} RF \xrightarrow{\sim} \operatorname{gr} RG$, $L^R F \xrightarrow{\sim} L^R G$ *and thus* $\hat{F}^R \xrightarrow{\sim} \hat{G}^R$. *In particular, if $H_1(G)$ is free, $H_2(G)$ is torsion and G is residually nilpotent then G is parafree.*

Proof. For any element $y \in H_1(G, R)$ there is an invertible element $r \in R$ such that $ry \in \operatorname{im}(H_1(G) \to H_1(G, R))$. Thus, to any R-basis Y of $H_1(G, R)$ we can find a linearly independent subset X of $H_1(G)$ such that $\operatorname{card} X = \operatorname{card} Y$. If F is the free group on the set X then there is a homomorphism $h: F \to G$ which induces a monomorphism $h_*: H_1(F) \rightarrowtail H_1(G)$ and an isomorphism $h_*: H_1(F, R) \xrightarrow{\sim} H_1(G, R)$. Since $H_2(G)$ is torsion it follows from 4.3 (ii) that $L(h): LF \to LG$, and thus $h: F \to G$ itself, is injective. By 4.3 (ii) or 4.3 (iii) both $L^R(h): L^R F \to L^R G$ and $\operatorname{gr} R(h): \operatorname{gr} RF \to \operatorname{gr} RG$ are isomorphisms. □

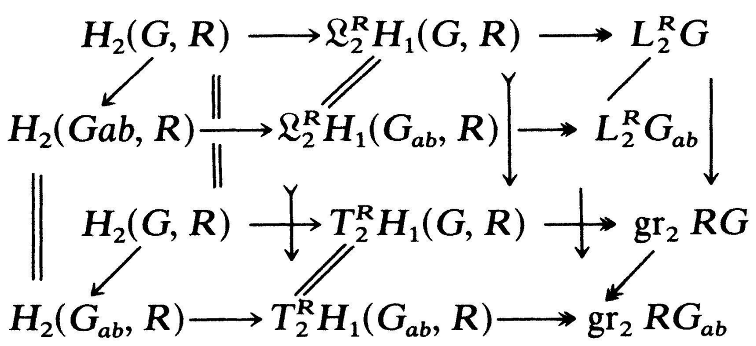
It follows now trivially that the conditions of the corollary are satisfied for a prenilpotent group G if and only if $G^R = R$. Also the following well-known result [2] is proved very efficiently in our present framework.

4.4. THEOREM: *For any group G the sequence*

$$0 \to L_2^R G \to \operatorname{gr}_2 RG \to \operatorname{gr}_2 RG_{ab} \to 0$$

is exact. If G is abelian then, $\operatorname{gr} RG \cong S_2^R(R \otimes G)$. *If G_{ab} is finitely generated then the sequence splits.*

Proof. By the functoriality of the spectral sequences $E(R, G)$, $\tilde{E}(R, G)$ and the naturality of $\kappa: E(R, G) \to \tilde{E}(R, G)$ the diagram



is commutative and has exact rows. Since $L_2^R G_{ab} = 0$ all our assertions except the last one follow immediately by diagram chasing. If $R \otimes G_{ab}$ is finitely generated R-module with R-basis $\{y_1, \ldots, y_n\}$ then the set $\{y_1 y_j \mid 1 \leq i \leq j \leq n\}$ is an R-basis for $S_2^R(R \otimes G_{ab})$. The R-module morphism $s: S_2^R(R \otimes G_{ab}) \to T_2^R(R \otimes G_{ab})$ defined by $s(y_i y_j) = y_i \otimes y_j$ is a section for the canonical projection $p: T_2^R(R \otimes$

$G_{ab}) \to S_2^R(R\otimes G_{ab})$. The required splitting can be constructed via the right hand bottom square in the diagram.

4.5. Let us turn to the natural map $\chi:\mathrm{gr}\, RG\underset{R}{\otimes}\mathrm{gr}\, RH \to \mathrm{gr}\, R(G\times K)$. For any group G the canonical Lie algebra homomorphism $\varphi: L^RG \to \mathrm{gr}\, RG$ induces a natural surjective map of graded algebras $\psi: UL^RG \to \mathrm{gr}\, RG$, where UL^RG is the universal envelope of L^RG. If L^RG is free as an R-module then ψ is an isomorphism. This is proved in [1] for $R=\mathbf{Z}$ using Quillen's result [8] for $R=\mathbf{Q}$. The more general case where R is any integral domain of characteristic 0 follows by exactly the same arguments if $\mathbf{Q}$ is replaced by the quotient field of R. The functor L^R of course preserves products while the functor U transforms products into tensorproducts. This implies that $UL^RG\underset{R}{\otimes} UL^RK$ is naturally isomorphic to $UL^R(G\times K)$ and that the natural map

$$\chi:\mathrm{gr}\, RG\underset{R}{\otimes}\mathrm{gr}\, RK \to \mathrm{gr}\, R(G\times K)$$

is surjective. It is of interest to know under what conditions χ is an isomorphism. Our first result is a trivial consequence of the remarks just made.

PROPOSITION. *If L^RG and L^RK are free as R-modules then*

$$\chi:\mathrm{gr}\, RG\underset{R}{\otimes}\mathrm{gr}\, RK \to \mathrm{gr}\, R(G\times K)$$

is an isomorphism.

The conditions of the proposition are by no means necessary. Take $G=\mathbf{Z}^n$ and $K=\mathbf{Z}/m\mathbf{Z}$. Then by [1], $\mathrm{gr}\,\mathbf{Z}G\otimes\mathrm{gr}\,\mathbf{Z}K \xrightarrow{\sim} S(G)\otimes S(K) \xrightarrow{\sim} S(G\oplus K) \xrightarrow{\sim} \mathrm{gr}\,\mathbf{Z}(G\times K)$, where S is the symmetric algebra functor. On the other hand χ is not always an isomorphism. If for example $G=\mathbf{Z}/r\mathbf{Z}$ and $H=\mathbf{Z}/s\mathbf{Z}$ with $(r,s)\neq 1$ then $[\mathrm{gr}\,\mathbf{Z}G\oplus\mathrm{gr}\,\mathbf{Z}H]_n \cong \mathbf{Z}/r\mathbf{Z}\oplus\mathbf{Z}/s\mathbf{Z}\oplus(\mathbf{Z}/(r,s)\mathbf{Z})^{n-1}$ while $\mathrm{gr}_n\,\mathbf{Z}(G\times H)\cong \mathrm{gr}_{n+1}\,\mathbf{Z}(G\times H)$ for large n. The proposition will nevertheless be useful to find weaker conditions under which χ is an isomorphism.

4.6. THEOREM: *If R is a* PID *of characteristic* 0 *then*

$$\chi:\mathrm{gr}\, RG\underset{R}{\otimes}\mathrm{gr}\, RK \to \mathrm{gr}\, R(G\times K)$$

is an isomorphism if either one of the following conditions is satisfied:

(a) gr RK *is torsionfree* (*in particular if* L^RK *is free as an* R-*module*),
(b) G_{ab} *and* K_{ab} *are both torsion and* $\mathrm{Tor}^R\,(H_1(G,R), H_1(H,R))=0$,
(c) G *and* K *are both finitely generated abelian and* $\mathrm{Tor}^R\,(H_1(G,R), H_1(K,R))=0$.

Proof. If $\varepsilon: X\to G$ is a free simplicial resolution of G then $\varepsilon\times\mathrm{id}: X\times K\to G\times K$ is a resolution (not free of course) of $G\times K$. This is because the functor $-xK:\mathfrak{G}\to\mathfrak{G}$ preserves simplicial kernels and epimorphisms. By 3.2 the spectral sequence $\tilde{E}(R, X\times K)$ converges weakly to gr $R(G\times K)$ on the n-axis, that is $\tilde{E}^{\infty}_{*,0}(R, X\times K)=\mathrm{gr}\,R(G\times K)$. If L^RK is free as an R-module then 4.5 and the universal coefficient theorem imply

$$\tilde{E}^1_{n,m}(R, X\times K)\cong\sum_{i+j=n}\tilde{E}^1_{i,m}(R,G)\underset{R}{\otimes}\mathrm{gr}_j\,RK\oplus\mathrm{Tor}^R\,(\tilde{E}^1_{i,m-1}(R,G),\mathrm{gr}_j\,RK),$$

i.e.:

$$\tilde{E}^1(R, X\times K)\cong\tilde{E}^1(R,G)\underset{R}{\otimes}\mathrm{gr}\,RK\oplus\mathrm{Tor}^R\,(\tilde{E}^1(R,G),\mathrm{gr}\,RK).$$

The second term in the direct sum is actually zero since gr RK is R-free. Going through the spectral sequence step by step the universal coefficient theorem shows that $\tilde{E}^r(R, X\times K)=\tilde{E}^r(R,G)\otimes_R\mathrm{gr}\,RK$ for all $r\geq 1$. In particular $\chi:\mathrm{gr}\,RG\otimes_R\mathrm{gr}\,RK\to\mathrm{gr}\,R(G\times K)$ is an isomorphism if L^RK is R-free. This in turn means that the above formula for $\tilde{E}^1(R, X\times K)$ holds for arbitrary G and K. If gr RK is torsion free (condition a)) procee b·as above to get the desired result. Condition b) of course implies that $\mathrm{Tor}^R\,(\tilde{E}^s_{*,0}(R,G),\mathrm{gr}\,RK)=0$ for all $s\geq 1$. But the obvious induction argument and the universal coefficient theorem show that

$$\tilde{E}^{r+1}_{*,0}(R, X\times K)=\tilde{E}^{r+1}_{*,0}(R,G)\otimes_R\mathrm{gr}\,RK$$

and

$$\tilde{E}^{r+1}_{n,1}(R, X\times K)\cong[E^{r+1}_{*,1}(R,G)\otimes_R\mathrm{gr}\,RK]_n\oplus\mathrm{Tor}^R\,(E^{r+1}_{*,0}(R,G),\mathrm{gr}\,RK)_{n+r}$$

if $\mathrm{Tor}^R\,(\tilde{E}^s_{*,0}(R,G),\mathrm{gr}\,RK)=0$ for $1\leq s\leq r$. Condition c) is a simple combination of a) and b).

4.7. COROLLARY: *If R is a* PID *of characteristic* 0, *then*

(i) $\operatorname{gr} R(\mathbf{Z}\times K)\cong\bigoplus_{i=0}^{n}\operatorname{gr}_i RK$ *for arbitrary K,*
(ii) $\operatorname{gr}_n R(G\times K)\cong\operatorname{gr}_n RG\oplus\operatorname{gr}_n RK$ *for all* $n\geqslant 1$ *if* G_{ab} *and* K_{ab} *are both torsion and* $R\otimes\operatorname{Tor}^{\mathbf{Z}}(G_{ab}, K_{ab})=0$.
(iii) $\operatorname{gr} RG\cong\otimes\operatorname{gr} R\bar{G}_p$ *if G is finite and* $\bar{G}_p$ *is the p-Sylow subgroup of the nilpotent residual* $\bar{G}$.

To prove (iii) use the fact that for a prenilpotent group G the graded algebra $\operatorname{gr} RG$ is isomorphic to $\operatorname{gr} R\bar{G}$, where $\bar{G}$ is the nilpotent residual of G. For abelian groups and $R=\mathbf{Z}$ the results of the corollary were first proved by J. B. S. Passi [10] using ideal theoretic methods. In low degrees the methods used to prove 4.6 yield a little more than stated in the theorem, namely:

4.8. PROPOSITION: *If R is a* PID *of characteristic* 0, *then*

$$\left[\operatorname{gr} RG\underset{R}{\otimes}\operatorname{gr} RK\right]_n \xrightarrow{\sim} \operatorname{gr}_n R(G\times K)$$

for (a) $n\leqslant 2$ *if G and K are arbitrary groups,*
(b) $n\leqslant 3$ *if* $R\otimes\operatorname{Tor}(G_{ab}, K_{ab})=0$,
(c) $n\leqslant 4$ *if* $\sum_{i+j=3}\operatorname{Tor}^R(\operatorname{gr}_i RG, \operatorname{gr}_j RK)=0$.

However, we do not know whether $\operatorname{Tor}^R(\operatorname{gr} RG, \operatorname{gr} RK)=0$ implies that χ is an isomorphism.

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Received May 9, 1979