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On cohomological periodicity for infinite groups

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Introduction

The phenomenon of cohomological periodicity for finite groups has long been understood. Here we introduce the notion of "a group G having period q after k-steps" so as to allow infinite groups to have periodic cohomology. The definition is given in terms of a projective resolution of G and, as expected, it is equivalent to having the functors $H^n(G, -)$ and $H^{n+q}(G, -)$ naturally isomorphic for all $n \geq k+1$. We then show that this definition coincides with the classical one for finite groups and moreover, we obtain that if an infinite group G has period q after k-steps then $k \geq 1$.

In §2 we investigate what it means for a countable locally finite group to have period q after k-steps. We obtain what one would expect, i.e. that a countable locally finite group G has period q after k-steps iff every finite subgroup of G has period q. Moreover, we have here that $k = 1$.

Then we show that there is an element $g \in H^q(G, Z)$ such that cup product with g

$$\bigcup g : H^i(G, -) \to H^{i+q}(G, -)$$

induces the natural isomorphism for all $i \geq 2$.

Finally, in §3 we characterize the infinite locally finite p-groups which have period q after k-steps. First, we point out two obvious candidates, i.e. the infinite locally cyclic p-group and the infinite locally quaternion group, and then we show that these are the only ones. This result depends heavily on the well known similar statement for periodic finite p-groups [Cartan+Eilenberg].

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§1. Periodicity after some "steps"

Let G be a group and ZG its integral group ring. We work in the category of left ZG-modules. If A is a ZG-module, by a resolution of A we shall always mean a projective resolution of A.

DEFINITION. *A group G is said to have period q after k-steps* if there is an exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & R_{k+q} \arrow[r, "\beta"'] & P_{k+q-1} \arrow[r] & \cdots \arrow[r] & P_k \arrow[rr, "\partial_k"] \arrow[dr, "\alpha"'] & & P_{k-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & Z \arrow[r] & 0 \\
& & & & & R_k \arrow[ur] & & & & &
\end{tikzcd}
\tag{1}
$$

where Z is regarded as a trivial ZG-module, $R_{k+q} = R_k$ and P_i $0 \le i \le k+q-1$ are projective ZG-modules. We take $R_0 = Z$. If $k = 0$ then G is said to have period q. Having (1) we can form a resolution of G

$$
\begin{tikzcd}
P'_* : \cdots \arrow[r] & P'_i \arrow[rr, "\partial_i"] \arrow[dr] & & P'_{i-1} \arrow[r] & \cdots \arrow[r] & P'_0 \arrow[r] & Z \arrow[r] & 0 \\
& & R'_i \arrow[ur] & & & & &
\end{tikzcd}
$$

by defining

$$
\begin{array}{llll}
P'_i = P_i & 0 \le i \le k-1 & \partial'_i = \partial_i & 1 \le i \le k
\end{array}
$$

and

$$
\begin{array}{llll}
P'_i = P_{k+\lambda_i} & i = k + nq + \lambda_i & \partial'_i = \partial_{k+\lambda_i} & i = k + nq + \lambda_i \\
& n \ge 0,\ 0 \le \lambda_i < q & & n \ge 0,\ 0 < \lambda_i < q \\
& & \partial'_{k+\mu q} = \beta\alpha & \mu \ge 1
\end{array}
$$

Such a resolution is naturally considered periodic after k-steps; we may refer to (1) itself as a resolution of G which is periodic after k-steps. If $k = 0$ we call such a resolution a periodic resolution of G.

Evidently the functors $H^n(G, -)$ and $H^{n+q}(G, -)$ are naturally isomorphic for all $n \ge k+1$.

Now let M, N be ZG-modules, $g \in \operatorname{Ext}^q_{ZG}(M, N)$ and consider the following diagram

$$
\begin{tikzcd}
P_* : \cdots \arrow[r] & P_{i+q} \arrow[r] \arrow[dd] & P_{i+q-1} \arrow[r] \arrow[dd] & \cdots \arrow[r] & P_{q+1} \arrow[r] \arrow[dd] & P_q \arrow[rr] \arrow[dd] \arrow[dr] & & P_{q-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0 \\
& & & & & & R_q \arrow[ur] \arrow[d, "\theta"] & & & & & \\
E_* : \cdots \arrow[r] & E_i \arrow[r] & E_{i-1} \arrow[r] & \cdots \arrow[r] & E_1 \arrow[r] & E_0 \arrow[r] & N \arrow[r] & 0 & & & &
\end{tikzcd}
$$

where $P_* \to M$, $E_* \to N$ are resolutions of M, N respectively and $\theta : R_q \to M$ a

q-cocycle representing g. Then θ lifts to a chain map $P_* \to E_*$ of degree $-q$ and this chain map induces cup product with g

$$\cup g : \operatorname{Ext}^i_{ZG}(N, -) \to \operatorname{Ext}^{i+q}_{ZG}(M, -) i \geq 0.$$

THEOREM 1.1. *Let G be a group. Then the following statements are equivalent:*

(i) *G has period q after k-steps*
(ii) *There is a resolution of G*

$$\begin{tikzcd} \cdots \arrow[r] & P_i \arrow[r] \arrow[dr] & P_{i-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & Z \arrow[r] & 0 \\ & & R_i \arrow[u] & & & & \end{tikzcd}$$

and an element $h \in \operatorname{Ext}^q_{ZG}(R_k, R_k)$ such that for every ZG-module A cup product with h

$$\cup h : \operatorname{Ext}^i_{ZG}(R_k, A) \longrightarrow \operatorname{Ext}^{i+q}_{ZG}(R_k, A)$$

is an isomorphism for $i \geq 1$ and an epimorphism for $i = 0$. Note that $\operatorname{Ext}^i_{ZG}(R_k, A) = H^{i+k}(G, A)$ for $i \geq 1$, $k \geq 0$.

Proof. (i)$\Rightarrow$(ii). Consider a resolution P'_* of G which is periodic, or period q, after k-steps

$$\begin{tikzcd} P'_* : \cdots \arrow[r] & P'_i \arrow[r] \arrow[dr] & P'_{i-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P'_0 \arrow[r] & Z \arrow[r] & 0 \\ & & R'_i & & & & \end{tikzcd}$$

and the resolution $P'_* \mid R'_k$ of R'_k

$$P'_* \mid_{R_k} : \cdots \longrightarrow P'_i \longrightarrow P'_{i-1} \longrightarrow \cdots \longrightarrow P'_k \xrightarrow{\alpha} R'_k \longrightarrow 0.$$

Now there is an element $h \in \operatorname{Ext}^q_{ZG}(R'_k, R'_k)$ defined by $P'_{k+q} \xrightarrow{\alpha} R'_k$ or in Yoneda's interpretation of $\operatorname{Ext}^i_{ZG}(-, -)$ by the multiple extension

$$0 \longrightarrow R'_k \longrightarrow P'_{k+q-1} \longrightarrow \cdots \longrightarrow P'_k \xrightarrow{\alpha} R'_k \longrightarrow 0.$$

Clearly, for any ZG-module A cup product with h

$$\cup h : \operatorname{Ext}^i_{ZG}(R_k, A) \longrightarrow \operatorname{Ext}^{i+q}_{ZG}(R_k, A)$$

is the identity for $i \geq 1$ and an epimorphism for $i = 0$. (ii) $\Rightarrow$ (i) can be easily deduced from the following Theorem. It is due to C. T. C. Wall [[7], Theorem. 1.2]; however, we give here a different proof.

THEOREM 1.2. *Let A, B be ZG-modules. If $g \in \operatorname{Ext}^n_{ZG}(A, B)$ is such that for any ZG-module E cup product with g*

$$\cup g : \operatorname{Ext}^i_{ZG}(B, E) \longrightarrow \operatorname{Ext}^{i+n}_{ZG}(A, E)$$

is injective for $i = 1$ and surjective for $i = 0$, then g is represented by a multiple extension

$$0 \longrightarrow B \longrightarrow K \longrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

with K, P_i $0 \leq i \leq n-2$ projective ZG-modules.

Proof. Let $P_* \to A$, $E_* \to B$ be resolutions of A and B respectively, and let $\theta : R_n \to B$ be a cocycle representing g

$$\begin{tikzcd}
P_* : \cdots \arrow[r] & P_{n+1} \arrow[r] \arrow[dd] & P_n \arrow[r] \arrow[dd] \arrow[dr] & P_{n-1} \arrow[r] & P_{n-2} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & A \arrow[r] & 0 \\
& & & R_n \arrow[u] \arrow[d, "\theta"] & & & & & \\
E_* : \cdots \arrow[r] & E_1 \arrow[r] & E_0 \arrow[r] & B \arrow[r] & 0. & & & &
\end{tikzcd}$$

Now θ lifts to a chain map $P_* \to E_*$ of degree $-n$ which induces cup product with g.

Let

$$\begin{tikzcd}
R_n \arrow[r, "\gamma"] \arrow[d, "\theta"] & P_{n-1} \arrow[d, "\alpha"] \\
B \arrow[r, "\beta"'] & K
\end{tikzcd}$$

be a push-out diagram. Then β is injective and coker $\beta \simeq$ coker $\gamma \simeq R_{n-1}$.

Now g is represented by $0 \to B \to K \to P_{n-2} \to \cdots \to P_0 \to A \to 0$. We shall show that K is a projective ZG-module.

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & R_n \arrow[r] \arrow[d, "\theta"] & P_{n-1} \arrow[r] \arrow[d, "\alpha"] & R_{n-1} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & B \arrow[r] & K \arrow[r] & R_{n-1} \arrow[r] & 0.
\end{tikzcd}$$

This gives rise to the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Hom}_{ZG}(B, E) \arrow[r] \arrow[d, "\theta^*"'] \arrow[dr] & \operatorname{Ext}_{ZG}(R_{n-1}, E) \arrow[r] \arrow[d, "id"'] & \operatorname{Ext}_{ZG}(K, E) \arrow[r] \arrow[d, "\alpha^*"'] & \operatorname{Ext}_{ZG}(B, E) \arrow[r] \arrow[d, "\theta^*"'] & \cdots \\
\cdots \arrow[r] & \operatorname{Hom}_{ZG}(R_n, E) \arrow[r, "\delta"'] & \operatorname{Ext}_{ZG}(R_{n-1}, E) \arrow[r] & \operatorname{Ext}_{ZG}(P_{n-1}, E) \arrow[r] & \operatorname{Ext}_{ZG}(R_n, E) \arrow[r] & \cdots
\end{tikzcd}$$

By hypotheses $\theta^* : \operatorname{Ext}_{ZG}(B, E) \to \operatorname{Ext}_{ZG}(R_n, E) \simeq \operatorname{Ext}_{ZG}^{1+n}(A, E)$ is injective and

$$\delta\theta^* : \operatorname{Hom}_{ZG}(B, E) \to \operatorname{Ext}_{ZG}(R_{n-1}, E) \simeq \operatorname{Ext}^n_{ZG}(A, E)$$

is surjective. Moreover, $\operatorname{Ext}_{ZG}(P_{n-1}, E) = 0$ since P_{n-1} is projective. Thus by the commutativity of the diagram we obtain that $\operatorname{Ext}_{ZG}(K, E) = 0$, and this holds for any ZG-module E. Hence K is a projective ZG-module.

Remark. If P_{n-1}, B are finitely generated ZG-modules then K is a finitely generated module since from the push-out diagram we have an epimorphism $P_{n-1} \oplus B \xrightarrow{\alpha+\beta} K \longrightarrow 0$.

PROPOSITION 1.3. *If G has period q after k-steps then so does every subgroup H of G.*

Proof. This follows from the fact that a projective ZG-module is a projective ZH-module.

PROPOSITION 1.4. *If G is an infinite group and has period q after k-steps then $k \geq 1$.*

Proof. Let H be a group which has period q and consider a periodic resolution of H

$$0 \longrightarrow Z \xrightarrow{\beta} P_{q-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow Z \longrightarrow 0.$$

Now im $\beta \rightarrowtail H^0(H, P_{q-1})$. But P_{q-1} is a direct summand of an induced ZH-module, and it is well known that if A is a non-trivial induced ZH-module then $H^0(H, A) \neq 0$ iff H is finite. The result follows.

LEMMA 1.5. *If G is a finite group and has period q after k-steps, then G has period q.*

Proof. By Theorem 1.1 (i)$\Rightarrow$(ii) there is a resolution of G

$$\begin{tikzcd}
\cdots \arrow[r] & P_i \arrow[r] \arrow[dr] & P_{i-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & Z \arrow[r] & 0 \\
& & R_i \arrow[u] & & & &
\end{tikzcd} \tag{2}$$

and an element $h \in \operatorname{Ext}^q_{ZG}(R_k, R_k)$ such that for every ZG-module A cup product with h

$$\bigcup h : \operatorname{Ext}^i_{ZG}(R_k, A) \to \operatorname{Ext}^{i+q}_{ZG}(R_k, A)$$

is an isomorphism for $i \geq 1$.

Now since G is a finite group, a projective ZG-module P is a direct summand of a coinduced ZG-module; hence $H^i(G, P) = 0$ for all $i \geq 1$. Thus from (2) we obtain an isomorphism

$$\delta : \operatorname{Ext}^q_{ZG}(R_k, R_k) \xrightarrow{\simeq} H^q(G, Z).$$

Let $h' \in H^q(G, Z)$ be the image of $h \in \operatorname{Ext}^q_{ZG}(R_k, R_k)$ under δ. Then clearly for any ZG-module A cup product with h'

$$\bigcup h' : H^i(G, A) \to H^{i+q}(G, A)$$

is an isomorphism for $i \geq k+1$.

Let $0 \to C \to P \to A \to 0$ be a short exact sequence with P a projective ZG-module. Then we obtain the following commutative diagram

$$\begin{array}{ccc} H^{n-1}(G, A) & \xrightarrow{\cup h'} & H^{n+q-1}(G, A) \\ \downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle \delta_{n+q}} \\ H^n(G, C) & \xrightarrow[\cup h']{} & H^{n+q}(G, C) \end{array}$$

where δ_n, δ_{n+q} are connecting homomorphisms. But since $H^i(G, P) = 0$ for all $i \geq 1$, δ_i is an isomorphism for $i > 1$ and an epimorphism for $i = 1$. Thus it follows that cup product with h'

$$\bigcup h' : H^i(G, A) \longrightarrow H^{i+q}(G, A)$$

is an isomorphism for $i \geq 1$ and an epimorphism for $i = 0$. The result now follows from Theorem 1.1 (ii) $\Rightarrow$ (i).

PROPOSITION 1.6. *If G has period q after k-steps then every finite subgroup H of G has period q.*

Proof. This follows from Proposition 1.3 and Lemma 1.5.

COROLLARY 1.7. *If G has period 2 after k-steps then every finite subgroup H of G is cyclic.*

Proof. It is known [[6], Lemma 5.2] that a finite group H has period 2 iff it is cyclic; hence the result follows from Proposition 1.6.

Let G be a group such that the functors $H^j(G,-)$ and $H^{j+q}(G,-)$ are naturally isomorphic for some $j \geq 1$. Consider a resolution of G

$$\begin{tikzcd} \cdots \arrow[r] & P_i \arrow[rr] \arrow[dr] & & P_{i-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & Z \arrow[r] & 0. \\ & & R_i \arrow[ur] & & & & & \end{tikzcd} \tag{3}$$

Then clearly $\mathrm{Ext}_{ZG}(R_{j-1},-) \overset{\text{nat.}}{\simeq} \mathrm{Ext}_{ZG}(R_{j+q-1},-)$. Thus by [[4], Thm. 2.6] there exist projective ZG-modules Q_1, Q_2 such that

$$R_{j-1} \oplus Q_1 \xrightarrow[\cong]{\alpha} R_{j+q-1} \oplus Q_2. \tag{$*$}$$

Now (3) gives rise to the following exact sequence

$$0 \longrightarrow R_{j+q-1} \oplus Q_2 \xrightarrow{\varrho} P_{j+q-2} \oplus Q_2 \longrightarrow P_{j+q-3} \longrightarrow \cdots \longrightarrow P_{j-2} \longrightarrow$$

$$\begin{tikzcd} \arrow[r] & P_{j-1} \oplus Q_1 \arrow[rr] \arrow[dr] & & P_{j-2} \oplus Q_1 \arrow[rr] \arrow[dr] & & P_{j-3} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & Z \arrow[r] & 0 \\ & & R_{j-1} \oplus Q_1 \arrow[ur] & & R_{j-2} \arrow[ur] & & & & & \end{tikzcd} \tag{4}$$

Hence it follows from (4) and $(*)$ that:

PROPOSITION 1.8. *If the functors $H^\lambda(G,-)$ and $H^{\lambda+q}(G,-)$ are naturally isomorphic for some $\lambda \geq 2$, then G has period q after λ-1-steps.*

Now let $j=1$ and $q>0$. Clearly it follows that $H^n(G,-) \overset{\text{nat}}{\cong} H^{n+q}(G,-)$ for all $n \geq 1$. Thus by Proposition 1.8 G has period q after 1-step. But G is finite since there is a monomorphism $\varrho\alpha : Z \oplus Q_1 \to P_{q-1} \oplus Q_2$ (same argument as for the proof of Proposition 1.4). Thus by Lemma 1.5 G has period q. Hence we have proved:

PROPOSITION 1.9. *If the functors $H^1(G,-)$ and $H^{1+q}(G,-)$ are naturally isomorphic for some $q>0$, then G is finite and has period q.*

2. Periodic countable locally finite groups

In this section we state and prove our main Theorem. The proof is based on direct limit arguments.

PROPOSITION 2.1 [[1]]. *Let $(P_i, \lambda_{ij})_I$ be a countable direct system of projective ZG-modules. Then there exists an exact sequence $0 \to Q \to Q \to P \to 0$ where $P = \varinjlim (P_i, \lambda_{ij})_I$ and Q is a projective ZG-module.*

LEMMA 2.2. *Let $\cdots \to A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots \to A_1 \xrightarrow{\alpha_1} A_0 \xrightarrow{\pi} \operatorname{coker} \alpha_1 \to 0$ be an exact sequence of ZG-modules such that for every $n \geq 0$ there is a resolution $P_{n*} \to A_n$ of the form $0 \to P_{n,1} \xrightarrow{\partial_1{}^n} P_{n,0} \xrightarrow{\partial_0{}^n} A_n \to 0$. If $\{\alpha_*^j\} : P_{j*} \to P_{j-1*}$ are chain maps lifting $\alpha_j : A_j \to A_{j-1}$ $j \geq 1$ then there is a resolution $P_* \to \operatorname{coker} \alpha_1$*

$$P_* : \cdots \longrightarrow P_k \xrightarrow{\partial_k} P_{k-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \operatorname{coker} \alpha_1 \longrightarrow 0$$

where

$$P_k = P_{k-1,1} \oplus P_{k,0} \quad k \geq 0$$

$$\partial_k = \begin{cases} P_{k,0} \xrightarrow[\alpha_0^k - (\partial_1^{k-2})^{-1} \alpha_0^{k-1} \alpha_0^k]{} P_{k-1,0} \oplus P_{k-2,1} \\ P_{k-1,1} \xrightarrow[\partial_1^{k-1} - \alpha_1^{k-1}]{} P_{k-1,0} \oplus P_{k-2,1} \end{cases} \quad k \geq 1$$

and $\varepsilon = \pi\, \partial_0^0$.

Note that P_{ij} is understood to be zero if $i < 0$ or $j < 0$.

Proof. We have the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "\alpha_n"] & A_{n-1} \arrow[r] & \cdots \arrow[r] & A_1 \arrow[r, "\alpha_1"] & A_0 \arrow[r, "\pi"] & \operatorname{coker} \alpha_1 \arrow[r] & 0 \\
\cdots \arrow[r] & P_{n,0} \arrow[r, "\alpha_0^n"] \arrow[u, "\partial_0^n"] & P_{n-1,0} \arrow[r] \arrow[u, "\partial_0^{n-1}"] & \cdots \arrow[r] & P_{1,0} \arrow[r, "\alpha_0^1"] \arrow[u, "\partial_0^1"] & P_{0,0} \arrow[u, "\partial_0^0"'] & & \\
\cdots \arrow[r] & P_{n,1} \arrow[r, "\alpha_1^n"] \arrow[u, "\partial_1^n"] & P_{n-1,1} \arrow[r] \arrow[u, "\partial_1^{n-1}"] & \cdots \arrow[r] & P_{1,1} \arrow[r, "\alpha_1^1"] \arrow[u, "\partial_1^1"] & P_{0,1} \arrow[u, "\partial_1^0"'] & &
\end{tikzcd}$$

Now it is easily seen that this gives rise to the resolution $P_* \to \operatorname{coker} \alpha_1$.

We shall need the following notion.

Let $(G_i, \gamma_{ij})_I$ be a direct system of groups. For $i \in I$ let (C_*^i, ∂_*^i) be a ZG_i-chain complex. If $i \leq j$ then let $c_{ij} : (C_*^i, \partial_*^i) \to (C_*^j, \partial_*^j)$ be a ZG_i-chain map, where (C_*^j, ∂_*^j) is considered as a ZG_i-chain complex via $\gamma_{ij} : G_i \to G_j$, such that

(1) c_{ii} is the identity chain map of (C_*^i, ∂_*^i), for all $i \in I$ and
(2) if $i \leq j \leq k$ then $c_{jk} c_{ij} = c_{ik}$ as ZG_i-chain maps.

We call $((C^i_*, \partial^i_*), c_{ij})_I$ a direct system of chain complexes over the direct system of groups $(G_i, \gamma_{ij})_I$. In particular let $(A_i, \alpha_{ij})_I$ be a direct system of modules over the direct system of groups $(G_i, \gamma_{ij})_I$. Clearly $(A_i, \alpha_{ij})_I$ is a direct system of abelian groups and if $A = \varinjlim (A_i, \alpha_{ij})_I$ then $G = \varinjlim (G_i, \gamma_{ij})_I$ acts on A in a natural way [[2], ch. 2, §6, nos 6, 7].

Moreover, if $B_i = ZG \underset{ZG_i}{\otimes} A_i$ and $\beta_{ij} : B_i \to B_j$

$$x \underset{ZG_i}{\otimes} a_i \to x \underset{ZG_j}{\otimes} \alpha_{ij} a_i$$

then it follows that $(B_i, \beta_{ij})_I$ is a direct system of ZG-modules and $\varinjlim (B_i, \beta_{ij})_I \simeq A$ as ZG-modules.

Recall that if X is a class of groups, then a group G is said to be locally an X-group if every finite set of elements of G is contained in some X-subgroup of G.

THEOREM 2.3. *Let G be a countable locally finite group all of whose finite subgroups have period q. Then G has period q after* 1-*step.*

Proof. It is easily seen that there is a direct system of finite subgroups of G over $I = \{1, 2, 3, \ldots\}$, $(G_i, e_{ij})_I$, with $e_{ij} : G_i \to G_j$ inclusions and $G = \varinjlim (G_i, e_{ij})_I$. By hypothesis for each $i \in I$ we have a periodic resolution $P^{i'}_*$ of G_i

$$P^{i'}_* : 0 \longrightarrow Z \longrightarrow P^{i'}_{q-1} \longrightarrow \cdots \longrightarrow P^{i'}_1 \longrightarrow ZG_i \xrightarrow{\varepsilon_i} Z \longrightarrow 0.$$

Now consider the following diagram

$$\begin{array}{ccccccccccccccc}
P^{i'}_* & & 0 \longrightarrow & Z & \longrightarrow & P^{i'}_{q-1} & \longrightarrow \cdots \longrightarrow & P^{i'}_1 & \longrightarrow & ZG_i & \xrightarrow{\varepsilon_i} & Z & \longrightarrow 0 \\
\downarrow{\scriptstyle \varrho'_{i\,i+1}} & & & \downarrow{\scriptstyle h} & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id_Z} & \\
P^{i+1'}_* & & 0 \longrightarrow & Z & \longrightarrow & P^{i+1'}_{q-1} & \longrightarrow \cdots \longrightarrow & P^{i+1'}_1 & \longrightarrow & ZG_{i+1} & \xrightarrow{\varepsilon_{i+1}} & Z & \longrightarrow 0.
\end{array}$$

Since $P^{i+1'}_*$ is evidently a resolution of G_i via $e_{i\,i+1} : G_i \to G_{i+1}$ we can lift id_Z to a ZG_i-chain map $\varrho'_{i\,i+1} : P^{i'}_* \to P^{i+1'}_*$.

Then $\varrho'_{i\,i+1}$ induces a map $h : Z \to Z$ (multiplication by h) which need not be the identity on Z.

Our aim is to find a periodic resolution P^{i+1}_* of G_{i+1} and a ZG_i-chain map $\varrho_{i\,i+1} : P^{i'}_* \to P^{i+1}_*$ which lifts id_Z on the right and induces id_Z on the left. Now the map $\varrho'_{i\,i+1}$ induces an isomorphism between the cohomology groups of G_i defined using $P^{i'}_*$ and those defined using $P^{i+1'}_*$. On $H^q(G_i, Z)$ this map is obviously multiplication by h. Since $H^q(G_i, Z) \simeq Z/|G_i|\, Z$ by the periodicity, it follows that $(h, |G_i|) = 1$; hence there are integers λ_1, λ_2 such that $1 = \lambda_1 h + \lambda_2 |G_i|$.

Let $\lambda = \lambda_1 + x\,|G_i|$ where $x = p_1 \cdots p_m$ with p_j $1 \le j \le m$ primes such that $p_j \mid |G_{i+1}|$ and $p_j \nmid |G_i|$, $p_j \nmid \lambda_1$ for all $1 \le j \le m$. If no such primes exist, take $x = 1$. Then

$$(\lambda, |G_{i+1}|) = 1 \quad \text{and} \quad 1 = \lambda h + (\lambda_2 - xh)\,|G_i|\ (*).$$

Consider $p_*^{i+1'}$ as a projective resolution of G_{i+1}. Then id_Z is a q-cocycle which defines an element $g_{i+1} \in H^q(G_{i+1}, Z)$. Clearly $\lambda g_{i+1} \in H^q(G_{i+1}, Z)$ is represented by the q-cocycle $\lambda : Z \to Z$ (multiplication by λ). We shall show that $\lambda g_{i+1} \in H^q(G_{i+1}, Z) = \mathrm{Ext}^q_{ZG_{i+1}}(Z, Z)$ is represented by a multiple extension

$$P_*^{i+1} : 0 \longrightarrow Z \longrightarrow P_{q-1}^{i+1} \longrightarrow P_{q-2}^{i+1} \longrightarrow \cdots \longrightarrow P_1^{i+1} \longrightarrow ZG_{i+1} \longrightarrow Z \longrightarrow 0$$

with P_k^{i+1} $1 \le k \le q-1$ projective ZG_{i+1}-modules. By Theorem 1.2 it is enough to show that for any ZG_{i+1}-module A cup product with λg_{i+1}

$$\cup \lambda g_{i+1} : H^k(G_{i+1}, A) \longrightarrow H^{k+q}(G_{i+1}, A)$$

is injective for $k = 1$ and surjective for $k = 0$. This follows easily since cup product with λg_{i+1} is multiplication by λ for $k = 0, 1$ and we have that $(\lambda, |G_{i+1}|) = 1$. Note that $|G_{i+1}|\, H^k(G_{i+1}, A) = 0$ for all $k \ge 1$ and by the periodicity $H^q(G_{i+1}, A) \simeq A^{G_{i+1}}/(\sum_{g \in G_{i+1}} g)A$ where $A^{G_{i+1}} \simeq \mathrm{Hom}_{ZG_{i+1}}(Z, A)$.

Moreover, from the proof of Theorem 1.2, we obtain the following commutative diagram

$$\begin{tikzcd}
P_*^{i'} \arrow[d, "\varrho'_{i\,i+1}"'] & 0 \arrow[r] & Z \arrow[r, "\mu_i"] \arrow[d, "h"'] & P_{q-1}^{i'} \arrow[rr, "\cdots"] \arrow[d, "\alpha"] \arrow[ddl, "\gamma"'] & & & P_1^{i'} \arrow[r] \arrow[d] & ZG_i \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "id_Z"] & 0 \\
P_*^{i+1'} \arrow[d, "L'"'] & 0 \arrow[r] & Z \arrow[r] \arrow[d, "\lambda"'] & P_{q-1}^{i+1'} \arrow[r] \arrow[d, "\beta"] & P_{q-2}^{i+1'} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & P_1^{i+1'} \arrow[r] \arrow[d, equal] & ZG_{i+1} \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, "id_Z"] & 0 \\
P_*^{i+1} & 0 \arrow[r] & Z \arrow[r, "\mu_{i+1}"'] & P_{q-1}^{i+1} \arrow[r] & P_{q-2}^{i+1} \arrow[r] & \cdots \arrow[r] & P_1^{i+1} \arrow[r] & ZG_{i+1} \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

Clearly $L'\varrho'_{i\,i+1} : P_*^{i'} \to P_*^{i+1}$ is a ZG_i-chain map.

Now consider $P_*^{i'}$ as a resolution of G_i. Then id_Z is a q-cocycle which defines an element $g_i \in H^q(G_1, Z)$.

Clearly the q-cocycle $\lambda h : Z \to Z$ represents $\lambda h g_i$. But from $(*)$ $\lambda h g_i = g_i$ since $|G_i|\, g_i = 0$. Hence the cocycles h and id_Z represent the same element. So there exists $\gamma: P'_{q-1} \to Z$ such that $id_Z - \lambda h = \gamma\mu_i$. If we now take $\beta\alpha + \mu_{i+1}\gamma$ instead of $\beta\alpha$ we still have a chain map $P^{i'}_* \to P^{i+1}_*$ which now induces id_Z on the left. We call this chain map $\varrho_{i\,i+1}$, i.e. we have the following commutative diagram

$$\begin{array}{cccccccccccccc}
P^{i'}_* & 0 & \longrightarrow & Z & \longrightarrow & P^{i'}_{q-1} & \longrightarrow \cdots \longrightarrow & P^{i'}_1 & \longrightarrow & ZG_i & \xrightarrow{\varepsilon_i} & Z & \longrightarrow & 0 \\
\downarrow{\scriptstyle \varrho_{i\,i+1}} & & & \downarrow{\scriptstyle id_Z} & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id_Z} & & \\
P^{i+1}_* & 0 & \longrightarrow & Z & \longrightarrow & P^{i+1}_{q-1} & \longrightarrow \cdots \longrightarrow & P^{i+1}_1 & \longrightarrow & ZG_{i+1} & \xrightarrow{\varepsilon_{i+1}} & Z & \longrightarrow & 0.
\end{array}$$

Take $P^{1'}_* = P^1_*$. Then given P^1_* and $P^{2'}_*$ we construct as above a periodic resolution P^2_* of G_2 and a ZG_1-chain map $\varrho_{1\,2}: P^1_* \to P^2_*$ which lifts id_Z on the right and induces id_Z on the left. In this way we construct inductively a direct system $(P^i_*, \varrho_{ij})_I$ of periodic resolutions over the direct system of groups $(G_i, e_{ij})_I$. Clearly $\varrho_{ij(i\le j)}: P^i_* \to P^j_*$ are given as $\varrho_{ij} = \varrho_{j-1j} \cdots \varrho_{i\,i+1}$ and they lift id_Z on the right and induce id_Z on the left.

Now since direct limit preserves exactness, taking the direct limit of $(P^i_*, \varrho_{ij})_I$ we obtain the following exact sequence of ZG-modules

$$0 \longrightarrow Z \xrightarrow{\theta} \varinjlim P^i_{q-1} \xrightarrow{\alpha_{q-1}} \varinjlim P^i_{q-2} \longrightarrow \cdots \longrightarrow \varinjlim P^i_1 \xrightarrow{\alpha_1} ZG \xrightarrow{\varepsilon} Z \longrightarrow 0 \qquad (*)'$$

By splicing together copies of $(*)'$ we obtain an exact sequence of ZG-modules

$$\cdots \xrightarrow{\alpha_1} ZG \xrightarrow{\alpha_q} \varinjlim P^i_{q-1} \xrightarrow{\alpha_{q-1}} \cdots \longrightarrow \varinjlim P^i_1 \xrightarrow{\alpha_1} ZG \longrightarrow$$
$$\xrightarrow{\alpha_q} \varinjlim P^i_{q-1} \xrightarrow{\alpha_{q-1}} \cdots \longrightarrow \varinjlim P^i_1 \xrightarrow{\alpha_1} ZG \xrightarrow{\varepsilon} Z \longrightarrow 0 \quad (1)$$

where $\alpha_q = \theta\varepsilon$.

By Proposition 2.1 we have resolutions $Q_{j*} \to \varinjlim P^i_j$ of the form

$$0 \longrightarrow Q_{j1} \longrightarrow Q_{j0} \longrightarrow \varinjlim P^i_j \longrightarrow 0 \quad \text{for all} \quad 1 \le j \le q-1.$$

Now the hypotheses of Lemma 2.2 hold for (1). Moreover, it is clear that we can choose here the chain maps $\{a^n_*\}$ of Lemma 2.2 so as

$$\{\alpha^{j+kq}_*\} = \{\alpha^j_*\}, \qquad \{\alpha^{\lambda q}_*\} = \{\alpha^q_*\} \qquad 1 \le j \le q-1, \qquad k \ge 0, \qquad \lambda \ge 1.$$

Thus by Lemma 2.2 we obtain a resolution of G

$$\begin{tikzcd}
\cdots \arrow[r] & Q_{2,0}\oplus Q_{1,1} \arrow[r, "\partial_{q+2}"] & Q_{1,0} \arrow[r, "\partial_{q+1}"] \arrow[dr] & ZG\oplus Q_{q-1,1} \arrow[r] & \cdots \\
& & & R_{q+1} \arrow[u] &
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r] & Q_{2,0}\oplus Q_{1,1} \arrow[r, "\partial_{2}"] & Q_{1,0} \arrow[r, "\partial_{1}"] \arrow[dr] & ZG \arrow[r, "\varepsilon"] & Z \arrow[r] & 0 \\
& & & R_{1} \arrow[u] & &
\end{tikzcd}$$

with $\partial_{q+2}=\partial_2$: $Q_{2,0}\oplus Q_{1,1}\to Q_{1,0}$. Hence $R_{q+1}\simeq R_1$; whence the result follows.

Remark. Note that in view of Proposition 1.4, Theorem 2.3 is the best possible we can obtain if G is an infinite countable locally finite group.

COROLLARY 2.4. *Let G be a countable locally finite group. Then the following statements are equivalent:*
(i) *G has period q after k-steps.*
(ii) *Every finite subgroup of G has period q.*
Moreover, $k=0$ if G is finite and $k=1$ if G is infinite.

Proof. (i) $\Rightarrow$ (ii) follows from Proposition 1.6, (ii) $\Rightarrow$ (i) is a consequence of Theorem 2.3.

Now if G is a finite group and P a projective ZG-module then $H^i(G,P)=0$ for all $i\geq 1$. We have an analogous result for a countable locally finite group:

PROPOSITION 2.5. *Let G be a countable locally finite group and P a projective ZG-module. Then $H^j(G,P)=0$ for all $j\geq 2$.*

Proof. It follows easily from the following result [[5], p. 158, Lemma 4.1].
If G is a countable locally finite group and A a ZG-module such that for some $n\geq 2$
$H^n(K,A)=0=H^{n-1}(K,A)$ for all finite subgroups K of G, then $H^n(G,A)=0$.

Now let G be a countable locally finite group which has period q after k-steps. Then by Corollary 2.4 G has period q after 1-step. Thus by Theorem 1.1 (i) $\Rightarrow$ (ii) there is a resolution of G

$$\begin{tikzcd}
\cdots \arrow[r] & P_i \arrow[r] \arrow[d] & P_{i-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & Z \arrow[r] & 0 \\
& R_i \arrow[ur] & & & & &
\end{tikzcd}$$

and an element $g' \in \mathrm{Ext}^q_{ZG}(R_1, R_1)$ such that for every ZG-module A cup product with g'

$$\cup g' : \mathrm{Ext}^i_{ZG}(R_1, A) \longrightarrow \mathrm{Ext}^{i+q}_{ZG}(R_1, A)$$

is an isomorphism for all $i \geq 1$.

By Proposition 2.5 if P is a projective ZG-module then $H^j(G, P) = 0$ for all $j \geq 2$. Having this result one obtains from (2) that $\mathrm{Ext}^q_{ZG}(R_1, R_1) \xrightarrow[\cong]{\delta} H^q(G, Z)$; note that $q \geq 2$ since by Proposition 1.6 q is a period of every finite subgroup of G, and that is known to be even [[3], ch. XII, p. 261].

Now if $g \in H^q(G, Z)$ corresponds to $g' \in \mathrm{Ext}^q_{ZG}(R_1, R_1)$ under δ then one clearly has that for any ZG-module A cup product with g

$$\cup g : H^i(G, A) \longrightarrow H^{i+q}(G, A)$$

is an isomorphism for all $i \geq 2$. Thus we have proved that:

PROPOSITION 2.6. *Let G be a countable locally finite group which has period q after k-steps. Then there is an element $g \in H^q(G, Z)$ such that for any ZG-module A cup product with g*

$$\cup g : H^i(G, A) \longrightarrow H^{i+q}(G, A)$$

is an isomorphism for all $i \geq 2$.

§3. Periodic locally finite p-groups

The following well known theorem characterizes the finite p-groups which have period $q > 0$.

THEOREM 3.1 [[3], ch. XII, Thm. 11.6]. *For a finite p-group G the following statements are equivalent:*
(i) *G has period $q > 0$*
(ii) G is either cyclic or is a (generalized) quaternion group.
Moreover, a cyclic group has period 2 and a (generalized) quaternion group has period 4.

We shall characterize the infinite locally finite p-groups which have period $q > 0$ after k-steps.

I. Consider an infinite locally cyclic p-group. It is easily seen that such a group is uniquely determined up to isomorphism by the prime p and is given by

$$C_{p^\infty} = \langle c_1, c_2, \ldots, c_i, \ldots ; c_1^p = 1, c_2^p = c_1, \ldots, c_{i+1}^p = c_i, \ldots \rangle.$$

II. Recall that a (generalized) quaternion group Q_{2^i} is given by

$$Q_{2^i} = \langle x, y ; x^{2^{i-2}} = y^2, xyx = y \rangle \quad i \geq 3.$$

One easily shows that a (generalized) quaternion group Q_{2^i} contains a normal cyclic subgroup C of index 2, which is characteristic if $i > 3$, and every element of $Q_{2^i} \setminus C$ is of order four and inverts every element of C. Having this result and following the proof of Proposition 1.I.2 [8], one shows:

PROPOSITION 3.2. *Let G be a locally quaternion group. Then G has a locally cyclic normal subgroup C of index 2, and every element of $G \setminus C$ is of order four and inverts every element of C.*

COROLLARY 3.3. *Up to isomorphism there exists only one infinite locally quaternion group, namely*

$$Q_{2^\infty} = \langle c_1, \ldots, c_n, \ldots, y;\ c_1^2 = 1, \ldots, c_{n+1}^2 = c_n, \ldots, y^2 = c_1,$$
$$c_i^y c_i = 1 \quad \text{all} \quad i \geq 1 \rangle.$$

Proof. This follows easily from Proposition 3.2 and the fact that up to isomorphism there exists only one infinite locally cyclic 2-group, namely, C_{2^∞}.

PROPOSITION 3.4. (i) *An infinite locally cyclic p-group has period 2 after 1-step.*
(ii) *An infinite locally quaternion group has period 4 after 1-step.*

Proof. (i) Clearly it is enough to consider C_{p^∞}. Now C_{p^∞} is a countable group. Moreover, every finite subgroup of C_{p^∞} is cyclic, hence by Theorem 3.1 it has period 2. The result now follows from Theorem 2.3.
(ii) By Corollary 3.3 it is enough to consider Q_{2^∞}. Clearly Q_{2^∞} is countable. Moreover, if K is a finite subgroup of Q_{2^∞} then K is contained in some (generalized) quaternion group; hence by Theorem 3.1 and Proposition 1.3 K has period 4. Now the result follows from Theorem 2.3.

THEOREM 3.5. *For an infinite locally finite p-group G the following statements are equivalent:*
(i) *G has period $q > 0$ after k-steps.*
(ii) *G is either $C_{p\infty}$ or $Q_{2\infty}$.*
Moreover, $k = 1$. If $G = C_{p\infty}$ then $q = 2$ and if $G = Q_{2\infty}$ then $q = 4$.

Proof. (i) $\Rightarrow$ (ii). Let G have period q after k-steps. By Proposition 1.6 every finite subgroup of G has period q and therefore by Theorem 3.1 every finite subgroup of G is either cyclic or is a (generalized) quaternion group. Thus G is either an infinite locally cyclic p-group or an infinite locally quaternion group, i.e. G is either $C_{p\infty}$ of by Corollary 3.3, G is $Q_{2\infty}$. (ii) $\Rightarrow$ (i) follows from Proposition 3.4.

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