Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 55 (1980)

Artikel: Generic finite schemes and Hochschild cocycles.
Autor: Mazzola, Guerino
DOI: https://doi.org/10.5169/seals-42375

Comment. Math. Helvetici. **55** (1980) 267–293 Birkhauser Verlag, Basel

Generic finite schemes and Hochschild cocycles

GUERINO MAZZOLA

Introduction

Let k be an algebraically closed field of characteristic different from 2 and 3. In this paper we investigate the schemes N_n, $n \in \mathbf{N}$, whose k-rational points are the k-algebra structures ξ on k^n which are commutative, associative and satisfy $a_1 \cdot a_2 \cdot \cdots \cdot a_{n+1} = 0$ for any $a_1, a_2, \ldots, a_{n+1} \in k^n$. Our main result is the following

THEOREM. *For $n = 1, 2, 3, 4, 5, 6$, the schemes N_n are irreducible and rational of dimension $n^2 - n$. The structures isomorphic to the maximal ideal of $k[T]/(T^{n+1})$ define a smooth, open subscheme of N_n.*

Hence every finite local k-scheme X of k-rank $n \leqq 7$ can be deformed to $\operatorname{Spec}(k[T]/(T^n))$. This implies that X admits a desingularization, i.e. a deformation to $\operatorname{Spec}(k^n)$.

For $n \geqq 7$, we show that there are structures $\xi_n \in N_n$ of embedding dimension $[n+1/2]$ which are not specializations of the maximal ideal of $k[T]/(T^{n+1})$. From this it follows that for $n \geqq 10$, there are finite schemes which cannot be "desingularized."

In contrast to the Hilbert-scheme method used by A. Iarrobino and J. Emsalem [2, 3, 4, 5], our technical tools are N_n-scheme S_n parametrizing the commutative Hochschild cocycles associated with structures in N_n. The description of S_n/N_n is discussed in §1 and in §2, where we list explicitely the cocycles we are interested in.

§3 is entirely devoted to the proof of the above theorem.

§4 presents the above structures ξ_n showing that for $n \geqq 7$, N_n admits at least two irreducible components.

§5 is an appendix, including two deformation criteria also valid for noncommutative, associative k-algebras, as well as the Hasse-diagram of the deformations of five-dimensional commutative, associative, unitary k-algebras.

I want to express my gratitude to P. Gabriel for careful reading and in particular for some suggestions concerning §3 which made it possible to avoid two very ugly deformations, one of which I include as a curiosity.

§1. Cocycles

Let k-Alg be the category of associative, commutative k-algebras with unit elements. We consider the following scheme N_n ($n \geqq 1$): for each $A \in k$-Alg, the A-points of N_n are the multiplications $\xi : A^n \times A^n \to A^n$ of commutative, associative A-algebra structures on A^n such that $a_1 a_2 \cdots a_{n+1} = 0$ for any $a_1, a_2, \ldots, a_{n+1} \in A^n$. Being bilinear, such a multiplication map ξ may clearly be identified with an element of $\mathrm{Hom}_A (A^n \otimes_A A^n, A^n) \xrightarrow{\sim} A^{n^3}$. In this way N_n is identified with a closed subscheme of the scheme underlying k^{n^3}.

We denote by $e_1, \ldots, e_n$ the canonical base of A^n and often write ξ instead of A^n, when the space is considered in relation with the multiplication ξ. For instance, we write $\xi^{\cdot i}$ for the i-th power of A^n under ξ. If $\xi \in N_n(k)$ is a k-rational point of N_n, $e(\xi)$ denotes the embedding dimension $\dim_k (\xi/\xi^{\cdot 2})$ of ξ.

By structural transport, $g \in \mathrm{GL}_n(A)$ acts on $N_n(A)$ from the right in such a way that $g : \xi^g \xrightarrow{\sim} \xi$ becomes an A-algebra isomorphism: $\xi^g(x, y) = g^{-1}(\xi(g(x), g(y)))$. If ξ and η are two k-rational structures on N_n, we shall write $\xi > \eta$ if η belongs to the Zariski-closure of the orbit ξ^{GL_n} *of* ξ.

In order to proceed from N_n to N_{n+1}, we set $C(\xi) = \{B \in \mathrm{Hom}_A (A^n \otimes_A A^n, A) : B = \text{symmetric and } \xi\text{-associative}\}$. For every $\xi \in N_n(A)$, this means that $B \in C(\xi)$ iff $B(x, y) = B(y, x)$ and $B(xy, z) = B(x, yz)$ for any $x, y, z \in A^n$, the products xy, yz being taken in ξ. We call such a B a symmetric Hochschild cocycle.

If $\xi \in N_n(A)$ and $\eta \in N_m(A)$, then a homomorphism $f : \xi \to \eta$ of A-algebras induces a homomorphism of A-modules $C(f) : C(\eta) \to C(\xi)$ by the usual formula $C(f)(B)(x, y) = B(f(x), f(y))$. In particular, if $\eta = \xi/\xi^{\cdot 2}$, f being the projection of ξ onto $\xi/\xi^{\cdot 2}$, then we may identify $C(\xi/\xi^{\cdot 2})$ with its $C(f)$-image in $C(\xi)$, the subspace of $C(\xi)$ consisting of all symmetric forms vanishing on $\xi \times \xi^{\cdot 2} + \xi^{\cdot 2} \times \xi$.

We define the N_n-scheme S_n of symmetric Hochschild cocycles over N_n by its functor $S_n(A) = \{(\xi, B) : \xi \in N_n(A), B \in C(\xi)\}$, the structural morphism $p : S_n \to N_n$ being the projection $(\xi, B) \mapsto \xi$. Observe that S_n is a commutative group scheme over N_n, the p-fibre $S(\xi) = \{\xi\} \times C(\xi)$ being "isomorphic" with $C(\xi)$. Again, GL_n acts on S_n from the right by $(\xi, B)^g = (\xi^g, C(g)(B))$.

EXAMPLES. (1) Let $\tau_n \in N_n(k)$ be the *uniserial* structure, for which $e_1^p = e_p$ if

$p \leqq n$ and $e_1^p = 0$ if $p > n$. Then $C(\tau_n) \xrightarrow{\sim} \bigoplus_{j=1}^n kI_j$, with

$$I_j = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \cdot & & 1 & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ 0 & & & & & & \cdot \\ 1 & & & & & & \cdot \\ 0 & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ 0 & & & & & & 0 \end{pmatrix}$$

(2) Let $\varphi_n \in N_n(k)$ be the *final* structure: $e_i e_j = 0$, all i, j. Then $C(\varphi_n) \xrightarrow{\sim} \mathbf{M}_n^s(k)$, the set of symmetric $n \times n$-matrices with coefficients in k.

Let $\mathrm{Ex}: S_n \to N_{n+1}$ be the morphism sending the couple $(\xi, B) \in S_n(A)$ to the structure $\eta = \mathrm{Ex}(\xi, B)$ on $A^{n+1} = \bigoplus_{i=1}^{n+1} Ae_i$ such that $e_{n+1} \underset{\eta}{\cdot} e_i = 0$ for $i = 1, 2, \ldots, n+1$, and $e_i \underset{\eta}{\cdot} e_j = e_i \underset{\xi}{\cdot} e_j + B(e_i, e_j)e_{n+1}$ for $i, j = 1, 2, \ldots, n$. Clearly, $\mathrm{Ex}: S_n \to N_{n+1}$ induces an isomorphism between S_n and the closed subscheme $\bar{S}_n$ of N_{n+1} formed by the structures ξ such that $\xi e_{n+1} = 0$ and $\xi^{\cdot(n+1)} \subset Ae_{n+1}$ (the last condition holds automatically if $\xi \in N_{n+1}(k)$ is k-rational). Moreover, Ex is equivariant with respect to the embedding $\mathrm{GL}_n \to \mathrm{GL}_{n+1}$, $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. Finally, the composed morphism $\mathrm{Exgl}: S_n \times \mathrm{GL}_{n+1} \xrightarrow{\mathrm{Ex} \times 1} N_{n+1} \times \mathrm{GL}_{n+1} \to N_{n+1}$ is surjective since every k-rational structure $\eta \in N_{n+1}(k)$ contains a one-dimensional ideal.

In order to show that N_{n+1} is irreducible for $n+1 = 1, 2, 3, 4, 5, 6$, we shall construct irreducible curves Γ in N_{n+1} such that $\eta^{\mathrm{GL}_{n+1}}$ contains some non empty open subset of Γ. If this holds, we shall say that Γ lies generically in $\eta^{\mathrm{GL}_{n+1}}$. This will imply $\eta > \xi$ whenever $\Gamma \cap \xi^{\mathrm{GL}_{n+1}} \neq \emptyset$.

PROPOSITION 1. Let $\xi, \eta \in N_{n+1}(k)$. Then $\eta > \xi$ iff there is a curve Γ in S_n whose image $\mathrm{Ex}(\Gamma)$ lies generically in $\eta^{GL_{n+1}}$ and satisfies $\mathrm{Ex}(\Gamma) \cap \xi^{GL_{n+1}} \neq \emptyset$.

Clearly the condition is sufficient. In order to prove the converse, consider the subscheme T_n of $N_{n+1} \times \mathbf{P}_n$ such that $T_n(A) = \{(\xi, \tau): \xi \cdot \tau = 0 \text{ and } \xi^{\cdot(n+1)} \subset \tau\}$; here $\xi \in N_{n+1}(A)$ is an algebra structure on A^{n+1} and τ is a direct summand of A^{n+1} of rank 1. Clearly, the canonical projection $v: T_n \to N_{n+1}$ is proper and surjective. Therefore we have $v(\overline{v^{-1}(\eta^{\mathrm{GL}_{n+1}})}) = \overline{\eta^{\mathrm{GL}_{n+1}}}$. If $\eta > \xi$, it follows that there is some $(\xi, \sigma) \in \overline{v^{-1}(\eta^{\mathrm{GL}_{n+1}})}$ lying over ξ. Let Δ be a curve in $v^{-1}(\eta^{\mathrm{GL}_{n+1}})$ running through

(ξ, σ) and cutting $v^{-1}(\eta^{GL_{n+1}})$. Replacing if necessary (ξ, σ) by some $(\xi^g, g^{-1}\sigma)$ with $g \in GL_{n+1}$, we may assume that $\sigma = ke_{n+1}$ and that $\Delta \subset N_{n+1} \times U$, where U is the open subscheme of $\mathbf{P}_n$ whose A-points are the supplements of $Ae_1 \oplus \cdots \oplus Ae_n$ in A^{n+1}. Replacing Δ by the image of $\Delta \to T_n$, $\delta \mapsto \delta^{\mu(\delta)}$ where $\mu: U \to GL_{n+1}$ is a morphism such that $u^{\mu(u)} = ke_{n+1}$ for all $u \in U(k)$, we are reduced to the case where $\Delta \subset N_{n+1} \times \{ke_{n+1}\}$. In that case we set $\Gamma = \mathrm{Ex}^{-1}(v(\Delta))$. QED.

For a k-rational $\xi \in N_n(k)$ we put $\mathrm{soc}(\xi) = \{x \in \xi : x\xi = 0\}$ to denote the socle of ξ.

COROLLARY 1. *Let* $\eta \in N_{n+1}(k)$ *and* $\xi \in \mathrm{Ex}(S_n(k))$ *be such that* $\eta > \xi$ *and* $\dim\mathrm{soc}(\eta) = \dim\mathrm{soc}(\xi)$. *Then there is a curve* Γ *in* S_n *such that* $\mathrm{Ex}(\Gamma)$ *runs through* ξ *and generically lies in* $\eta^{GL_{n+1}}$.

Proof. Keeping the notations of proof of proposition 1, we only have to show that the curve $\Delta \subset v^{-1}(\eta^{GL_{n+1}})$ constructed in that proof may be chosen in such a way that $(\xi, ke_{n+1}) \in \Delta$. For that it suffices to prove that $(\xi, ke_{n+1}) \in \overline{v^{-1}(\eta^{GL_{n+1}})}$. In fact, consider any irreducible component V of $\overline{v^{-1}(\eta^{GL_{n+1}})}$ which dominates $\overline{\eta^{GL_{n+1}}}$. Consider any point $(\zeta, \tau) \in V$ which is contained in no other irreducible component and lies over $\eta^{GL_{n+1}}$. Then $\dim(v^{-1}(\zeta) \cap V) = \dim v^{-1}(\zeta) = \dim\mathrm{soc}(\eta)$. As $v(V)$ is closed, $v^{-1}(\xi) \cap V$ is not empty; hence $\dim(v^{-1}(\xi) \cap V) \geqq \dim(v^{-1}(\zeta) \cap V) = \dim\mathrm{soc}(\eta) = \dim\mathrm{soc}(\xi) = \dim v^{-1}(\xi)$ and $v^{-1}(\xi) \cap V = v^{-1}(\xi)$. We infer that $v^{-1}(\xi) \subset V \subset \overline{v^{-1}(\eta^{GL_{n+1}})} = v^{-1}(\overline{\eta^{GL_{n+1}}})$.

Remark. The preceding proposition applies in particular to the case where $\eta = \tau_{n+1}$ and $e(\xi) \leqq 2$. This follows from a theorem of Briançon [1] stating that $\mathrm{Hilb}^{n+2}\, k\{x, y\}$ is irreducible (for the density we refer also to theorem 1 below). In fact, let the ideal $I \subset k\{x, y\}$ in $\mathrm{Hilb}^{n+2}\, k\{x, y\}$ define a local algebra isomorphic to $k \oplus \xi$. Then the theorem implies that I deforms to a "generic" ideal I_0 defining a local algebra isomorphic to $k \oplus \tau_{n+1}$. Consequently, in a neighbourhood of I, this deformation may be projected to a deformation of ξ to τ_{n+1}.

PROPOSITION 2. *Let* $\xi, \eta \in N_n(k)$ *be such that* $\dim C(\xi) = \dim C(\eta)$ (*resp.* $e(\xi) = e(\eta)$). *If* Γ *is a curve of* N_n *through* ξ *lying generically in* η^{GL_n} (*so that* $\eta > \xi$), *and if* $B \in C(\xi)$ (*resp. if* $B \in C(\xi/\xi^2)$) *then there is a curve* Δ *in* S_n *through* (ξ, B) *lying over* Γ.

Proof. We may suppose that $\dim C(\gamma) = \dim C(\eta)$ for all $\gamma \in \Gamma$. Let $p: S_n \to N_n$ be the canonical projection. The first statement to be proved is equivalent to $S(\xi) = p^{-1}(\xi) \subset \overline{p^{-1}(\Gamma \cap \eta^{GL_n})}$. In fact we shall prove that $p^{-1}(\Gamma)$ is irreducible. For this purpose consider an irreducible component V of $p^{-1}(\Gamma)$ containing the zero-section $\Gamma \times \{0\} \subset p^{-1}(\Gamma)$. Let (ζ, σ) be a point of V, which is contained in no other irreducible component and where $p \mid V$ has minimal fibre dimension. Then

$p^{-1}(\zeta) \rightrightarrows C(\zeta)$ is contained in V; hence the minimal fibre dimension of $p \mid V$ is $\dim C(\zeta) = \dim C(\eta)$. It follows that $\dim (p \mid V)^{-1}(\gamma) \geqq \dim C(\eta)$ for all $\gamma \in \Gamma$, hence that $(p \mid V)^{-1}(\gamma) = S(\gamma)$ and $V = p^{-1}(\Gamma)$. (In fact we prove that a morphism of algebraic varieties which has a section, and whose fibres are irreducible of constant dimension, is universally open.) A similar proof holds for the second part of proposition 2.

PROPOSITION 3. *Let $\eta \in N_m(k)$, $\xi \in N_n(k)$ and $B \in C(\eta \times \xi)$ such that $B \mid \eta \times \eta$ is non degenerate. Then there is an automorphism of $\eta \times \xi$ which maps B into $C(\eta) \oplus C(\xi) \subset C(\eta \times \xi)$.*

Proof. Let H be the orthogonal projection of ξ onto η with respect to B, and set $g = \begin{pmatrix} \mathbf{1} & -H \\ 0 & \mathbf{1} \end{pmatrix} \in \mathrm{GL}(\eta \oplus \xi)$. Then g maps η identically onto η and maps ξ bijectively onto the orthogonal supplement of η with respect to B. The formula $B^g(x, y) = B(gx, gy)$ shows that η and ξ are orthogonal with respect to B^g. If we can prove that g is an automorphism of the algebra structure, it will follow that $B^g \in C(\eta) \oplus C(\xi)$.

In order to prove that g is an automorphism, we first prove that the socle of η is the orthogonal subspace of $\eta^{\cdot 2}$ in η with respect to B: in fact, we have $sx = 0$ for all $x \in \eta$ iff $B(sx, y) = 0$ for all $x, y \in \eta$, and this holds iff $B(s, xy) = 0$.

Then we prove that H maps ξ into the socle of η: indeed, if $x, y \in \eta$ and $z \in \xi$ we have $B(Hz, xy) = B(z, xy) = B(zx, y) = 0$. Finally we observe that $H(\xi^{\cdot 2}) = 0$. In fact, if $x, y \in \xi$, we have $B(z, H(xy)) = B(z, xy) = B(zx, y) = 0$, for all $z \in \eta$.

Now take $x \in \eta$ and $y \in \xi$. Then $(gx)(gy) = x(y - Hy) = -xHy = 0 = g(0) = g(xy)$. Similarly, if $x, y \in \xi$, we have $(gx)(gy) = (x - Hx)(y - Hy) = (Hx)(Hy) + xy = xy = xy - H(xy) = g(xy)$. Finally, if $x, y \in \eta$, we have $(gx)(gy) = xy = g(xy)$.

Remark. Call an algebra-structure $\zeta \in N_n(k)$ *colocal* if it has a socle of dimension 1. This is equivalent to saying that the algebra with unit $k \oplus \zeta$ is symmetric (= self-injective). Clearly, if $\eta \in N_n(k)$, a form $A \in C(\eta)$ is non degenerate iff $\mathrm{Ex}(\eta, A) \in N_{n+1}(k)$ is colocal.

We therefore say that η is *presymmetric* if there exists a non-degenerate $A \in C(\eta)$. The presymmetric algebras are obtained by dividing the maximal ideal of a local symmetric algebra by its socle.

COROLLARY 2. *Suppose $\xi \in N_n(k)$ is such that $S(\xi) \subset \overline{S(\tau_n)^{\mathrm{GL}_n}}$. Then $S(\tau_m \times \xi) \subset \overline{S(\tau_{n+m})^{\mathrm{GL}_{n+m}}}$.*

Proof. By proposition 3, it suffices to show that

$$\{\tau_m \times \xi\} \times (C(\tau_m) \oplus C(\xi)) \subset \overline{S(\tau_{n+m})^{\mathrm{GL}_{n+m}}}.$$

But our assumption implies that $\{\tau_m \times \xi\} \times (C(\tau_m) \oplus C(\xi)) \subset \overline{S(\tau_m \times \tau_n)^{\mathrm{GL}_{n+m}}}$, and a general member ζ of $\mathrm{Ex}\,(S(\tau_m \times \tau_n))$ has $e(\zeta) = 2$ and is colocal. So by the remark following corollary 1, $S(\tau_m \times \tau_n) \subset \overline{S(\tau_{m+n})^{\mathrm{GL}_{m+n}}}$ (apply corollary 1 to ζ and $\eta = \tau_{m+n+1} \in \mathrm{Ex}\,(S(\tau_{m+n})))$.

COROLLARY 3. *We have* $S(\varphi_n) \subset \overline{S(\tau_n)^{\mathrm{GL}_n}}$.

This results from $\varphi_n \xrightarrow{\sim} \tau_1 \times \tau_1 \times \cdots \times \tau_1$, n times.

Remark. This corollary more directly follows from the fact that the curve $(\lambda\tau_n, I_n)$, $\lambda \in k$, is in S_n (think of $\tau_n \in \mathrm{Hom}_k\,(k^n \otimes_k k^n, k^n)$ to define $\lambda\tau_n$), and for $\lambda = 0$, this is (φ_n, I_n), which has an open orbit in $S(\varphi_n)$ under $\mathrm{Aut}\,(\varphi_n) = \mathrm{GL}_n$.

PROPOSITION 4. *Suppose* $\eta \in N_{n+1}(k)$. *Let* $\eta = \mathrm{Ex}\,(\xi, B)$, $(\xi, B) \in S_n$. *Then* $e(\xi) \leqq e(\eta) \leqq e(\xi) + 1$, *and* $e(\eta) = e(\xi) + 1$ *iff the algebra extension* $0 \to ke_{n+1} \to \eta \to \xi \to 0$ *is trivial, i.e. iff* $B(x, y) = f(xy)$ *for some* $f: \xi \to k$.

Proof. The only point is the implication $e(\eta) = e(\xi) + 1 \Rightarrow \eta \xrightarrow{\sim} \mathrm{Ex}\,(\xi, 0)$. Now, since $e(\eta) = e(\xi) + 1$, $\eta^{\cdot 2} \cap ke_{n+1} = (0)$. Take a supplement U of ke_{n+1} in k^{n+1} containing $\eta^{\cdot 2}$. Then U is a subalgebra of η which is isomorphic to ξ and $\eta \xrightarrow{\sim} U \times ke_{n+1}$, QED.

In general, the uniserial structure furnishes one irreducible component for N_n and for S_n according to the following

THEOREM 1. *Let* $\tau_n \in N_n(k)$ *be the uniserial structure.*

(i) *The orbit* $\tau_n^{\mathrm{GL}_n}$ *is an open, smooth, rational subscheme of* N_n *with dimension* $n^2 - n$.

(ii) *Let* $p: S_n \to N_n$ *be the canonical projection. Then* $p^{-1}(\tau_n^{\mathrm{GL}_n})$ *is a smooth open subscheme of* S_n *with dimension* n^2.

(iii) *Let* $\Omega = \{g \in \mathrm{GL}_n$: *all diagonal minors of* g *invertible*$\}$ *be the big cell of* GL_n *with respect to the Borel group* $B(n)$ *of upper triangular matrices and the torus* $T(n)$ *of diagonal matrices. Call* f (*resp.* f_0) *the orbit morphism* $\Omega \to S_n : g \mapsto (\tau_n, I_n)^g$ (*resp.* $\Omega \to N_n : g \mapsto \tau_n^g$) *restricted to* Ω, *the notation* I_n *being that of example* (1). *Then* f_0 *admits a section* s *such that the multiplication* $\mathrm{Aut}\,(\tau_n) \times \mathrm{Im}\,(s \circ f_0) \to \Omega$ *is an isomorphism. If* $\mathrm{char}\,(k) = p \geqq n+1$ *or* $p = 0$, *then* f *is quasi-finite and the orbit of* (τ_n, I_n) *is dense in* $p^{-1}(\tau_n^{\mathrm{GL}_n})$.

Proof. We first show that the orbit morphism $q: \mathrm{GL}_n \to N_n : g \mapsto \tau_n^g$ is smooth. We verify the functorial criterion (formal smoothness). Consider a commutative square

$$\begin{array}{ccc} \mathrm{Spec}\,(\bar{A}) & \xrightarrow{s} & \mathrm{GL}_n \\ \big\downarrow & \overset{t}{\nearrow} & \big\downarrow q \\ \mathrm{Spec}\,(A) & \xrightarrow[m]{} & N_n \end{array}$$

where $\bar{A} = A/I$, $I^2 = 0$, and m is a multiplication on A^n. We have to find $t: \mathrm{Spec}\,(A) \to \mathrm{GL}_n$ such that both triangles become commutative. The datum of s is equivalent to that of a basis $\bar{s}_1, \ldots, \bar{s}_n$ of $\bar{A}^n$ such that $\bar{s}_p = \bar{s}_1^p$. We have to lift this basis to an appropriate basis of A^n: lift $\bar{s}_1$ to s_1 and set $s_p = s_1^p$!

From this the first two assertions of (i) follow. For the third assertion of (i), note that a functorial description of $U = \tau_n^{\mathrm{GL}_n}$ is this: for any $A \in k$-Alg, $U(A)$ is formed by the A-algebra-structures on A^n which are isomorphic to $\omega \oplus \omega^{\otimes 2} \oplus \cdots \oplus \omega^{\otimes n}$, ω an invertible A-module. To see that $p^{-1}(U)$ is smooth of dimension n^2, let $\mathrm{Spec}\,(A) \to U$ be any morphism. We describe $\mathrm{Spec}\,(A) \times_{N_n} S_n$ as follows. Let ω be an invertible direct summand of A^n such that (A^n, m) is isomorphic to $\omega \oplus \omega^{\otimes 2} \oplus \cdots \oplus \omega^{\otimes n}$. Then $\mathrm{Spec}\,(A) \times_{N_n} S_n$ is the scheme over $\mathrm{Spec}\,(A)$ attached to the A-module of Hochschild cocycles relative to $\omega \oplus \omega^{\otimes 2} \oplus \cdots \oplus \omega^{\otimes n}$. This module is identified with $\bigoplus_{i=1}^n \mathrm{Hom}_A\,(\omega \otimes_A \omega^{\otimes i}, A) = \omega^{\otimes -2} \oplus \cdots \oplus \omega^{\otimes -n-1}$. To see that $\dim\,(U) = n^2 - n$ and hence $\dim\,(p^{-1}(U)) = n^2$, observe that

$$\mathrm{Aut}\,(\tau_n) = \left\{ \begin{pmatrix} a_1 & & \\ a_2 & a_1^2 & 0 \\ \vdots & \vdots & \ddots \\ \vdots & *_{ij} & \ddots \\ a_n & & a_1^n \end{pmatrix} : \begin{array}{l} a_1 \text{ invertible, } a_2, \ldots, a_n \text{ arbitrary} \\ *_{ij} = \text{polynomial in } a_1, \ldots, a_n \end{array} \right\}$$

$\mathrm{Aut}\,(\tau_n)$ is a subgroup of $B^-(n)$, the Borel group opposite to $B(n)$ relative to $T(n)$. Identify $B^-(n-1)$ with $\begin{pmatrix} 1 & 0 \\ 0 & B^-(n-1) \end{pmatrix} \subset B^-(n)$. Then the multiplication $\mathrm{Aut}\,(\tau_n) \times B^-(n-1) \times B_u(n) \to \Omega$ is an isomorphism, where $B_u(n)$ is the unipotent part of $B(n)$. The restriction of f_0 to $B^-(n-1) \times B_u(n)$ is an isomorphism onto U and its inverse s is the section we are looking for in assertion (iii). The rationality of U follows from this isomorphism.

Finally $\mathrm{Aut}\,(\tau_n, I_n) = G \rtimes \mu_{n+1}$, where G is a smooth unipotent group of dimension $[n/p]$, $p = \text{char}(k)$, with $[n/p] = 0$ for $p = 0$. Here we embedd the group μ_{n+1} of $(n+1)$-th roots of unity in GL_n by

$$x \mapsto \begin{pmatrix} x & & & \\ & x^2 & & 0 \\ & & \ddots & \\ 0 & & & x^n \end{pmatrix}$$

The subgroup G of $\text{Aut}(\tau_n)$ is identified with $k^{[n/p]}$ by the map

$$g = \begin{pmatrix} 1 & & & \\ a_2 & 1 & & \\ \vdots & & \ddots & \\ \cdot & *_{ij} & & \ddots \\ a_n & & & 1 \end{pmatrix} \mapsto (a_q)_{p|n+2-q},$$

whereas a_r is a polynomial in the a_q, $q < r$ and $p \mid n+2-q$, whenever $p \nmid n+2-r$, as is easily verified inductively with decreasing indices. This implies that the orbit of (τ_n, I_n) has dimension $n^2 - [n/p]$. QED.

§2. Description of S_n and N_n for $n \leqq 5$.

For $n \leqq 5$, N_n contains a finite number of orbits. We are going to list one (k-rational) structure α for each orbit, writing α as quotient of the maximal ideal $I = (X_1, \ldots, X_e)$ of $k[X_1, \ldots, X_e]$ plus basis $\langle X_1, \ldots, X_e, \ldots \rangle$, $e = e(\alpha)$. Let $J = (f_1, \ldots, f_s) \subset I$ be an ideal defining α as quotient, and suppose that $\{f_1, \ldots, f_s\}$ is a minimal set of generators for J. Then the numbers n, e, s, $\dim C(\alpha)$ are related by the equation

$$\dim C(\alpha) = n + s - e.$$

This follows from the exact sequence ($V^* = k$-dual of V)

$$0 \to (\alpha/\alpha^{\cdot 2})^* \to \alpha^* \to C(\alpha) \to H^2_s(\alpha) \to 0$$

of k-vectorspaces and from the k-linear isomorphism $(J/IJ)^* \xrightarrow{\sim} H^2_s(\alpha, k)$ sending a form $f: J/IJ \to k$ to the class of the extension $0 \to k \to k \oplus_J I \to \alpha \to 0$, where $k \oplus_J I$ denotes the fibre sum defined by the maps $J \to J/IJ \xrightarrow{f} k$ and $J \hookrightarrow I$. Observe that $s \geqq e$ by the theorem of Krull–Chevalley–Samuel, equality holding iff α is a complete intersection. It follows that $\dim C(\alpha) \geqq n$ for all $\alpha \in N_n(k)$. In each N_n, $n \leqq 5$, we order the structures by increasing cocycle-space dimension.

	Structure	Space of cocycles
N_1	$\alpha_1 = \tau_1$	kI_1
N_2	$\beta_1 = \tau_2$	$kI_1 \oplus kI_2$
	$\beta_2 = \varphi_2$	$\mathbf{M}_2^s(k)$
N_3	$\gamma_1 = \tau_3$	$\bigoplus_{j=1}^{3} kI_j$
	$\gamma_2 = (X, Y)/(X^2, Y^2)$; $\langle X, Y, XY\rangle$	$\begin{pmatrix} \mathbf{M}_2^s(k) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix}$
	$\gamma_3 = (X, Y)/(X^3, XY, Y^2)$; $\langle X, X^2, Y\rangle$	$\left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & 0 & 0 \\ a_{13} & 0 & a_{33} \end{pmatrix} \middle| a_{ij} \in k \right\}$
	$\gamma_4 = \varphi_3$	$\mathbf{M}_3^s(k)$
N_4	$\delta_1 = \tau_4$	$\bigoplus_{j=1}^{4} kI_j$
	$\delta_2 = (X, Y)/(XY, Y^2 + X^3)$; $\langle X, X^2, X^3, Y\rangle$	$\left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{14} & 0 & 0 & a_{44} \end{pmatrix} \middle| a_{ij} \in k \right\}$
	$\delta_3 = (X, Y)/(XY, X^3, Y^3)$; $\langle X, X^2, Y, Y^2\rangle$	$\left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{12} & 0 & 0 & 0 \\ a_{13} & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{34} & 0 \end{pmatrix} \middle| a_{ij} \in k \right\}$
	$\delta_4 = (X, Y)/(XY, Y^2, X^4)$; $\langle X, X^2, X^3, Y\rangle$	$\left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{12} & a_{13} & 0 & a_{24} \\ a_{13} & 0 & 0 & 0 \\ 0 & a_{24} & 0 & 0 \end{pmatrix} \middle| a_{ij} \in k \right\}$

	Structure	Space of cocycles
	$\delta_5=(X,Y)/(Y^2,X^3,X^2Y)$; $\langle X, X^2, Y, XY\rangle$	$\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & 0 & a_{14} & 0 \\ a_{13} & a_{14} & a_{33} & 0 \\ a_{14} & 0 & 0 & 0 \end{pmatrix} \middle\vert\, a_{ij}\in k\right\}$
	$\delta_6=(X,Y,Z)/(XY,XZ,YZ, X^2-Y^2, X^2-Z^2)$; $\langle X, Y, Z, X^2\rangle$	$\begin{pmatrix} \mathbf{M}_3^s(k) & \begin{matrix}0\\0\\0\end{matrix} \\ 0\ 0\ 0 & 0 \end{pmatrix}$
	$\delta_7=(X,Y,Z)/(XY,XZ,YZ,X^2, Y^2-Z^2)$; $\langle X, Y, Z, Y^2\rangle$	$\begin{pmatrix} \mathbf{M}_3^s(k) & \begin{matrix}0\\0\\0\end{matrix} \\ 0\ 0\ 0 & 0 \end{pmatrix}$
	$\delta_8=(X,Y,Z)/(XY,XZ,YZ,Y^2, Z^2,X^3)$; $\langle X, Y, Z, X^2\rangle$	$\begin{pmatrix} \mathbf{M}_3^s(k) & \begin{matrix}0\\0\\0\end{matrix} \\ 0\ 0\ 0 & 0 \end{pmatrix} \oplus k\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$
	$\delta_9=\varphi_4$	$\mathbf{M}_4^s(k)$
N_5	$\varepsilon_1=\tau_5$	$\bigoplus_{j=1}^{5} kI_j$
	$\varepsilon_2=(X,Y)/(XY,X^4-Y^2)$; $\langle X, X^2, X^3, X^4, Y\rangle$	$\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & a_{15} \\ a_{12} & a_{13} & 0 & 0 & 0 \\ a_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_{15} & 0 & 0 & 0 & a_{55} \end{pmatrix} \middle\vert\, a_{ij}\in k\right\}$
	$\varepsilon_3=(X,Y)/(XY,X^3-Y^3)$; $\langle X, X^2, X^3, Y, Y^2\rangle$	$\left\{\begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} & 0 \\ a_{12} & 0 & 0 & a_{24} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_{14} & a_{24} & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle\vert\, a_{ij}\in k\right\}$

	Structure	Space of cocycles
	$\varepsilon_4=(X,Y)/(X^3,Y^2)$; $\langle X,Y,X^2,XY,X^2Y\rangle$	$\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ a_{12} & a_{22} & a_{14} & 0 & 0 \\ a_{13} & a_{14} & 0 & 0 & 0 \\ a_{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle\vert\, a_{ij}\in k\right\}$
	$\varepsilon_5=(X,Y)/(X^5,XY,Y^2)$; $\langle X,X^2,X^3,X^4,Y\rangle$	$\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{12} & a_{13} & a_{14} & 0 & 0 \\ a_{13} & a_{14} & 0 & 0 & 0 \\ a_{14} & 0 & 0 & 0 & 0 \\ a_{15} & 0 & 0 & 0 & a_{55} \end{pmatrix} \middle\vert\, a_{ij}\in k\right\}$
	$\varepsilon_6=(X,Y)/(X^4,XY,Y^3)$; $\langle Y,Y^2,X,X^2,X^3\rangle$	$\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{12} & 0 & 0 & 0 & 0 \\ a_{13} & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & a_{34} & a_{35} & 0 \\ 0 & 0 & a_{35} & 0 & 0 \end{pmatrix} \middle\vert\, a_{ij}\in k\right\}$
	$\varepsilon_7=(X,Y,Z)/(X^2,Y^2,Z^2,$ $XY-XZ-YZ)$; $\langle X,Y,Z,XZ,YZ\rangle$	$\begin{pmatrix} \multicolumn{3}{c}{} & 0 & 0 \\ & & & 0 & 0 \\ & \mathbf{M}^s_3(k) & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
	$\varepsilon_8=(X,Y)/(X^4,X^2Y,Y^2-X^3)$; $\langle X,Y,X^2,X^3,XY\rangle$	$\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{23} \\ a_{12} & a_{22} & a_{23} & 0 & a_{14} \\ a_{13} & a_{23} & a_{14} & 0 & 0 \\ a_{14} & 0 & 0 & 0 & 0 \\ a_{23} & a_{14} & 0 & 0 & 0 \end{pmatrix} \middle\vert\, a_{ij}\in k\right\}$

	Structure	Space of cocycles
	$\varepsilon_9=(X, Y, Z)/(X^2, Y^2, Z^2, YZ+XZ)$; $\langle X, Y, Z, XY, XZ\rangle$	$\begin{pmatrix} & & & 0 & 0 \\ & \mathbf{M}_3^s(k) & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
	$\varepsilon_{10}=(X, Y)/(X^4, X^2Y, Y^2)$; $\langle X, Y, X^2, X^3, XY\rangle$	$\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{23} \\ a_{12} & a_{22} & a_{23} & 0 & 0 \\ a_{13} & a_{23} & a_{14} & 0 & 0 \\ a_{14} & 0 & 0 & 0 & 0 \\ a_{23} & 0 & 0 & 0 & 0 \end{pmatrix} \middle\vert\, a_{ij}\in k\right\}$
	$\varepsilon_{11}=(X, Y, Z)/(Y^2, Z^2, XY, X^2-YZ)$; $\langle X, Y, Z, X^2, XZ\rangle$	$\begin{pmatrix} & & & 0 & 0 \\ & \mathbf{M}_3^s(k) & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
	$\varepsilon_{12}=(X, Y, Z)/(X^2, Y^2, Z^2, YZ)$; $\langle X, Y, Z, XY, XZ\rangle$	$\begin{pmatrix} & & & 0 & 0 \\ & \mathbf{M}_3^s(k) & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
	$\varepsilon_{13}=(X, Y, Z)/(X^2, Y^2, XZ, YZ, XY-Z^3)$; $\langle X, Y, Z, Z^2, Z^3\rangle$	$\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 \\ a_{13} & a_{23} & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{34} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle\vert\, a_{ij}\in k\right\}$

	Structure	Space of cocycles
	$\varepsilon_{14}=(X, Y, Z)/(Y^2, XY, YZ, XZ, X^2+Z^3)$; $\langle X, Y, Z, Z^2, Z^3\rangle$	$\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 \\ a_{13} & a_{23} & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{34} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle\vert\, a_{ij} \in k \right\}$
	$\varepsilon_{15}=(X, Y, Z)/(Z^2, Y^2, XY, XZ, X^3)$; $\langle X, Y, Z, X^2, YZ\rangle$	$\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 \\ a_{13} & a_{23} & a_{33} & 0 & 0 \\ a_{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle\vert\, a_{ij} \in k \right\}$
	$\varepsilon_{16}=(X, Y)/(X^3, X^2Y, XY^2, Y^3)$; $\langle X, Y, X^2, XY, Y^2\rangle$	$\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{12} & a_{22} & a_{14} & a_{15} & a_{25} \\ a_{13} & a_{14} & 0 & 0 & 0 \\ a_{14} & a_{15} & 0 & 0 & 0 \\ a_{15} & a_{25} & 0 & 0 & 0 \end{pmatrix} \middle\vert\, a_{ij} \in k \right\}$
	$\varepsilon_{17}=(X, Y, Z)/(Y^2, YZ, XZ, Z^2-XY, X^3)$; $\langle X, Y, Z, X^2, Z^2\rangle$	$\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 \\ a_{13} & a_{23} & a_{33} & 0 & 0 \\ a_{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle\vert\, a_{ij} \in k \right\}$
	$\varepsilon_{18}=(X, Y, Z)/(X^2, Y^2, XY, XZ, YZ, Z^4)$; $\langle X, Y, Z, Z^2, Z^3\rangle$	$\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} \\ 0 & 0 & a_{34} & a_{35} & 0 \\ 0 & 0 & a_{35} & 0 & 0 \end{pmatrix} \middle\vert\, a_{ij} \in k \right\}$

	Structure	Space of cocycles
	$\varepsilon_{19} = (X, Y, Z)/(Z^2, XZ, YZ, XY, X^3, Y^3)$; $\langle X, Y, Z, X^2, Y^2 \rangle$	$\left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ a_{12} & a_{22} & a_{23} & 0 & a_{25} \\ a_{13} & a_{23} & a_{33} & 0 & 0 \\ a_{14} & 0 & 0 & 0 & 0 \\ 0 & a_{25} & 0 & 0 & 0 \end{pmatrix} \middle\| a_{ij} \in k \right\}$
	$\varepsilon_{20} = (X, Y, Z)/(Y^2, Z^2, YZ, XZ, X^3, X^2Y)$; $\langle X, Y, Z, X^2, XY \rangle$	$\left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{12} & a_{22} & a_{23} & a_{15} & 0 \\ a_{13} & a_{23} & a_{33} & 0 & 0 \\ a_{14} & a_{15} & 0 & 0 & 0 \\ a_{15} & 0 & 0 & 0 & 0 \end{pmatrix} \middle\| a_{ij} \in k \right\}$
	$\varepsilon_{21} = (X, Y, Z, W)/(X^2, Y^2, Z^2, W^2, XY, XZ, YW, ZW, XW - YZ)$; $\langle X, Y, Z, W, XW \rangle$	$\begin{pmatrix} \mathbf{M}_4^s & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} & 0 \end{pmatrix}$
	$\varepsilon_{22} = (X, Y, Z, W)/(X^2, Y^2, XZ, XW, YZ, YW, ZW, W^2, XY - Z^2)$; $\langle X, Y, Z, W, Z^2 \rangle$	$\begin{pmatrix} \mathbf{M}_4^s & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} & 0 \end{pmatrix}$
	$\varepsilon_{23} = (X, Y, Z, W)/(X^2, Y^2, Z^2, W^2, XZ, XW, YZ, YW, ZW)$; $\langle X, Y, Z, W, XY \rangle$	$\begin{pmatrix} \mathbf{M}_4^s(k) & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} & 0 \end{pmatrix}$

	Structures	
	$\varepsilon_{24}=(X, Y, Z, W)/(Y^2, Z^2, W^2, XY, XZ, XW, YZ, YW, ZW, X^3)$; $\langle X, Y, Z, W, X^2\rangle$	$\left\{\left(\begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{12} & a_{22} & a_{23} & a_{24} & 0 \\ a_{13} & a_{23} & a_{33} & a_{34} & 0 \\ a_{14} & a_{24} & a_{34} & a_{44} & 0 \\ a_{15} & 0 & 0 & 0 & 0 \end{matrix}\right) \middle\vert\, a_{ij} \in k\right\}$
	$\varepsilon_{25}=\varphi_5$	$\mathbf{M}_5^s(k)$

§3. The irreducibility of N_1, N_2, N_3, N_4, N_5, N_6.

We proceed in three steps. In the first, we show that S_1, S_2, S_3 are irreducible, and hence so are N_1, N_2, N_3, N_4. In the second, resp. third we show that N_5, resp. N_6 are irreducible.

First step. It is clear that S_1, S_2 are irreducible since $\alpha_1=\tau_1$, $\beta_1=\tau_2$, $\beta_2=\varphi_2$, and corollary 3 applies. In S_3, observe $\tau_3 > \gamma_2$ and $\dim C(\tau_3)=\dim C(\gamma_2)$, hence by proposition 2, every cocycle over γ_2 is a specialization of a cocycle over τ_3. Since a general member $B\in C(\gamma_3)$ is non-degenerate, $\mathrm{Ex}\,(\gamma_3, B)$ is colocal with embedding dimension two. Hence by corollary 1 and the remark following this corollary, there is a curve Γ in S_3 through (γ_3, B) and generically over $\tau_3^{\mathrm{GL}_3}$. Finally, $\gamma_4=\varphi_3$, so by corollary 3, we conclude that S_3 is irreducible.

Second step. The cocycles of $S(\delta_2)$ are specializations of cocycles over τ_4 since by the first step $\tau_4 > \delta_2$ and $\dim C(\delta_2)=\dim C(\tau_4)$ and proposition 2 applies.

Each $C(\delta_3)$ and $C(\delta_5)$ contain non-degenerate forms, so the argument used for $S(\gamma_3)$ above works again: The cocycles of $S(\delta_3)$ and of $S(\delta_5)$ are specializations of those over τ_4.

Observe that $\delta_4 \twoheadrightarrow \tau_3\times\tau_1$, so corollary 2 applies to $S(\delta_4)$. The cocycles in $S(\delta_7)$ are specializations of those in $S(\delta_6)$. We have $\delta_8 \twoheadrightarrow \varphi_3\times\tau_2$, so the cocycles in $S(\delta_8)$ are specializations of those lying over τ_4 by corollary 2. We are left with $C(\delta_6)$. We shall show within the third step that the structure $\varepsilon_7 \twoheadrightarrow \mathrm{Ex}\,(\delta_6, B)$ for general $B\in C(\delta_6)$ is a specialization of τ_5. From this it follows that N_5 is irreducible.

Third step. Let $\xi\in N_5(k)$ be of embedding dimension $\leqq 2$. Then either an extension $\mathrm{Ex}\,(\xi, B)$, $B\in C(\xi)$, is trivial or its embedding dimension is still $\leqq 2$. In the latter case, by the remark following corollary 1, $\mathrm{Ex}\,(\xi, B)$ is a specialization of τ_6; in the first case, this is trivial. Since by corollary 3, all cocycles over φ_5 are specializations of cocycles over τ_5, we are left with the investigation of cocycles lying over structures ε_i with $e(\varepsilon_i)=3$. or 4.

In embedding dimension four, note that $C(\varepsilon_{21})=C(\varepsilon_{22})=C(\varepsilon_{23})\overset{\sim}{\rightarrow}\mathbf{M}_4^s(k)$. So by proposition 2 and since $\varepsilon_{21}>\varepsilon_{22}>\varepsilon_{23}$ for trivial reasons, it is sufficient to consider a general cocycle in $C(\varepsilon_{21})$. Look at the specialization $\tau_5\rightarrow\varepsilon_{21}$ defined by the base change $X=e_1$, $Y=e_2/\lambda$, $Z=e_3/\lambda^2$, $W=e_4/\lambda^3$, $XW=e_5/\lambda^3$. Call this variable structure $\tau_5(\lambda)$, so $\tau_5(1)=\tau_5$ and $\tau_5(0)=\varepsilon_{21}$. We have

$$C(\tau_5(\lambda))\overset{\sim}{\rightarrow}\left\{\begin{pmatrix} b_1 & b_2 & b_3 & b_4 & \lambda b_5\\ b_2 & b_3 & b_4 & b_5 & 0\\ b_3 & b_4 & b_5 & 0 & 0\\ b_4 & b_5 & 0 & 0 & 0\\ \lambda b_5 & 0 & 0 & 0 & 0\end{pmatrix}\middle|\, b_i\in k\right\},$$

whence all the structures $\mathrm{Ex}\,(\varepsilon_{21}, B)$, where

$$B=\begin{pmatrix} b_1 & b_2 & b_3 & b_4 & 0\\ b_2 & b_3 & b_4 & b_5 & 0\\ b_3 & b_4 & b_5 & 0 & 0\\ b_4 & b_5 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix},$$

are specializations of τ_6. They are described as follows: Let X, Y, Z, W, be the canonical basis of $E=k^4$ and S, T the canonical basis of $F=k^2$. Identify I_4 and B with the bilinear forms they define on $E\times E$ with respect to X, Y, Z, W. Then $\mathrm{Ex}\,(\varepsilon_{21}, B)$ is this multiplication:

(i) $EF=FF=0$,

(ii) For $x, y\in E$, we have $xy=I_4(x,y)S+B(x,y)T$.

Write $B(x,y)=I_4(\sigma_B(x),y)$, $\sigma_b\in\mathrm{GL}\,(E)$. With respect to the basis X, Y, Z, W, σ_B has the matrix

$$\sigma_B=\begin{pmatrix} b_4 & b_5 & 0 & 0\\ b_3 & b_4 & b_5 & 0\\ b_2 & b_3 & b_4 & b_5\\ b_1 & b_2 & b_3 & b_4\end{pmatrix}$$

whose characteristic polynomial is

$$\chi_B = \det(\sigma_B - \mu\mathbf{1}) = (b_4-\mu)^4 - 3b_3b_5(b_4-\mu)^2 + 2b_2b_5^2(b_4-\mu) + b_3^2b_5^2 - b_1b_5^3.$$

Let Z be the 20-dimensional affine space consisting of pairs of symmetric 4×4-matrices. Consider the morphism $z:\mathbf{A}^5\times \mathrm{GL}_4 \to Z:(b_1, b_2, b_3, b_4, b_5; g) \mapsto (I_4^g, B^g)$. We show that z is dominant. This implies that for general B we get the general extension $\mathrm{Ex}(\varepsilon_{21}, B)$ of ε_{21}. Now, if $(I_4^g, B^g) = (I_4, B')$, B' being defined by $b_1', b_2', b_3', b_4', b_5'$ (like B), then $\chi_B = \chi_{B'}$. Hence, for fixed B, the possible B' define a one-dimensional variety in $\mathbf{A}^5$. On the other hand, the stabilizers of I_4 and of B have a finite intersection, if B is sufficiently general. So the generic fibre of z is one-dimensional, and z is dominant.

Remarks

(1) With the above notation, it is easily seen that the multiplication

(i) $EF = FF = 0$,

(ii) For $x, y \in E$, $xy = B_1(x, y)S + B_2(x, y)T$ with

$$B_1 = \begin{pmatrix} 1 & \lambda & 0 & 1 \\ \lambda & \lambda^2 - i\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i\lambda & 0 & \lambda \\ 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \end{pmatrix}$$

defines a one-parameter family $(\beta_\lambda)_{\lambda\in k}$ of structures which is generic among the extensions of ε_{21}. By an elementary but very long calculus, one finds the following curve $\Gamma_\lambda = \{\tau_6(t) : t \in k\backslash\{0\}\}$ in N_6 which defines a specialization $\tau_6 \to \beta_\lambda$: If $e_1, e_2, e_3, e_4, e_5, e_6$ is the canonical basis of k^6 we derive $\tau_6(t)$ from τ_6 by the new basis

$$\begin{aligned}
X &= a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 \\
Y &= b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5 \\
Z &= c_3e_3 + c_4e_4 + c_5e_5 \\
W &= d_4e_4 + d_5e_5 \\
&XW \\
&Z^2
\end{aligned}$$

where

$$a_1 = t^{24}$$
$$a_2 = \lambda^2(\lambda - i)t^3$$
$$a_3 = \tfrac{1}{4}\lambda(2\lambda - i)t^{-14} - \tfrac{1}{2}\lambda^4(\lambda - i)^2 t^{-18}$$
$$a_4 = \tfrac{8}{3}\lambda^4(1 + i\lambda)^2 t^{-25} + \tfrac{1}{3}\lambda t^{-31} + \tfrac{5}{6}\lambda^3(\lambda - i)(2\lambda - i)t^{-35} + \tfrac{2}{3}\lambda^6(\lambda - i)^3 t^{-39}$$
$$a_5 = \tfrac{8}{3}\lambda^6(\lambda - i)^3 t^{-46} + \tfrac{1}{96}\lambda^2(148\lambda^2 - 148i\lambda + 3)t^{-52}$$
$$\quad + \tfrac{17}{24}\lambda^5(\lambda - i)^2(i - 2\lambda)t^{-56} - \tfrac{19}{24}\lambda^8(\lambda - i)^4 t^{-60}$$
$$b_2 = t^7$$
$$b_3 = 2\lambda^2(\lambda - i)t^{-14}$$
$$b_4 = 2\lambda^2(\lambda - i)t^{-21} + \tfrac{1}{2}i\lambda t^{-31}$$
$$b_5 = \tfrac{2}{3}\lambda^4(\lambda - i)^2 t^{-42} - \tfrac{1}{3}\lambda t^{-48} + \tfrac{1}{6}\lambda^3(\lambda - i)(8\lambda + 5i)t^{-52} + \tfrac{1}{3}\lambda^6(\lambda - i)^3 t^{-56}$$
$$c_3 = t^{-12}$$
$$c_4 = 2\lambda^2(\lambda - i)t^{-33}$$
$$c_5 = 2\lambda^2(\lambda - i)t^{-40} + \tfrac{1}{4}\lambda(i - 2\lambda)t^{-50} - \tfrac{3}{2}\lambda^4(\lambda - i)^2 t^{-54}$$
$$d_4 = \lambda t^{-31}$$
$$d_5 = 3\lambda^3(\lambda - i)t^{-52}$$

(Check!)

(2) In contrast to this complicated specialization, it is easy to desingularize the local k-algebras $k[\beta_\lambda]$ having β_λ as maximal ideal.

Call a k-algebra A *weakly coupled* iff $A \xrightarrow{\sim} k[X_1, \ldots, X_s]/I + J + (X_1^{m_1+1}, \ldots, X_s^{m_s+1})$, where

(j) all the m_i satisfy $m_i > 1$,

(jj) the ideal I is contained in the ideal I_{mix} generated by the monomials in several variables,

(jjj) for $i \neq l$, $X_l X_i^{m_i - 1} \in I$,

(jv) the vectorspace J is contained in $\sum_{i=1}^s kX_i^{m_i}$.

PROPOSITION 5. *A weakly coupled k-algebra A with $e(A) = s$ is a specialization of the direct product of $s + 1$ algebras. In particular, if $I = I_{\text{mix}}$, then A is desingularizable.*

Proof. Write A as set of the k-linear combinations formed by $1_A, X_i, X_i^2, \ldots, X_i^{m_i}$, $i = 1, \ldots, s$, by mixed monomials $f_1, \ldots, f_r$ defining a basis for I_{mix}/I. The relations among these generators are determined by J.

Choose in $C = k^s \times k[X_1, \ldots, X_s]/(X_1^{m_1}, \ldots, X_s^{m_s}) + I$ the system of generators 1_C, $X_{i,1} = \lambda 1_i + X_i$, $X_{i,2} = X_i^2 - \lambda X_i, \ldots, X_{i,m_i-1} = X_i^{m_i-1} - \lambda X_i^{m_i-2}$, $X_{i,m_i} = -\lambda X_i^{m_i-1}$, $i = 1, \ldots, s$, and $f_1, \ldots, f_r$, where $\lambda \in k\backslash\{0\}$ and 1_i denotes the i-th primitive idempotent of C. The relations from J are transported into this system of generators by the isomorphism $X_i^{m_i} \mapsto X_{i,m_i}$. Now it is clear that the structure A_λ gotten from C by dividing through these relations among the X_{i,m_i} tends to A if $\lambda \to 0$. For $I = I_{\text{mix}}$, either $m_i > 2$, all i, and the $(s+1)^{\text{st}}$ factor of A_λ is again weakly coupled with $I = I_{\text{mix}}$. Or else, we have $m_1 = 2$, without loss of generality. Then either X_1 is linearly dependent of $X_2, \ldots, X_s$, and the embedding dimension diminishes, or X_1 is independent, and the $(s+1)^{\text{st}}$ factor can be deformed to a non-local structure by deforming the subalgebra $k[X_1]/(X_1^2)$ to $k \times k$. In either case, the induction works since new weakly coupled algebras with $I = I_{\text{mix}}$ are produced. Finally, we get a specialization of k^n to A, $n = $ rank of A, if $I = I_{\text{mix}}$. QED.

In particular, the generic extensions of ε_{21} which may be defined by the two bilinear forms

$$B_1 = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_4 \end{pmatrix}$$

as above, are desingularizable.

In embedding dimension four we are left with the cocycles $B \in S(\varepsilon_{24})$. We have $\varepsilon_{24} \xrightarrow{\sim} \tau_2 \times \varphi_3$, so corollary 2 solves this case. This concludes the discussion of embedding dimension four.-

The most interesting case is embedding dimension three. We first discuss the algebras ε_{13}, ε_{14}, ε_{18} having non-vanishing third powers.

An extension $\sigma = \text{Ex}(\varepsilon_{13}, B)$ on $\varepsilon_{13} \oplus ke_6$ has multiplication $\sigma e_6 = 0$, $a \underset{\sigma}{\cdot} b = a \underset{\varepsilon_{13}}{\cdot} b + B(a, b)e_6$ for $a, b \in \varepsilon_{13}$. Choose the basis X, Y, Z, $Z \underset{\sigma}{\cdot} Z$, $Z \underset{\sigma}{\cdot} Z \underset{\sigma}{\cdot} Z$, e_6 in σ. Now, this new structure σ' has $Z^{\cdot 3} := Z \underset{\sigma}{\cdot} Z \underset{\sigma}{\cdot} Z$ in its socle, so $\sigma' \xrightarrow{\sim} \text{Ex}(\sigma'/kZ^{\cdot 3}, \gamma)$ where $e(\sigma'/kZ^{\cdot 3}) = 3$, and $(\sigma'/kZ^{\cdot 3})^{\cdot 3} = 0$. So the algebras lying over ε_{13} (i.e. coming from $S(\varepsilon_{13})$) are structures coming from cocycles lying over algebras of embedding dimension three and having vanishing third powers. These are discussed below.

Since $\varepsilon_{14} \tilde{\rightarrow} \delta_2 \times \tau_1$, by corollary 2, and because δ_2-cocycles are specializations of τ_4-cocycles (cf. 2nd step) we recognize the cocycles over ε_{14} as specializations of cocycles over τ_5.

As $\varepsilon_{18} \tilde{\rightarrow} \varphi_2 \times \tau_3$, corollary 2 applies to view cocycles over ε_{18} as specializations of cocycles over τ_5.

We are left with the structures of embedding dimension three and having vanishing third powers (together with their cocycles). There are two subsets 1st set $= \{\varepsilon_{15}, \varepsilon_{17}, \varepsilon_{19}, \varepsilon_{20}\}$, and 2nd set $= \{\varepsilon_7, \varepsilon_9, \varepsilon_{11}, \varepsilon_{12}\}$ of this set of algebras which we treat differently.

The first set is easy, because $\varepsilon_{15} \tilde{\rightarrow} \tau_3 \times \gamma_2$, $\varepsilon_{19} \tilde{\rightarrow} \tau_1 \times \delta_3$, $\varepsilon_{20} \tilde{\rightarrow} \tau_1 \times \delta_5$. The cocycles in $S(\delta_3)$, $S(\delta_5)$ are specializations of $S(\tau_4)$ by the discussion of S_4. As S_3 is irreducible $S(\tau_3)$ specializes to $S(\delta_3)$. Hence by corollary 2, $S(\varepsilon_{15})$, $S(\varepsilon_{19})$, $S(\varepsilon_{20})$ are specializations of $S(\tau_5)$. As to ε_{17}, note that $\dim C(\varepsilon_{17}) = \dim C(\varepsilon_{15})$, so if we show that $\varepsilon_{15} > \varepsilon_{17}$, proposition 2 applies to get the cocycles over ε_{17}. For any $\lambda \in k \backslash \{0\}$, consider the structure $(X, Y, Z)/((Y^2, X^3, XZ, \lambda Z^2 - YZ, Z^2 - YX)$ with basis $\langle X, Y, Z, X^2, Z^2 \rangle$. If one puts $X' = Y + \lambda^2 X - \lambda Z$, $Y' = Y$, $Z' = Z - (1/2\lambda) Y$, one sees that this structure is isomorphic to ε_{15}. But for $\lambda = 0$ we get ε_{17}, as desired. This ends the discussion of the first set of structures.

In view of $\mathbf{M}_3^s(k) \tilde{\rightarrow} C(\varepsilon_7/\varepsilon_7^2) \tilde{\rightarrow} C(\varepsilon_7) = C(\varepsilon_9) = C(\varepsilon_{11}) = C(\varepsilon_{12})$ and by proposition 2, it suffices to show that

$$\begin{array}{ccc} & \varepsilon_7 & \\ & \vee & \\ & \varepsilon_9 & \\ \swarrow & & \searrow \\ \varepsilon_{11} & & \varepsilon_{12} \end{array}$$

holds, and that $\operatorname{Ex}(S(\tau_5))$ specializes to $\operatorname{Ex}(S(\varepsilon_7))$ in order to handle this last set of structures.

Consider the family $\varepsilon_7(\lambda) \tilde{\rightarrow} (X, Y, Z)/(X^2, Y^2, Z^2, \lambda XY - XZ - YZ)$ with $\varepsilon_7(\lambda) \tilde{\rightarrow} \varepsilon_7$ for $\lambda \neq 0$ and $\varepsilon_7(0) \tilde{\rightarrow} \varepsilon_9$, thus $\varepsilon_7 > \varepsilon_9$. The family $\varepsilon_9(\lambda) \tilde{\rightarrow} (X, Y, Z)/(X^2, Y^2, Z^2, \lambda YZ + XZ)$ specializes to $\varepsilon_9(0) \tilde{\rightarrow} \varepsilon_{12}$, and $\varepsilon_9(\lambda) \tilde{\rightarrow} \varepsilon_9$ for $\lambda \neq 0$. To get $\varepsilon_9 > \varepsilon_{11}$, note that $\varepsilon_9 \tilde{\rightarrow} (X, Y, Z)/(Y^2, Z^2, XY, X^2 - XZ)$ which clearly specializes to ε_{11}.

To handle the structures in $\operatorname{Ex}(S(\varepsilon_7))$, consider the specialization $\tau_5 \rightarrow \varepsilon_7$ given by the family $x = e_1$, $y = (1/\lambda)e_2$, $z = (1/\lambda^2)e_3 - (1/\lambda^4)e_5$, $u = (1/\lambda^2)e_4$, $v = (1/\lambda^3)e_5$ of bases which define a family $(\tau_5(\lambda))_{\lambda \in k}$ of structures isomorphic to τ_5

for $\lambda \neq 0$, and such that $\tau_5(0) = \varepsilon_7$. The cocycle-spaces are

$$C(\tau_5(\lambda)) \rightrightarrows \left\{ \begin{pmatrix} b_1 & b_2 & b_3-b_5 & \lambda b_4 & \lambda b_5 \\ b_2 & b_3 & b_4 & \lambda b_5 & 0 \\ b_3-b_5 & b_4 & b_5 & 0 & 0 \\ \lambda b_4 & \lambda b_5 & 0 & 0 & 0 \\ \lambda b_5 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \begin{array}{l} b_i \in k \\ \text{all } i = 1, 2, 3, 4, 5 \end{array} \right\}$$

Hence we can lift the curve $(\tau(\lambda))_{\lambda \in k}$ in N_5 to a curve in S_5 passing through every couple

$$\left(\varepsilon_7, \begin{pmatrix} b_1 & b_2 & b_3-b_5 & 0 & 0 \\ b_2 & b_3 & b_4 & 0 & 0 \\ b_3-b_5 & b_4 & b_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right), \ b_i \in k, \quad \text{all} \quad i = 1, 2, 3, 4, 5,$$

in $S(\varepsilon_7)$. The algebra-extension defined by such a couple has the following description. Set $E = ke_1 \oplus ke_2 \oplus ke_3$ and $F = ke_4 \oplus ke_5 \oplus ke_6$, such that $k^6 = E \oplus F$. Call A, B, C the three symmetric bilinear forms on $E \times E$ defined by the matrices

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} b_1 & b_2 & b_3-b_5 \\ b_2 & b_3 & b_4 \\ b_3-b_5 & b_4 & b_5 \end{pmatrix}$$

with respect to $\langle e_1, e_2, e_3 \rangle$. Then the multiplication is $FE = FF = 0$, $xy = A(x, y)e_4 + B(x, y)e_5 + C(x, y)e_6$ for $x, y \in E$. We want to show that it is sufficient for our purpose to consider the coefficients $b_2 = b_3 = 0$, $b_4 = 1$ and $b_5 = b_1$. Call this structure $\alpha(b_1)$.

We now investigate the structures $\alpha(A, B, C)$ defined by an arbitrary triplet (A, B, C) of symmetric bilinear forms on $E \times E$ in the above way. Since Ex $(S(\varepsilon_7))$ is contained in this 18-dimensional irreducible set X of structures, we shall show that the set $X \cap (\bigcup_{b. \in k^5} \alpha(b.)^{\mathrm{GL}_6})$ is dense in X. Now, $\dim(\alpha(b.)^{\mathrm{GL}_6} \cap X) = 17$ if $b. \in k^5$ is sufficiently general. In fact, for general $b.$ we have $\alpha(b.)^{\mathrm{GL}_6} \cap X = \alpha(b.)^{\mathrm{GL}_3 \times \mathrm{GL}_3}$. Viewing a structure $\alpha(A, B, C)$ as a three-dimensional vectorspace V of symmetric bilinear forms on $E \times E$ plus a basis of V, the action of $\mathrm{GL}_3 \times \mathrm{GL}_3$ on $\alpha(A, B, C)$ becomes this: the first factor acts canonically on V. For general V, its orbit in the Grassmannian of all 3-dimensional subspaces of the space of the symmetric bilinear forms of $E \times E$ is 8-dimensional. The second factor simply acts as base-change.- Clearly, the subspace V defined by $\alpha(b.)$ also has an 8-dimensional orbit for general $b.$, whence $\alpha(b.)^{GL_3 \times GL_3} = 8 + 9 = 17$. Hence it suffices to find a GL_6-invariant rational function on X which is not constant on the set $\{\alpha(b_1) : b_1 \in k\}$. If $\alpha = \alpha(A, B, C) \in X$, consider the equation

$$0 = f_\alpha(\lambda, \mu, \nu) = \det(\lambda M_A + \mu M_B + \nu M_C),$$

where M_A, M_B, M_C are 3×3-matrices representing A, B, C in the basis $\langle e_1, e_2, e_3 \rangle$. For general α, this is the homogeneous equation of an elliptic curve $E_\alpha \subset \mathbf{P}_2$. Clearly, all $\xi \in X \cap \alpha^{GL_6}$ define isomorphic curves. So "the" modular invariant $j(E_\alpha)$ is a GL_6-invariant rational function. We calculate this function as a rational function of b_1 for structures $\alpha(b_1)$ in the following way: we have the cubic equation

$$0 = f_{\alpha(b_1)}(\lambda, \mu, \nu) = 2\lambda\mu^2 - 4b_1\mu^2\nu + 2\lambda\mu\nu - 4b_1\mu\nu^2 + 2b_1\lambda^2\nu - b_1\nu^3 - \lambda^3$$

For $b_1 \neq 0$, the point P with homogeneous coordinates $(0, 1, 0)$ is not a point of inflection of $E_{\alpha(b_1)}$. Hence there are four projective lines through P which are tangent to $E_{\alpha(b_1)}$ in points different from P. Call P_1, P_2, P_3, P_4 the four points on the line $\mu = 0$ cut out by the four tangents. Let $\Lambda = \Lambda(P_1, P_2, P_3, P_4)$ be the cross ratio of these four points, then the rational function $j = (\Lambda^2 - \Lambda + 1)^3/\Lambda(\Lambda - 1)^2$ is a well-known parameter for the four-points set $\{P_1, P_2, P_3, P_4\}$ on $\mu = 0$ yielding "the" modular invariant of $E_{\alpha(b_1)}$. The homogeneous coordinates $(\lambda_i, 0, 1)$, $i = 1, 2, 3, 4$ of P_i stem from the solution λ_i of the equation

$$0 = \lambda^4 - 4b_1\lambda^3 + (4b_1^2 + \tfrac{1}{2})\lambda^2 - b_1\lambda$$

which means the vanishing of the discriminant of the quadratic equation $0 = f_{\alpha(b_1)}(\lambda, \mu, 1)$ in μ. Putting $u = -4b_1$, $v = 4b_1^2 + \frac{1}{2}$, $w = -b_1$, we get

$$j = \frac{(v^2 - 3uw)^3}{w((uv)^2 - 4(v^3 + u^3w) - 27w^2 + 18uvw)}$$

which clearly is non-constant in b_1. QED.

Together with theorem 1, we conclude:

THEOREM 2. *The schemes N_n, $n = 1, 2, 3, 4, 5, 6$ are irreducible, rational of* *dimension $n^2 - n$, the orbit of the uniserial structure τ_n forming a smooth subscheme of N_n.*

COROLLARY 4. *Let* Alg_n *be the scheme of associative, unitary k-algebra-structures on k^n (§5.). Let* Alcom_n *be the closed subscheme of commutative structures, and denote by* $\mathrm{Alcomloc}_n \subset \mathrm{Alcom}_n$ *the reduced subscheme of local, commutative structures. Then for $n \leqq 7$,* $\mathrm{Alcomloc}_n$ *and (a fortiori)* Alcom_n *is irreducible.*

§4. Counterexamples

For $n > 6$, the schemes N_n are no longer irreducible. In fact, fix a subspace $E \subset k^n$ of dimension e. Let $S \subset k^n$ be any linear supplement (= complement) of E. Suppose $e(e+1)/2 \geqq n-e$, and pick a surjective linear map $B: \mathrm{Sym}_2(E) \to S$, where $\mathrm{Sym}_2(E)$ denotes the second symmetric power of E. Then we get a structure $E(S, B)$ in N_n by the rules:

(i) The product Sk^n vanishes.

(ii) If $x, y \in E$, then $xy = B(x \circ y)$, where $x \circ y$ is the class of $x \otimes y$ in $\mathrm{Sym}_2(E)$.

Since $E(S, B)^{\cdot 2} = S$, the morphism

$$E(?, ?): G(E) \to N_n : (S, B) \mapsto E(S, B)$$

is injective, where $G(E)$ denotes the irreducible scheme whose k-points are the above couples. Because of $\dim G(E) = \frac{1}{2}e(e+1)(n-e) + e(n-e)$, $\dim(E(?, ?)(G(E))) \geqq n^2 - n$ means that we consider couples $(e, n) \in \mathbf{N} \times \mathbf{N}$ satisfying

(i) the linear inequality $n - e \geqq 0$,

(ii) the parabolic inequality $e^2 + 3e - 2n \geqq 0$,

(iii) the elliptic inequality $ne^2 - e^3 + 3ne - 3e^2 - 2n^2 + 2n \geqq 0$.

These inequalities are clearly satisfied for any couple $(e, n) = (e, 2e)$ and $(e, n) = (e, 2e-1)$ for $e \geqq 4$. Hence for any $n \geqq 7$, the irreducible subset $E(?, ?)(G(E))$ of N_n is not dominated by the $(n^2 - n)$-dimensional orbit of τ_n. So:

PROPOSITION 6. *For $n \geqq 7$, N_n, and hence* $\mathrm{Alcomloc}_{n+1}$ *is not irreducible.*

PROPOSITION 7. *For $n \geqq 10$,* Alcom_n *is not irreducible.*

Proof. Choose the fundamental affine neighbourhood $U_n \subset \mathrm{Grass}_{n-1,n}$ consisting of the supplements of $\{0\} \times \cdots \times \{0\} \times k$ in k^n. This induces an algebraic choice of a basis for any $R \in U_n$. Hence every R bears the nilpotent structure $E(S, B)(R)$ defined by $E(S, B)$ and by the base-choice. Finally, pick a vector $\mathbf{1} \in k^n \backslash R$. These dates define a unique local structure $(S, B)(R, \mathbf{1})$ having $\mathbf{1}$ as unity and $E(S, B)(R)$ as maximal ideal. The irreducible subset $L(E)$ of Alcom_n consisting of these structures has dimension $n+(n-1)+\frac{1}{2}e(e+1)(n-1-e)+e(n-1-e)$. The condition $\dim L(E) \geqq n^2$ is the singular cubic inequality

$$n^2e - e^3 + 3ne - 4e^2 - 2n^2 - 3e + 4n - 2 \geqq 0 \tag{*}$$

So $L(E)$ is not dominated by the orbit of $k^{\times n}$ as soon as the following hold:

(i) the linear inequality $n - e - 1 \geqq 0$,

(ii) the parabolic inequality $e^2 + 3e - 2n + 2 \geqq 0$,

(iii) the cubic inequality (*) above.

It is clear that all couples $(e, n) = (e, e+4)$ for $e \geqq 6$ satisfy these inequalities, and that $(e, n) = (5, 11)$ is a solution of minimal embedding dimension five. QED.

§5. Two criteria for deformation of finite-dimensional algebras and the Hasse-diagram of the deformations of commutative algebras of dimension five.

In this paragraph, we are dealing with the scheme Alg_n whose functor on the category k-Alg takes the values

$$\mathrm{Alg}_n(A) = \left\{ \begin{array}{l} \xi \in (A^n) * \otimes_A (A^n),\ \xi \text{ defines on } A^n \text{ the structure} \\ \text{of an associative, unitary } A\text{-algebra} \end{array} \right\}$$

where $(A^n)* = A$-dual of A^n.

Like in §1. GL_n acts upon Alg_n by structural transport from the right. We carry over to Alg_n the notations of §1 concerning this action.

The first deformation criterion is concerned with central idempotents. Let Zip_n be the scheme whose functor on k-Alg takes the values

$$\mathrm{Zip}_n(A) = \left\{ \begin{array}{l} (\xi, i),\ \xi \in \mathrm{Alg}_n(A),\ i \in A^n, \text{ and } i \text{ is central} \\ \text{and idempotent for the structure } \xi \end{array} \right\}$$

LEMMA (P. Gabriel). *The projection $p: \mathrm{Zip}_n \to \mathrm{Alg}_n$ is an étale morphism.* (For the definition of an étale morphism, cf. [8; (IV, 17.1.1)].)

Idea of proof. The only non-trivial point is the verification that p is formally smooth, Let B be local, artinian in k-Alg. Take an ideal $I \subset B$ with $I^2 = 0$, and let ξ be a B-valued structure in Alg_n. The undirected graph of ξ has vertices S_i representing a complete system of simple ξ-modules. For $i \neq j$, there is an edge between S_i and S_j iff either $\mathrm{Ext}_\xi(S_i, S_j)$ or $\mathrm{Ext}_\xi(S_j, S_i)$ doesn't vanish. The connected components of this graph correspond one-to-one to the primitive central idempotents of ξ. The lemma now follows from the fact that $\mathrm{Ext}_{\xi/I\xi}(S_i, S_j)$ doesn't vanish if $\mathrm{Ext}_\xi(S_i, S_j)$ doesn't. QED.

THEOREM. *Let ξ, η be two k-rational structures in* Alg_n. *Let $\xi \cong \xi_1 \times \xi_2$, ξ_i being k-rational in* Alg_{n_i}, $i = 1, 2$. *Then $\eta > \xi$, iff there are k-rational structures η_i in* Alg_{n_i}, $i = 1, 2$, *satisfying $\eta_i > \xi_i$, $i = 1, 2$, and such that $\eta \cong \eta_1 \times \eta_2$.*

Idea of proof. Let the structures η_1, η_2 have the required properties. Then trivially $\eta_1 \times \eta_2 > \xi_1 \times \xi_2$. For the converse, observe that there is a GL_n-action on Zip_n by $(\zeta, i)^g := (\zeta^g, g^{-1}(i))$ for $g \in \mathrm{GL}_n(A)$ and $(\zeta, i) \in \mathrm{Zip}_n(A)$ such that the symbol $>$ of dominance makes sense on Zip_n too.- Let $\eta > \xi$. Call i_ξ the central idempotent corresponding to the factor ξ_1. From the lemma it follows that there is a central idempotent i_η in η with $(\eta, i_\eta) > (\xi, i_\xi)$. Let ηi_η denote the structure of the direct factor of η generated by the central idempotent i_η. Then it follows by a standard argument that $\eta i_\eta > \xi i_\xi$ and that $\eta(1_\eta - i_\eta) > \xi(1_\xi - i_\xi)$. QED.

The following criterion is concerned with semi-simple modules. It is quite useful while deforming non-commutative structures and has been used in [7]. We omit the proof since it is routine work in deformation theory.

THEOREM. *Let ξ, ξ' be two k-rational structures in* Alg_n. *Suppose that* (i) *to* (iii) *hold:*

(i) *We have $\xi > \xi'$.*

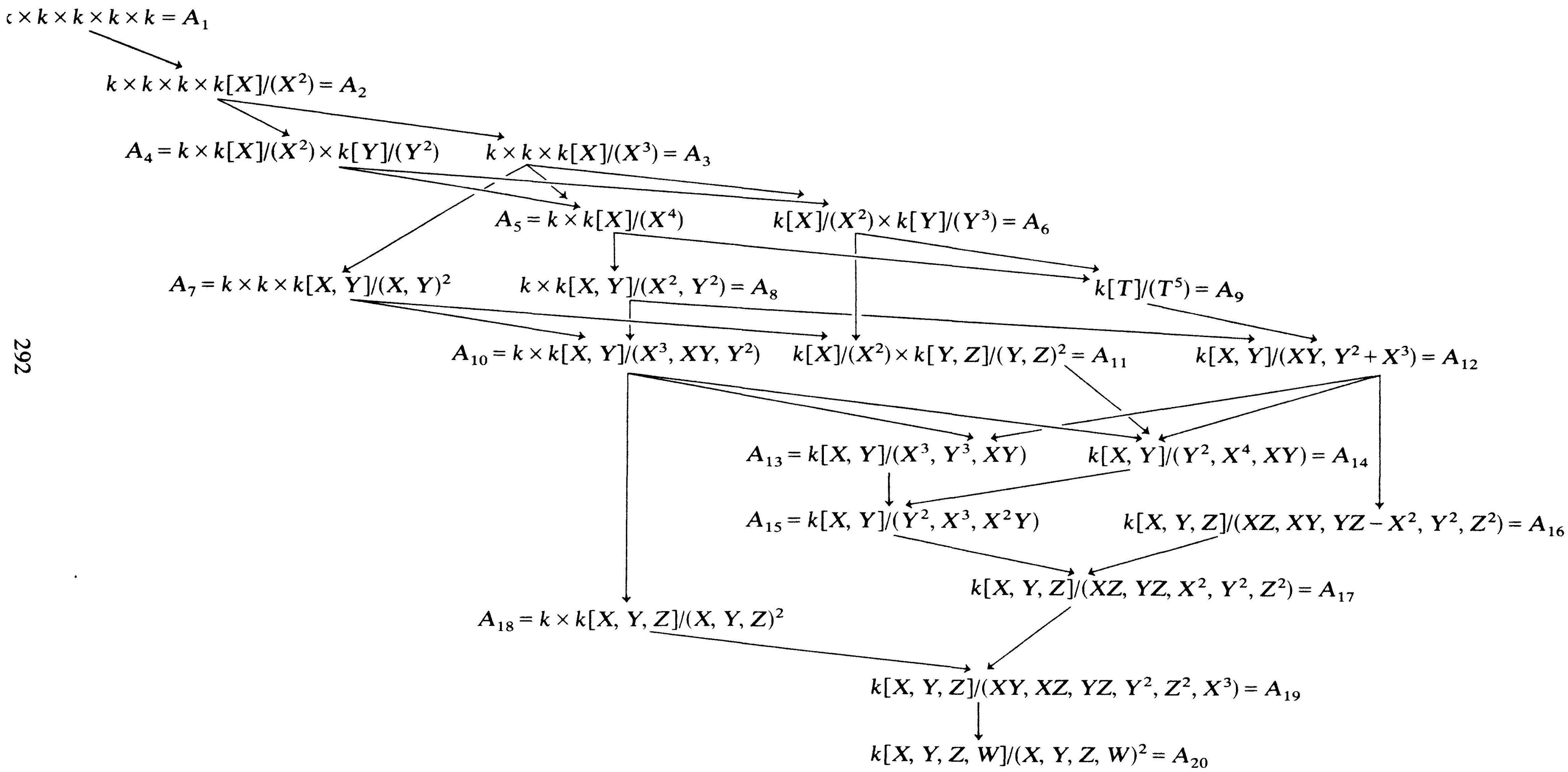
(ii) *Both structures ξ resp. ξ' have subalgebras L resp. L' which are isomorphic to k^r. Here we don't require coincidence of unities of L and ξ resp. of L' and ξ'.*

(iii) *There is only one equivalence class of subalgebras of ξ isomorphic to k^r under the action of* $\mathrm{Aut}(\xi)$. *Under these conditions, for every left-sub-L-module M of ξ there is a left-sub-L'-module M' of ξ' which is di-isomorphic to M.*

To finish this paragraph, we would like to include the Hasse-diagram of the deformations of commutative algebras of dimension five. Here an arrow $X \to Y$ means that Y deforms to X. Most of the deformations in the diagram are trivial. Let us merely point out two non-trivial ones:

(1) $A_8 \to A_{12}$. For $\lambda \in k\backslash\{0\}$, take the A_8-base 1, $\hat{X} = \lambda^2(1, 0) + X + Y$, $\hat{X}^2$,

The Hasse-diagram of deformations of commutative structures in Alg_5.



$\hat{X}^3$, $\hat{Y}=\lambda(X-Y)$. The relations among $\hat{X}$ and $\hat{Y}$ are defined by the singular cubic $\hat{X}^3+\lambda^2\hat{X}^2-\hat{Y}^2=0$ and the union of two lines $\hat{X}\hat{Y}=0$.

(2) $A_9 \to A_{12}$. For $\lambda \in k\backslash\{0\}$ we take the A_9-base 1, $X=T$, X^2, X^3, $Y=\lambda^3((T/\lambda^2)^2+(T/\lambda^2)^3+(T/\lambda^2)^4)$. The relations among X and Y are defined by the singular cubic $Y^2+X^3-\lambda XY=0$ and by the hyperbola $XY-\lambda^2Y+\lambda X^2=0$.

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Kapfsteig 31
CH-8032 Zürich

Received May 4, 1979