Open books and h-cobordisms.

Comment. Math. Helvetici 55 (1980) 330–346 Birkhäuser Verlag, Basel

Open books and h-cobordisms

JEAN-CLAUDE HAUSMANN

Let (W^{n+1}, M, N) be an h-cobordism between the closed manifolds M and N. Suppose that M is given with an open book decomposition with page V and monodromy α (see section 2 for the definition). If $n \geqslant 5$ and if the Whitehead torsion $\tau^0(W, M) \in \mathrm{Wh}(M)$ is in the image of $\mathrm{Wh}(\partial V) \to \mathrm{Wh}(M)$, our main result (Theorem 3.1) provides an open book decomposition for the manifold N, whose page T and monodromy β are closely related to V and α. (In fact, one has $\operatorname{int} V \approx \operatorname{int} T$). Although this resemblance between the two open book decomposition, the manifold N has, in general, not the simple homotopy type of an open book with page homeomorphic to V (Theorem 6.3).

It has been proven that any closed odd dimensional manifold M^n ($n \geqslant 5$) has an open book decomposition with page V satisfying $\pi_1(V) \cong \pi_1(M)$. For such an open book decomposition, any h-cobordism will satisfy the hypothesis of our theorem, which can thus be applied, in theory, to any h-cobordism over any closed odd dimensional manifold.

Our other main results concern the set $I_{\mathrm{CAT}}(M)$ of torsions of CAT-inertial h-cobordisms from M, when the manifold M is an open book. (An h-cobordism (W, M, N) is CAT-inertial if N is CAT-homeomorphic to M; CAT = DIFF or PL). This is done in Section 5. The results are the most complete when M is a stable open book (definition in Section 2). In this case, $I_{\mathrm{CAT}}(M)$ is strongly related to the set $E_{\mathrm{CAT}}(V)$ of torsion of CAT-tangential self-homotopy equivalences of V (V is the page of the stable open book M). Some aspects of this relationship have already been noticed in [La 1 and 2], [H-L], and [Ha 1].

We apply these results in Section 6 to perform concrete computations on $I_{\mathrm{CAT}}(L^3_{p,q} \times S^{2k})$, where $L^3_{p,q}$ is the 3-dimensional lens space and S^{2k} is the standard sphere. Among the interesting examples are two manifolds M and N which are h-cobordant but such that $I_{\mathrm{CAT}}(M) \neq \{0\}$ and $I_{\mathrm{CAT}}(N) = \{0\}$. We also construct examples where $\sigma \in I_{\mathrm{CAT}}(M)$ but $-\sigma \notin I_{\mathrm{CAT}}(M)$.

Section 1 contains the general definitions used throughout this paper. Section 2 is devoted to defininig open books and proving the basic facts we need about them. Although this paper is practically self-contained with regard to techniques involving open books, we would like to mention References [Wi], [Qu] and [La 3].

In these papers, the problem of finding an open book decomposition on a given manifold of high dimension is solved. In addition the reader will find there several historical remarks about open books and the role they play in the theory of manifolds. Concerning the Whitehead torsion and h-cobordism theories, we assume that the reader has some familiarity with them. References are [Co], [Mi 2] and [Ke].

The abbreviation CAT stands for PL or DIFF (= smooth C^∞). Strictly speaking, the arguments we give work for CAT = PL. We leave to the reader the standard adjustments needed to obtain valid proofs in the smooth category. Most of our results are likely to be true for topological manifolds as well.

1. Definitions and notations

(1.1) As was said in the introduction, the abbreviation CAT means "PL or DIFF" where "DIFF" is used for "smooth C^∞". The notation $M \simeq_{\mathrm{CAT}} N$ stands for: M is CAT-homeomorphic to N.

(1.2) BCAT is the classifying space BO if CAT = DIFF or BPL if CAT = PL.

(1.3) If V is a manifold, ∂V denotes its boundary. If a CAT-manifold M bounds CAT-manifolds A and B, we denote by $A \cup_\partial B$ the manifold constructed by gluing A to B along their common boundary M, with the identity of M as gluing map.

(1.4) If K is a connected finite complex, we denote by Wh (K) the Whitehead group Wh $(\pi_1(K))$ ([Co, Chapter III] or [Mi 2]). This coincides with the geometric definition of Wh (K) [Co, Chapter II, §6]. The abelian group Wh (K) has a $\mathbf{Z}[\mathbf{Z}_2]$-module structure generated by the involution $\sigma \to \bar{\sigma}$ [Mi 2, p. 373].

(1.5) If V is a compact CAT-manifold, Wh (V) must be understood as the abelian group Wh $(\pi_1(V))$ endowed with the $\mathbf{Z}[\mathbf{Z}_2]$-module structure for which we take into account the orientation character $\omega_V : \pi_1(V) \to \mathbf{Z}_2$ in the definition of the involution $\sigma \to \bar{\sigma}$ [Mi 2, p. 398].

(1.6) As in [Mi 2], the torsion $\tau(f)$ of a homotopy equivalence $f: X \to Y$ is measured in Wh (Y). We also use the "source torsion" $f_*^{-1}(\tau(f)) \in$ Wh (X) which simplifies the exposition in places and is denoted by $\tau^0(f)$. A homotopy equivalence is simple if $\tau(f) = 0$ (or equivalently $\tau^0(f) = 0$). Recall the composition formulas: for $X \xrightarrow{f} Y \xrightarrow{g} Z$:

$$\tau(g \circ f) = g_*(\tau(f)) + \tau(g)$$
$$\tau^0(g \circ f) = \tau^0(f) + f_*^{-1}(\tau^0(g)).$$

(1.7) Let X be a subcomplex of a finite complex Y. We denote by $i_{X,Y}: \mathrm{Wh}(X) \to \mathrm{Wh}(Y)$ the homomorphism induced by the inclusion $X \subset Y$. If this inclusion is a homotopy equivalence, we denote by $\tau(Y; X) \in \mathrm{Wh}(Y)$ its torsion, and $\tau^0(Y, X) = i_{X,Y}^{-1}(\tau(Y; X)) \in \mathrm{Wh}(X)$.

(1.8) A cobordism (W^{n+1}, M, N) (i.e. a compact CAT-manifold with $\partial W = M \bigcup_\partial N$) is called an h-cobordism if both inclusions $M \subset W$ and $N \subset W$ are homotopy equivalence. An s-cobordism is an h-cobordism with $\tau(W, M) = 0$ (and consequently with $\tau(W, N) = 0$ by the duality formula [Mi 2, p. 395]). If $n \geq 5$, the s-cobordism theorem ([Ke] for CAT = DIFF, [Hn] for CAT = PL) asserts that any CAT-homeomorphism $h: M \times 0 \cup \partial M \times I \to M \cup \partial M \times I \subset \partial W$ extends to a CAT-homeomorphism $H: M \times I \to W$.

(1.9) Let V be a compact CAT-manifold. Define $\mathrm{Aut}_{\mathrm{CAT}}(V \operatorname{rel} \partial V)$ as the group of CAT-homeomorphisms $\alpha: V \to V$ such that the restriction of α to a neighborhood of ∂V is the identity. The group structure is given by composition of automorphisms. We denote by $\mathrm{Aut}^c_{\mathrm{CAT}}(V \operatorname{rel} \partial V)$ the group of concordance classes of elements of $\mathrm{Aut}_{\mathrm{CAT}}(V \operatorname{rel} \partial V)$ (the concordances are relative to ∂V). If $\alpha \in \mathrm{Aut}_{\mathrm{CAT}}(V \operatorname{rel} \partial V)$, we denote by α^c its class in $\mathrm{Aut}^c_{\mathrm{CAT}}(V \operatorname{rel} \partial V)$.

2. Open books: definitions and basic facts

(2.1) Let V be a compact CAT-manifold of dimension $n-1$ with $\partial V \neq \emptyset$. Let $\alpha \in \mathrm{Aut}_{\mathrm{CAT}}(V, \operatorname{rel} \partial V)$ (Definition (1.9)). Let $\mathcal{M}_\alpha$ be the mapping torus of α, i.e. $\mathcal{M}_\alpha = V \times [-1, 1]/(x, -1) = (\alpha(x), 1)$. The boundary $\partial \mathcal{M}_\alpha$ admits a natural identification with $\partial V \times S^1$. The open book $\mathrm{OB}(V, \alpha)$ with page V and monodromy α is the closed CAT-manifold of dimension n defined by:

$$\mathrm{OB}(V, \alpha) = \mathcal{M}_\alpha \bigcup_\partial \partial V \times D^2.$$

To complete the terminology, ∂V is called the binding and $\operatorname{int} V$ the detached page of the open book. Observe that $\alpha \mid \operatorname{int} V$ is a CAT-automorphism of $\operatorname{int} V$ with compact support.

(2.2) Let $I = [0, 1]$. One has a canonical inclusion $V \times I \subset \mathrm{OB}(V, \alpha)$ which gives a decomposition $\mathrm{OB}(V, \alpha) = V \times I \cup_\partial W$ where W is the image of $V \times [0, -1] \cup \partial V \times D^2$ in $\mathrm{OB}(V, \alpha)$. Clearly, $(W, V \times 0, V \times 1)$ is an s-cobordism.

(2.3) LEMMA. *Let α and $\beta \in \mathrm{Aut}_{\mathrm{CAT}}(V^{n-1} \operatorname{rel} \partial V)$, $(n \geq 5)$. Then $\mathrm{OB}(V, \alpha)$ and $\mathrm{OB}(V, \beta)$ are s-cobordant relative $V \times I$ if and only if $\alpha^c = \beta^c$ in* $\mathrm{Aut}^c_{\mathrm{CAT}}(V \operatorname{rel} \partial V)$.

Proof. Let $\gamma : V\times I \to V\times I$ be a CAT-concordance relative ∂V from α to β. Then the manifold $R = \mathcal{M}_\gamma \cup \partial V \times I \times D^2$ is an s-cobordism rel $V\times I$ from $\mathrm{OB}(V, \alpha)$ to $\mathrm{OB}(V, \beta)$. Conversely, any such s-cobordism can be realized in this way, using the s-cobordism theorem.

(2.4) For a closed CAT-manifold M^n, a CAT-homeomorphism $h : \mathrm{OB}(V, \alpha) \to M$ is called an open book decomposition of M. Two open books decomposition $h_i : \mathrm{OB}(V, \alpha_i) \to M$ $(i = 0, 1)$ with page V are called concordant if there is an s-cobordism $(R, \mathrm{OB}(V, \alpha_0), \mathrm{OB}(V, \alpha_1))$ relative to $V\times I$ and a CAT-homeomorphism $H : (R, \mathrm{OB}(V, \alpha_0), \mathrm{OB}(V, \alpha_1)) \to (M\times I, M\times 0, M\times 1)$ such that $H \mid \mathrm{OB}(V, \alpha_i) = h_i$. By (2.3), this implies that $\alpha_0^c = \alpha_1^c$ in $\mathrm{Aut}^c_{\mathrm{CAT}}(V, \mathrm{rel}\, \partial V)$.

(2.5) Let $h : \mathrm{OB}(V, \alpha) \to M$ be an open book decomposition. The restriction of h to $V\times I$ defines a CAT-embedding $\mu : V\times I \to M$ such that $(M - \mathrm{int}\, \mu(V\times I), \mu(V\times 0), \mu(V\times 1))$ is an s-cobordism. On the other hand, if $\mu : V\times I \to M$ is such an embedding, then $\mu \mid V\times 0 : V\times 0 \to M$ extends to a CAT-homeomorphism $\bar{\mu} : V\times [0, -1] \to M - \mathrm{int}\, \mu(V\times I)$ and $\mu \cup \bar{\mu}$ essentially provides an open book decomposition $h : \mathrm{OB}(V, \alpha) \to M$ where α is defined by $(\alpha(x), 1) = \mu^{-1}(\bar{\mu}(x; -1))$. Concordant open book decompositions give rise to concordant embeddings and conversely. This proves the following lemma.

LEMMA. *There is a bijection (given above) between concordance classes of open book decompositions with page V on a* CAT-*manifold M^n $(n \geqq 6)$ and concordance classes of* CAT-*embeddings $\mu : V\times I \to M$ such that $(M - \mathrm{int}\, \mu(V\times I), \mu(V\times 0), \mu(V\times 1))$ is an s-cobordism.*

In order to simplify our notations we usually identify $V\times I$ with $\mu(V\times I)$. Thus, an open book decomposition of M^n is viewed as a decomposition $M = V\times I \cup W$ where $(W, V\times 0, V\times 1)$ is an s-cobordism. The use of this point of view can be seen at once ·ŋ the proof of the following proposition.

(2.6) PROPOSITION. *Let M^n $(n \geq 7)$ be a closed* CAT-*manifold which has an open book decomposition with page V. Suppose that the homomorphism $\pi_1(\partial V) \to \pi_1(V)$ induced by the inclusion is an isomorphism. Then, any manifold N^n is the same simple homotopy type as M has an open book decomposition with page T, where T (respectively ∂T) has the same simple homotopy type as V (respectively ∂V).*

Proof. Let $f : N \to M$ be a simple homotopy equivalence. Using (2.5), write $M = V\times I \cup_\partial W$, where $(W, V\times 0, V\times 1)$ is an s-cobordism. By Van Kampen Theorem, the condition $\pi_1(\partial V) \simeq \pi_1(V)$ implies that the homomorphisms $\pi_1(\partial(V\times I)) \to \pi_1(V\times I)$ and $\pi_1(\partial(V\times I)) \to \pi_1(W)$ are isomorphisms. By [Wa 2, Theorem 12.1], there is a decomposition $N = U_0 \cup U_1$ and f is homotopic to g, where g is a simple homotopy equivalence of 4-ads:

$$g:(N;\, U_0,\, U_1,\, \partial U_1) \to (M;\, V \times I,\, W,\, \partial W).$$

Now $\partial W = \partial(V \times I) \cong V \cup_\partial V$. Using [Wa 2, Theorem 12.1] again we get a decomposition $\partial U_0 = T_0 \cup_\partial T_1$ and $g \mid \partial U_0$ is homotopic to a simple equivalence g' of 4-ads:

$$g':(\partial U_0;\, T_0,\, T_1,\, \partial T_0) \to (\partial(V \times I),\, V \times 0,\, V \times 1,\, \partial(V \times 0)).$$

By the homotopy extension property, g' can be extended to a map from N to M having the same properties as g. The cobordisms (U_0, T_0, T_1) and (U_1, T_0, T_1) are then s-cobordism so we can identify one of them with $T_0 \times I$. The result now follows from (2.5).

(2.7) If $h: V \to V'$ is a CAT-homeomorphism, then $h \times id: V \times I \to V' \times I$ extends to a CAT-homeomorphism $H: \mathrm{OB}(V, \alpha) \to \mathrm{OB}(V', h \cdot \alpha \cdot h^{-1})$.

(2.8) An open book $L = \mathrm{OB}(V, \alpha)$ is called a *stable open book* if V^{n-1} has a handle decomposition with handles of index $\leq (n/2) - 1$. We write $M = \mathrm{SOB}(V, \alpha)$.

If $M = \mathrm{SOB}(V, \alpha)$, then $\pi_1(\partial V) = \pi_1(V)$ and so, by Proposition 2.6, a manifold of the same simple homotopy type as a stable open book is a stable open book. This is not true if we replace "simple homotopy type" by "homotopy type" [Ha 2], so the two words "simple" cannot be crossed out in Proposition 2.6. Stable open books play an important role in the understanding of manifolds without middle dimensional handles [Ha 2]. The problem of the existence and classification of stable open book structures on a manifold will be solved in this forthcoming paper [Ha 2].

3. Open book decomposition for manifolds h-cobordant to $\mathrm{OB}(V, \alpha)$

This section is devoted to the proof of Theorem 3.1 below and to some of its applications. We first need some definitions:

Let V^n be a CAT-manifold, with $\partial V \neq \emptyset$. Let $\sigma \in \mathrm{Wh}(\partial V)$. A σ-enlargement of V is a manifold V_σ such that $V \subset \mathrm{int}\, V_\sigma$ and $U = V_\sigma - \mathrm{int}\, V$ is an h-cobordism from ∂V to ∂V_σ with $\tau^0(U, \partial V) = \sigma$.

For any σ-enlargement V_σ of V, one has a map $e(V_\sigma): \mathrm{Aut}_{\mathrm{CAT}}(V \,\mathrm{rel}\, \partial V) \to \mathrm{Aut}_{\mathrm{CAT}}(V_\sigma \,\mathrm{rel}\, \partial V_\sigma)$ defined by

$$e(V_\sigma)(\alpha) = \begin{cases} \alpha & \text{on } V \\ id & \text{elsewhere.} \end{cases}$$

Clearly $e(V_\sigma)$ is a monomorphism of groups. It induces a homomorphism $e^c(V_\sigma)$: $\mathrm{Aut}^c_{\mathrm{CAT}}(V \,\mathrm{rel}\, \partial V) \to \mathrm{Aut}^c_{\mathrm{CAT}}(V_\sigma \,\mathrm{rel}\, \partial V_\sigma)$. As $V_\sigma - V$ is contained in a collar along ∂V_σ, the homomorphism $e^c(V_\sigma)$ is an isomorphism.

(3.1) THEOREM. *Let* $M^n = \mathrm{OB}(V, \alpha)$ *with* $n \geq 5$. *Let* (R, M, M') *be an* h-*cobordism with* $\tau(R, M) = i_{\partial V,M}(\sigma)$ *for* $\sigma \in \mathrm{Wh}(\partial V)$. *Then, for any* σ-*enlargement* V_σ *of* V, *the manifold* M' *has an open book decomposition*

$$\mathrm{OB}(V_\sigma, e(V_\sigma)(\alpha)) \xrightarrow{\simeq} M'.$$

Proof. Let $M = \mathcal{M}_\alpha \cup_\partial \partial V \times D^2$ be our open book decomposition. Let V_σ be a σ-enlargement of V and write $W = V_\sigma - \mathrm{int}\, V$. The h-cobordism (R, M, M') with $\tau^0(R, M) = i_{\partial V,M}(\sigma)$ is homeomorphic relative to M to $M \times I \bigcup_{\partial V \times D^2} W \times D$. Indeed, one checks by excision [Co, (20.3)] that the torsion of the latter is $i_{\partial V,M}(\sigma)$. Thus $M' \simeq [\mathcal{M}_\alpha \bigcup_{\partial V \times S^1} W \times S^1] \bigcup_{\partial V_\sigma \times S^1} \partial V_\sigma \times D^2$. Clearly $\mathcal{M}_\alpha \cup_{\partial V \times S^1} W \times S' = \mathcal{M}_{e(V_\sigma)(\alpha)}$; thus $M' \simeq \mathrm{OB}(V_\sigma, e(V_\sigma)(\alpha))$.

(3.2) COROLLARY. *Let* $M^n \simeq \mathrm{OB}(V, \alpha)$ *with* $n \geq 7$ *and* $i_{\partial V,M} : \mathrm{Wh}(\partial V) \to \mathrm{Wh}(M)$ *surjective. Then any manifold* M' *which is* h-*cobordant to* M *admits an open book decomposition* $M' \simeq \mathrm{OB}(T, \beta)$ *with* int T CAT-*homeomorphic to* int V.

Proof. This is a direct consequence of (3.1) together with the fact that h-cobordisms are characterized by their torsions [Mi 2, Theorem 11.3] and that V and V_σ have CAT-homeomorphic interior (an h-cobordism (U, P, Q) satisfies $U - Q \simeq_{\mathrm{CAT}} P \times [0, \infty)$, [Ke, p. 41]).

(3.3) *Remarks.* The homomorphism $i_{\partial V,M}$ is surjective in the following situations:

(1) $i_{\partial V,V} : \mathrm{Wh}(\partial V) \to \mathrm{Wh}(V)$ is surjective and $\pi_1(\alpha) = id$. Indeed, one has $\pi_1(M) = \pi_1(V) \times_{\pi_1(\alpha)} \langle t \rangle / \{t = 1\}$. So, if $\pi_1(\alpha) = id$, one has $\pi_1(V) \simeq \pi_1(M)$.

(2) $\pi_1(\partial V) \to \pi_1(V)$ is surjective and $i_{\partial V,V}$ is surjective. Indeed, as $\alpha \mid \partial V = \mathrm{id}$, the surjectivity of $\pi_1(\partial V) \to \pi_1(V)$ implies $\pi_1(\alpha) = id$ and we are in Case 1).

(3) $\pi_1(\partial V) \to \pi_1(V)$ is split surjective. So we are in Case 2).

(4) $V = V_0 \times I$. This is a particular case of Case 3), since the inclusion $V_0 \subset \partial(V_0 \times I) \subset V_0 \times I$ makes $\pi_1(\partial V) \to \pi_1(V)$ split surjective.

(5) Another interesting situation for which $i_{\partial V,V}$ is surjective is when $\pi_1(V)$ is a finite group and $\mathrm{Im}\,(\pi_1(\partial V) \to \pi_1(V))$ contains a conjugate of any maximal hyperelementary subgroup H of $\pi_1(V)$ such that $\mathrm{Wh}(H) \neq 0$ (see [St]). Example: $\pi_1(V)$ is the alternating group A_5 and $\pi_1(\partial V)$ is the dihedral group D_{10}.

(3.4) COROLLARY. *Let V^{n-1} be a compact* CAT-*manifold with $n \geq 5$ and $\partial V \neq \emptyset$. Let σ_1 and $\sigma_2 \in \mathrm{Wh}(\partial V)$ be such that $i_{\partial V,V}(\sigma_1) = i_{\partial V,V}(\sigma_2)$ in* Wh(V). *Then, for any σ_i-enlargement V_{σ_i} of V and any $\alpha \in \mathrm{Aut}_{\mathrm{CAT}}(V \operatorname{rel} \partial V)$ one has a* CAT-*homeomorphism*:

$$\mathrm{OB}(V_{\sigma_1}, e(V_{\sigma_1})(\alpha)) \simeq_{\mathrm{CAT}} \mathrm{OB}(V_{\sigma_2}, e(V_{\sigma_2})(\alpha)).$$

Proof. Let $M = \mathrm{OB}(V, \alpha)$ and let (R, M, M') be an h-cobordism with $\tau^0(R, M) = i_{\partial V,M}(\sigma_1) = i_{\partial V,M}(\sigma_2)$. Then, by (4.1), $\mathrm{OB}(V_{\sigma_i}; e(V_{\sigma_i})(\alpha)) \simeq_{\mathrm{CAT}} M$ for $i = 1, 2$.

(3.5) *Remark.* The condition $i_{\partial V,V}(\sigma_1) = i_{\partial V,V}(\sigma)$ in (4.6) does not imply in general that $V_{\sigma_1} \simeq_{\mathrm{CAT}} V_{\sigma_2}$ (∂V_{σ_1} could be not homeomorphic to ∂V_{σ_2}).

4. Pre-open book decompositions

The material of this section will be used in Section 5. It plays also an important role in the classification of manifolds without middle dimensional handles [Ha 2].

Let V^{n-1} be a compact CAT-manifold with $\partial V \neq \emptyset$. Let M^n be a closed manifold.

(4.1) DEFINITION. A decomposition $M \simeq V \times I \cup_\partial W$ is a pre-openbook decomposition of M if $(W, V \times 0, V \times 1)$ is an h-cobordism.

By (2.5), an open book decomposition of M is a pre-open book decomposition.

Let $M = V \times I \cup_\partial W$ be a pre-open book decomposition. Let $\sigma \in \mathrm{Wh}(\partial V)$ and let V_σ be a σ-enlargement (Definition in §3). i.e. $V_\sigma = V \cup U$ where $(U, \partial V, \partial V_\sigma)$ is an h-cobordism with $\tau^0(U, \partial V) = \sigma$. Choose an embedding of $U \times I$ into a collar of $\partial V \times I$ in W. This gives a decomposition $M = V_\sigma \times I \cup W_\sigma$ where $W_\sigma = W - \operatorname{int}(V_\sigma \times I)$.

(4.2) LEMMA. *$M = V_\sigma \times I \cup W_\sigma$ is a pre-open book decomposition.*

Proof. For $i = 0$, 1, one has the following diagram of inclusions:

$$\begin{array}{ccc} V \times \{i\} & \xrightarrow{j_1} & W \\ \Big\downarrow{\scriptstyle j_2} & & \Big\uparrow{\scriptstyle j_4} \\ V_\sigma \times \{i\} & \xrightarrow{j_3} & W_\sigma \end{array}$$

As j_1, j_2 and j_4 are homotopy equivalences, so is j_3, which implies the lemma.

The torsions $\tau(W, V\times 0)$ and $\tau(W_\sigma, V_\sigma\times 0)$ are related by the following *main formula:*

(4.3) PROPOSITION.

$$i_{W_\sigma,W}(\tau(W_\sigma, V_\sigma\times 0)) = \tau(W, V\times 0) - i_{\partial V,W}(\sigma) + (-1)^n\overline{i_{\partial V,W}(\sigma)}.$$

Proof. First fix some notation. Let $X\subset Y\subset Z$ be inclusions between finite subcomplexes of a complex N. If $X\subset Y$ and $Y\subset Z$ are homotopy equivalences, we define

$$\tau^N(Y, X) = i_{Y,N}(\tau(Y; X)) \in \mathrm{Wh}(N).$$

The composition formula implies that

$$\tau^N(Z, X) = \tau^N(Y; X) + \tau^N(Z, Y)$$

so τ^N behaves as a homomorphism.

The main formula is now equivalent to

$$\tau^W(W_\sigma, V_\sigma\times 0) = \tau(W, V\times 0) - \tau^W(U, \partial V) + \tau^W(U, \partial V_\sigma)$$

since $(-1)^n\ \overline{\tau(U, \partial V)} = \tau(U, \partial V_\sigma)$ by the duality formula [Mi 2, p. 394]. The commutativity of the diagram in the proof of (4.1) and the fact that τ^W is a homorphism for the composition of inclusions between subspaces of W give:

$$\tau^W(W_\sigma, V_\sigma\times 0) = \tau(W, V\times 0) - \tau^W(V_\sigma, V) - \tau^W(W, W_\sigma).$$

By the excision principle for Whitehead torsion [Co, (20.3)], one has $\tau^W(V_\sigma, V) = \tau^W(U, \partial V)$ and $\tau^W(W, W_\sigma) = \tau^W(U\times I, Z)$ for $Z = \partial(U\times I) - \operatorname{int}(\partial V\times I)$. The commutativity of the following diagram

$$\begin{tikzcd} & U\times 1\cup \partial V_\sigma\times I \arrow[dl, hook'] \arrow[dr, hook] & \\ U\times I & & Z \arrow[ll, hook'] \end{tikzcd}$$

implies that $\tau^W(U\times I, Z) = -\tau^W(Z, U\times 1\cup\partial V_\sigma\times I)$. By excision again, $\tau^W(Z, U\times 1\cup\partial V_\sigma\times I) = \tau^W(U\times 0, \partial V_\sigma\times 0)$. This proves our formula.

5. Inertial h-cobordisms for openbooks

An h-cobordisms (W, M, N) is called CAT-inertial (or just "inertial") if M is CAT-homeomorphic to N. Define $I_{\mathrm{CAT}}(M) \subset \mathrm{Wh}(M)$ as the set of torsions $\tau^0(W, M)$ of CAT-inertial h-cobordisms (W, M, N). In this section we study the set $I_{\mathrm{CAT}}(M)$ when M is an open book. We need some definitions:

A homotopy equivalence $f: A^n \to B^n$ between CAT-manifolds is called CAT-tangential (or, by abuse of language, just "tangential") if there is an isomorphism of stable vector bundles over A between T^sA and $f^*(T^sB)$, where T^s denotes the stable tangent bundle. It is equivalent to say that f commutes up to homotopy with the classifying maps for T^sA and T^sB:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "T^sB"'] & & B \arrow[dl, "T^sA"] \\ & \mathrm{BCAT} & \end{tikzcd}$$

Thus, a homotopy inverse of a tangential homotopy equivalence is a tangential homotopy equivalence.

Let K be a finite complex and let V be a compact CAT-manifold. The torsions $\tau^0(f) \in \mathrm{Wh}(K)$ of self-homotopy equivalences $f: K \to K$ form a subset of $\mathrm{Wh}(K)$ which is denoted by $E^0(K)$. The set $E^0_{\mathrm{CAT}}(V)$ is the set of torsions $\tau^0(f) \in \mathrm{Wh}(V)$ of CAT-tangential self-homotopy equivalences $f: V \to V$. One has $E^0_{\mathrm{CAT}}(V) \subset E^0(V)$ and $E^0(K) \simeq E^0_{\mathrm{CAT}}(V^n(K))$ for $V^n(K)$ a regular neighborhood of an embedding of K in R^n. (If V is parallelizable, then $E^0(V) = E^0_{\mathrm{CAT}}(V)$.) The corresponding sets for $\tau(f)$ are denoted by $E(K)$ and $E_{\mathrm{CAT}}(V)$.

For a subset X of $\mathrm{Wh}(V)$, we define $-X = \{\sigma \in \mathrm{Wh}(V) \mid -\sigma \in X\}$ and $\bar{X} = \{\sigma \in \mathrm{Wh}(V) \mid \bar{\sigma} \in X\}$. The precise relationship between $E^0_{\mathrm{CAT}}(V)$ and $E_{\mathrm{CAT}}(V)$ is as follows:

(5.1) LEMMA. *For any compact* CAT-*manifold V or finite complex K, one has:*

$$E^0_{\mathrm{CAT}}(V) = -E_{\mathrm{CAT}}(V) \quad \textit{and} \quad E^0(K) = -E(K).$$

(5.2) *Remark.* In general, $E^0_{\mathrm{CAT}}(V) \neq E_{\mathrm{CAT}}(V)$ and $E^0(K) \neq E(K)$, see (6.2).

Proof. Let $f: K \to K$ be a self homotopy equivalence and $\varphi: K \to K$ be a homotopy inverse of f. One has

$$0=\tau(f\circ\varphi)=\tau(f)+f_*\tau(\varphi)=\tau(f)+\varphi_*^{-1}\tau(\varphi)$$
$$=\tau(f)+\tau^0(\varphi)$$

whence $-E^0(K)\subset E(K)$ and also $-E(K)\subset E^0(K)$. Thus $E^0(K)=-E(K)$. The same argument holds for $E^0_{\mathrm{CAT}}(V)$ and $E_{\mathrm{CAT}}(V)$, since a homotopy inverse of a tangential homotopy equivalence is a tangential homotopy equivalence.

Let $S_n(\mathrm{Wh}(V))$ denote the subgroup of $\mathrm{Wh}(V)$ constituted by the elements σ with $\sigma=(-1)^n\bar{\sigma}$ (n-symmetric elements of $\mathrm{Wh}(V)$). For $\sigma\in\mathrm{Wh}(V)$ we denote by $[\sigma]_n$ its image in $\mathrm{Wh}(V)/S_n(\mathrm{Wh}(V))$. When $X\subset\mathrm{Wh}(V)$, we denote by $[X]_n$ the set $\{[\sigma]_n\mid\sigma\in X\}\subset\mathrm{Wh}(V)/S_n(\mathrm{Wh}(V))$. For instance, $[\mathrm{Wh}(V)]_n=\mathrm{Wh}(V)/S_n(\mathrm{Wh}(V))$.

(5.3) PROPOSITION. *Let $M^n=\mathrm{SOB}(V,\alpha)$ with $n\geq 5$ and let (R^{n+1},M,M') be an h-cobordism. If M is* CAT*-homeomorphic to* $\mathrm{SOB}(V,\beta)$ *for some β, then*

$$[\tau^0(R,M)]_n\in[i_{\mathrm{V,M}}(E^0_{\mathrm{CAT}}(V))]_n\cap[i_{\mathrm{V,M}}\overline{(E^0_{\mathrm{CAT}}(V))}]_n.$$

Proof. A stable open book satisfies $\pi_1(\partial V)\simeq\pi_1(V)\simeq\pi_1(M)$. Let $\sigma=i^{-1}_{\mathrm{V,M}}(\tau^0(R,M))\in\mathrm{Wh}(V)$. Let $M\simeq V\times I\cup_\partial W$ be the open book decomposition under consideration viewed as in (2.5).

By the characterization of h-cobordisms by their torsion [Mi 2, Theorem 11.3] and the sum theorem for Whitehead torsion [Co, 23.1], R is CAT-homeomorphic to $V\times I\times I\cup R_0$, where $V\times I\times I\cap R_0=\partial(V\times I)\times I$ and (R_0,W,W') is an h-cobordism with $\tau(R_0,W)=i_{\mathrm{V,R_0}}(\sigma)$. Hence $M'\simeq V\times I\cup_\partial W'$ is a pre-open book decomposition (see §4). Let $\rho:R_0\to W'$ be a deformation retraction.

We want to compute $\nu=\tau(W',V\times 0\times 1)$. Identifying V with $V\times 0\times 0$ or $V\times 0\times 1$ (both inclusions are isotopic in R_0), one gets the following equation in $\mathrm{Wh}(R_0)$:

$$i_{\mathrm{W',R_0}}(\nu)+\tau(R_0,W')=\tau(R_0,W)$$

whence:

$$i_{\mathrm{W',R_0}}(\nu)=\tau(R_0,W)-\tau(R_0,W')$$
$$=i_{\mathrm{V,R_0}}(\sigma)+(-1)^{n+1}\overline{i_{\mathrm{V,R_0}}(\sigma)}.$$

As $\rho_*\circ i_{\mathrm{W',R_0}}=id_{\mathrm{Wh(W')}}$, and $\rho_*\circ i_{\mathrm{V,R_0}}=i_{\mathrm{V,W'}}$, we get

$$\omega=i_{\mathrm{V,W'}}(\sigma)+(-1)^{n+1}i_{\mathrm{V,W'}}(\bar{\sigma}).\tag{1}$$

Now use the fact that $M' \simeq \mathrm{SOB}(V, \beta)$ which corresponds to $M' = (V \times I)_0 \cup W_0$. Observe that W' has the homotopy type of V and $\pi_1(\partial W') \simeq \pi_1(W')$. Therefore, the classical procedure for eliminating handles [Ke] implies that W' has a handlebody decomposition with handles of index $\leq (n/2) - 1$. By general position and unicity of regular neighborhoods, one can move the decomposition $M' = V \times I \cup W'$ by an ambiant isotopy so that

$$V \times 0 \subset \operatorname{int}(V \times 0)_0 \quad \text{and} \quad (V \times I)_0 = V \times I \cup U \times I$$

where $U = (V \times 0)_0 - \operatorname{int}(V \times 0)$. By connexity of the inclusions $V \times I \subset M'$ and $(V \times I)_0 \subset M'$, the inclusion $j: V \times 0 \subset (V \times 0)_0$ is a homotopy equivalence, which is CAT-tangential since j is a CAT-embedding. Thus $\eta = \tau^0(j) \in E^0_{\mathrm{CAT}}(V)$. Also, as $\pi_1(\partial V) \cong \pi_1(V)$, the region U is an h-cobordism. By excision [Co, (23.1)], one has $i^{-1}_{\partial V, V}(\eta) = \tau^0(U, \partial(V \times 0))$, so $(V \times 0)_0$ is an $i^{-1}_{\partial V, V}(\eta)$-enlargement of $V \times 0$. As $\tau(W_0, (V \times 0)_0) = 0$, the main formula (4.3) together with formula (1) gives:

$$0 = i_{V,M'}(\sigma) + (-1)^{n+1} i_{V,M'}(\bar{\sigma}) - i_{V,M'}(\eta) + (-1)^n i_{V,M'}(\bar{\eta}). \tag{1}$$

As $i_{V,M'}$ is an isomorphism, this leads us to

$$0 = \sigma + (-1)^{n+1}\bar{\sigma} - \eta + (-1)^n \bar{\eta}. \tag{2}$$

From formula (2) one deduces:

$$\sigma - \eta = (-1)^n (\bar{\sigma} - \eta) \tag{3}$$

$$(-1)^{n+1}\bar{\sigma} - \eta = -\sigma - (-1)^n \bar{\eta}. \tag{4}$$

Formula (3) implies that $[\sigma]_n = [\eta]_n$. One checks in both cases (n odd and even) that formula (4) makes $[\bar{\sigma}]_n = [\eta]_n$. As $\eta \in E^0_{\mathrm{CAT}}(V)$, Proposition (5.3) is proved.

Let $\mathrm{Em}_{\mathrm{CAT}}(V)$ be the set of torsions $\tau(F) \in \mathrm{Wh}(V)$ for which $F: V \to V$ is a homotopy equivalence which is a CAT-embedding of V into int V. Denote by $\mathrm{Em}^0_{\mathrm{CAT}}(V)$ the set of torsions $\tau^0(F) \in \mathrm{Wh}(V)$ for such maps F. If F is such an embedding, then $V = F(V) \cup U$ where $(U, F(\partial V), \partial V)$ is an h-cobordism. Gluing to V an h-cobordism $(U', \partial V, Z)$ inverse to U, one checks that F has a homotopy inverse Φ which is also a CAT-embedding of $V \to$ int V. Therefore, as in Lemma (5.1), one proves that $\mathrm{Em}^0_{\mathrm{CAT}}(V) = -\mathrm{Em}_{\mathrm{CAT}}(V)$.

(5.4) PROPOSITION. *Let* $M^n = \mathrm{OB}(V, \mathrm{id})$ *with* $n \geq 5$. *Then*

$$i_{V,M}[\mathrm{Em}^0_{\mathrm{CAT}}(V \times D^2) \cup (-1)^{n+1}\overline{\mathrm{Em}^0_{\mathrm{CAT}}(V \times D^2)}] \subset I_{\mathrm{CAT}}(M).$$

Proof. Let $\sigma \in \mathrm{Em}^0_{\mathrm{CAT}}(V \times D^2)$ and let $F: V \times D^2 \to \mathrm{int}(V \times D^2)$ be a CAT-embedding which is a homotopy equivalence with $\tau(F) = -\sigma$. Recall that $M = \partial(V \times D^2)$ and that the Van-Kampen Theorem applied to the decomposition $M = V \times I \cup_\partial V \times I$ implies that $\pi_1(M) \to \pi_1(V \times D^2)$ is an isomorphism. Therefore, the fact that $\sigma \in I_{\mathrm{CAT}}(M)$ and $(-1)^{n+1}\bar{\sigma} \in I_{\mathrm{CAT}}(M)$ comes from Lemma (5.5) and (5.6) below.

(5.5) LEMMA. *Let T^m be a compact* CAT*-manifold with $m \geq 6$ and $\pi_1(\partial T) \simeq \tau_1(T)$ an isomorphism. Let $F: T \to \mathrm{int}\, T$ be a* CAT*-embedding which is a homotopy equivalence with $\tau(F) = i_{\partial T, T}(\sigma)$, for $\sigma \in \mathrm{Wh}(\partial T)$. Then any $(-\sigma)$-enlargement of T is* CAT*-homeomorphic to T. (In consequence, $-\sigma \in I_{\mathrm{CAT}}(\partial T)$.)*

Proof. Write $Z = F(\partial T)$ and $U = T - \mathrm{int}\, F(T)$. As $\pi_1(\partial T) \simeq \pi_1(T)$, the cobordism $(U, \partial T, Z)$ is an h-cobordism. Let $(U', \partial T, Z')$ be an inverse h-cobordism for U, i.e. $(U \cup U', Z) \simeq_{\mathrm{CAT}} (Z \times I, Z \times 0)$. Define $N = T \cup_{\partial T} U'$.

The excision principle [Co, (20.3)] gives

$$\tau^N(U', \partial T) = \tau^N(N, T)$$

(see proof of (4.3) for the definition of τ^N), and

$$\tau^N(U, Z) = \tau^N(T, F(T)) = i_{T,N}(\tau(F)).$$

One also has

$$\tau^N(U', \partial T) + \tau^N(U, Z) = \tau^N(U \cup U', Z) = 0$$

whence $\tau^N(U', \partial T) = -i_{T,N}(\tau(F))$. Thus $\tau^0(U', \partial T) = -\sigma$. By construction, N is a $(-\sigma)$-enlargement of T and is CAT-homeomorphic to T. As any two $(-\sigma)$-enlargements of T are CAT-homeomorphic, this proves Lemma (5.5).

(5.6) LEMMA. *Let M^n be a compact manifold with $n \geq 5$. Then $(-1)^{n+1}\overline{I_{\mathrm{CAT}}(M)} = I_{\mathrm{CAT}}(M)$.*

Proof. Let $\eta \in I_{\mathrm{CAT}}(M)$. Let (W, M, N) and $(\bar{W}, M, \bar{N})$ be h-cobordisms with $\tau^0(W, M) = \sigma$ and $\tau^0(\bar{W}, M) = (-1)^{n+1}\bar{\sigma}$. One checks that $(W \cup_M \bar{W}, N, \bar{N})$ is an s-cobordism (use excision [Co, (23.1)] and the duality formula [Mi, p. 394]). Therefore $N \simeq \bar{N}$ and $N \simeq M$ by hypothesis. Thus $(-1)^{n+1}\overline{I_{\mathrm{CAT}}(M)} \subset I_{\mathrm{CAT}}(M)$. The reverse inclusion is obtained by symmetry.

Observe that $\mathrm{Em}^0_{\mathrm{CAT}}(V \times D^2) \subset E^0_{\mathrm{CAT}}(V \times D^2) = E^0_{\mathrm{CAT}}(V)$. If V has a handle decomposition with handles of index $\leq n/2$, then $E^0_{\mathrm{CAT}}(V) \subset \mathrm{Em}^0_{\mathrm{CAT}}(V \times D^2)$.

Indeed, $V\times D^2$ is then a stable thickening [Wa 1, §5] and the proof of Proposition 5.1 of [Wa 1] shows that a CAT-tangential homotopy equivalence $f: V\times D^2 \to V\times D^2$ is homotopic to a CAT-embedding. Therefore, Propositions (5.3) and (5.4) imply the following theorem:

(5.7) THEOREM. *Let $M^n = \mathrm{SOB}(V, id)$ with $n \geq 5$. Then*

$$[I_{\mathrm{CAT}}(M)]_n = [i_{V,M}E^0_{\mathrm{CAT}}(V)]_n.$$

Moreover, if $n = 2k$, then

$$[I_{\mathrm{CAT}}(M)]_n = -[I_{\mathrm{CAT}}(M)]n.$$

(5.8) *Remark.* Theorem (5.7) is of special interest when $S_n(\mathrm{Wh}(M))$ is small. For instance, if $n = 2k+1$, $\pi_1(M)$ is finite abelian and $\omega_1(M) = 0$, $S_n(\mathrm{Wh}(M))$ is the subgroup of $\mathrm{Wh}(M)$ of the elements of order 2. Also, if $x = \bar{x}$ and $\mathrm{Wh}(M)$ has no 2-torsion, then Theorem (5.7) says that $I_{\mathrm{CAT}}(M) = i_{V,M}(E^0_{\mathrm{CAT}}(V))$ $(n = 2k+1)$. This happens in the following cases:

(1) M is orientable, $\pi_1(M)$ is finite abelian with cyclic 2-torsion.

(2) M orientable, $\pi_1(M)$ is a dihedral group, or an alternating group A_r $(r \leq 6)$ or the binary icosaedral group [St].

In our applications we shall use the following corollary of Theorem (5.7):

(5.9) COROLLARY. *Let N^n be a closed* CAT-*manifold. If $k \geq n$ and $n+k \geq 5$, one has*

$$[I_{\mathrm{CAT}}(N\times S^k)]_{n+k} = [E^0_{\mathrm{CAT}}(N)]_{n+k}$$

In particular, if N is parallelizable, one has $[I_{\mathrm{CAT}}(N\times S^k)]_{n+k} = [E^0(N)]_{n+k}$.

Proof. As $N\times S^k = \mathrm{SOB}(N\times D^{k-1}, id)$, this follows from Theorem (5.7).

Finally, as a consequence of the techniques of this section, we have the following result:

(5.10) PROPOSITION. *Let M_0^n and M_1^n be two closed* CAT-*manifolds, and let k be an integer $\geq n+2$. Suppose that $S_{n+k}(\mathrm{Wh}(M)) = 0$. Then $M_0\times S^k$ is* CAT-*homeomorphic to $M_1\times S^k$ if and only if M_0 and M_1 have the same* CAT-*tangential simple homotopy type.*

Proof. Write $V_i = M_i\times D^{k-1}$, so $M\times S^k = \mathrm{SOB}(V_i, \mathrm{id})$ if $k \geq n+2$. Let

$h: M_0 \times S^k \to M_1 \times S^k$ be a CAT-homeomorphism. By the general position and connectivity arguments of the proof of (5.3), one may assume that $h(V_0 \times I) \subset \operatorname{int}(V_1 \times I)$ and that $h_0 = h \mid V_0 \times I$ is a tangential homotopy equivalence from $V_0 \times I$ to $V_1 \times I$.

Let $F = h_0 \times id: V_0 \times I \times I \to (V_1 \times I \times I)$. As in the proof of Lemma (5.5), this embedding F gives rise to an h-cobordism $(W, M_1 \times S^k, Z)$ with $\tau^0(W, M) = -i_{M_1, M_2 \times S^k}(\tau(F))$ and $Z \simeq_{\mathrm{CAT}} F(M_0 \times S^k) \simeq_{\mathrm{CAT}} M_0 \times S^k$. Thus W is an inertial cobordism and by Proposition (6.3), $-\tau(F) \in E^0_{\mathrm{CAT}}(M_1)$. Let $\varphi: M_2 \to M_1$ be a CAT-tangential homotopy equivalence such that $\tau^0(\varphi) = -\tau(F)$. Then:

$$\tau(\varphi \cdot F) = \tau(\varphi) + \varphi_*(\tau(F)) = \tau(\varphi) - \varphi_*(\tau^0(\varphi)) = 0.$$

Hence $\varphi \circ F: M_0 \to M_1$ is a CAT-tangential simple homotopy equivalence.

To prove the converse, observe that a CAT-tangential homotopy equivalence $f: M_0 \to M_1$ produces a CAT-embedding $F: M_0 \times D^{k+1} \to M_1 \times D^{k+1}$ for $k+1 \geq n+1$. If F is a simple homotopy equivalence, the manifold $M_1 \times D^{k+1} - \operatorname{int} F(M_0 \times D^{k+1})$ is an s-cobordism from $F(M_0 \times S^k)$ to $M_1 \times S^k$. The result follows from the s-cobordism theorem.

(5.11) *Remark.* As we have just seen, $M_0 \times S^k$ and $M_1 \times S^k$ are h-cobordant if M_0 and M_1 have the same tangential homotopy type and $k \geq n$. But, under the assumptions of Proposition (5.10), M_0 and M_1 need to have the same *simple* tangential homotopy type in order for $M_0 \times S^k$ and $M_1 \times S^k$ to be homeomorphic. This generalizes the technique of [Mi 1] to get non-inertial h-cobordisms.

6. Applications and examples

Let $L_{p,q}$ be the 3-dimensional lens space for p prime $\neq 2$ as it is defined classically [Co, §27]. Its fundamental group is C_p, the cyclic of order p with generator t. Let S^n denote the standard sphere.

(6.1) THEOREM. *For* $2k \geqslant 6$ *and* CAT = DIFF *or* PL, *one has:*
(1) $I_{\mathrm{CAT}}(L_{p,q} \times S^{2k})$ *is either* $\{0\}$ *or* $\{0, \sigma\}$ *for some* $\sigma \in \mathrm{Wh}(L_{p,q})$.
(2) $I_{\mathrm{CAT}}(L_{p,q} \times S^{2k}) = \{0\}$ *if* $p \equiv 3 \pmod 4$
(3) $I_{\mathrm{CAT}}(L_{p,1} \times S^{2k}) \neq \{0\}$ *if* $p \equiv 1 \pmod 4$.

Proof. As $L_{p,q}$ is parallelizable, one has $[I_{\mathrm{CAT}}(L_{p,q} \times S^{2k})]_{2k+3} = [E^0(L_{p,q})]_{2k+3}$ by Theorem (5.7). But $S_{\mathrm{odd}}(\mathrm{Wh}(L_{p,q})) = 0$ by [Ba, (4.2)]. Thus $I_{\mathrm{CAT}}(L_{p,q} \times S^{2k}) = E^0(L_{p,q})$ and it suffices to prove 1), 2) and 3) for $E^0(L_{p,q})$.

The set of homotopy classes of self-homotopy equivalence of $L_{p,q}$ is a finite set which is in bijection with $\{a \in N \mid 0 < a < p \text{ and } a^2 \equiv \pm 1 \bmod p\}$ [Co, (29.6)]. The bijection sends $f: L_{p,q} \to L_{p,q}$ to $a(f)$ such that $\pi_1(f)(t) = t^{a(f)}$.

If $a(f) = \pm 1$, the homomorphism $\mathrm{Wh}(f)$ is the identity (it sends $x \to x$ or $x \to \bar{x}$, but $\bar{x} = x$ [Ba, Proposition 4.2]). This implies that the set $\{\tau(f) \mid a(f) = \pm 1\}$ is a subgroup of $\mathrm{Wh}(L_{p,q})$. But any finite subgroup of $\mathrm{Wh}(L_{p,q})$ is trivial [Ba, Proposition 4.2]. Thus $\tau(f) = 0$ if $a(f) = \pm 1$, which proves Point 2).

If $-1 \equiv b^2 \bmod p$, for $0 < b < p$, write f_+, $f_-: L_{p,q} \to L_{p,q}$ the self-homotopy equivalence with $a(f_\pm) = \pm b$. Then $a(f_+ \cdot f_-) = 1$ and $a(f_+ \cdot f_+) = -1$. Thus:

$$\begin{cases} \tau(f_+) + (f_+)_* \tau(f_-) = 0 \\ \tau(f_+) + (f_+)_* \tau(f_+) = 0 \end{cases}$$

which implies $\tau(f_+) = \tau(f_-)$. Point 1) is thus proved.

Point (3) is proved using the special torsion $\Delta(L_{p,1}) \in \mathbb{Q}C_p$ defined in [Mi 2, §13]. One has $\Delta(L_{p,1}) = (t-1)^2$. Let $f: L_{p,1} \to L_{p,1}$ a self-homotopy equivalence with $a(f) = b \neq \pm 1$. One checks easily that there is no integer k satisfying

$$f_*(\Delta(L_{p,1})) = (t^b - 1)^2 = \pm t^k (t-1)^2 = \pm t^k \Delta(L_{p,1}).$$

This implies that $\tau(f) \neq 0$ by [Mi 2, Lemma 12.5].

(6.2) *Remark.* As $\mathrm{Wh}(L_{p,q})$ has no torsion, it follows that $I_{\mathrm{CAT}}(L_{p,1} \times S^{2k})$ is not a subgroup of $\mathrm{Wh}(L_{p,q})$. Also, if $0 \neq \sigma \in I_{\mathrm{CAT}}(L_{p,q} \times S^{2k})$, then $-\sigma \notin I_{\mathrm{CAT}}(L_{p,q} \times S^{2k})$. Also $E(L_{p,1}) \neq E^0(L_{p,1})$.

(6.3) THEOREM. *Let $M = \mathrm{SOB}(V, \alpha)$, with $V = L_{p,q} \times D^{2k-1}$, $k \geq 3$. Then. $I_{\mathrm{CAT}}(M)$ contains at most 2 elements. If (W, M, N) is an h-cobordism with $\tau^0(W, M) \notin I_{\mathrm{ACT}}(M)$, then N has not the same simple homotopy type as an open book with page homeomorphic to V.*

Proof. By Proposition 5.3 and the fact that $S_{\mathrm{odd}}\mathrm{Wh}(L_{p,q}) = 0$ [Ba, Proposition 4.2], one has $I_{\mathrm{CAT}}(M) \subset E^0(L_{p,q})$. The first assertion thus follows from (6.1) and its proof. For the last affirmation, let us suppose that N has the same simple homotopy type as $\mathrm{OB}(V, \beta)$. By Proposition (2.6) N is CAT-homeomorphic to $\mathrm{OB}(T, \gamma)$ with $(T; \partial T)$ of the same simple homotopy type as $(V; \partial V)$. Observe that $[L_{p,q}; \mathrm{BCAT}] = \{\text{constant map}\}$, by obstruction theory and the fact that $\pi_i(\mathrm{BCAT}) = Z_2$ for $i \leq 2$ and $\pi_3(\mathrm{BCAT}) = 0$. Therefore, T is a trivial thickening of $L_{p,q}$ [Wa 1, Proposition 5.1] and is homeomorphic to V. Thus $M \simeq \mathrm{SOB}(V, \gamma)$ and, by Proposition (5.3) and the discussion above concerning $S_{\mathrm{odd}}(\mathrm{Wh}(L_{p,q}))$, one has $\tau^0(W, M) \in E^0_{\mathrm{CAT}}(L_{p,q}) = I_{\mathrm{CAT}}(M)$.

(6.4) *Remark.* Theorem (6.3) contrasts with Corollary (3.2), which asserts that M' has an open book decomposition with detached page int V. This shows that the data of the detached page and the restriction of the monodromy to it does not even determine the simple homotopy type of $\mathrm{OB}(V, \gamma)$.

The following lemma proves that the condition $q = 1$ is essential in Theorem (6.1):

(6.5) LEMMA. *For* CAT = DIFF *or* PL *and* $k \geq 3$, *one has* $I_{\mathrm{CAT}}(L_{5,2} \times S^{2k}) = \{0\}$

Proof. (One uses the results and notations of the proof of (6.1)). It suffices to show that $E^0(L_{5,2}) = \{0\}$. This can be done by using the Milnor's special torsion $\Delta(L_{5,2}) = (t-1)(t^2-1) \in \mathbb{Q}C_p$. The only possible non-zero element of $E^0(L_{5,2})$ would occur as $\tau(g)$ for the self-homotopy equivalence g of $L_{5,2}$ characterized by $a(g) = 2$. But one checks that $g_*(\Delta(L_{5,2})) = -t^4\, \Delta(L_{5,2})$, which implies that $\tau(g) = 0$ by [Mi 2, Lemma 12.5].

The situation of Lemma (6.5) gives rise to an interesting phenomenon:

(6.6) THEOREM. *Let* $p \equiv 1 \pmod 4$ *and* q *such that* $I_{\mathrm{CAT}}(L_{p,q} \times S^{2k}) = \{0\} (k \geq 3)$. *Then, there exists a manifold N which is h-cohordant to* $L_{p,q} \times S^{2k}$ *such that* $I_{\mathrm{CAT}}(N) \neq \{0\}$.

Proof. Again, one uses throughout this proof the results and the notations of the proof of (6.1). The non-zero torsion of $E^0(L_{p,1})$ is realized by the self-homotopy equivalence g of $L_{p,1}$ characterized by $a(g) = b$, for b a square root of -1 in $\mathbb{Z}/pZ$. Thus, the homomorphism g_* induced on $\mathrm{Wh}(L_{p,1})$ is different from the identity. Chose $\sigma \in \mathrm{Wh}(L_{p,q}) = \mathrm{Wh}(L_{p,1})$ such that $g_*^{-1}(\sigma) \neq \sigma$.

On the other hand, the self-homotopy equivalence f of $L_{p,q}$ characterized by $a(f) = b$, although satisfying $f_* = g_*$, is simple, since $E^0(L_{p,q}) = I_{\mathrm{CAT}}(L_{p,q} \times S^{2k})) = 0$.

By [Co, 20.7 and 8.5], there exists a finite complex X obtained by attaching 2 and 3 cells to $L_{p,q}$ such that the inclusion $i: L_{p,q} \subset X$ is a homotopy equivalence satisfying $\tau^0(i) = \sigma$. Let $r: X \to L_{p,q}$ be a homotopy inverse to i. Then $i \circ f \circ r$ is a self-homotopy equivalence of X. As $\tau \circ (f) = 0$, one has:

$$\begin{aligned}\tau^0(i \circ f \circ r) &= \tau^0(r) + r_*^{-1} f_*^{-1}(\tau^0(i)) = -r_*^{-1}(\tau^0(i)) + r_*^{-1} f_*^{-1}(\tau^0(i)) \\ &= r_*^{-1}(-\sigma + f_*^{-1}(\sigma)) \neq 0.\end{aligned}$$

Now let V_X be a regular neighborhood of X embedded in $\mathbb{R}^{2k+2}$. Then V_x is an enlargement of $L_{p,q} \times D^{2k-1}$. Thus $L_{p,q} \times S^{2k} = \mathrm{SOB}(L_{p,q} \times D^{2k-1}, \mathrm{id})$ is h-cobordant to $\mathrm{SOB}(V_X, \mathrm{id})$ by Theorem (3.1), and $I_{\mathrm{CAT}}(\mathrm{SOB}(V_X, \mathrm{id}) = E^0(X) \neq 0$.

REFERENCES

[Ba] BASS, H., *L_3 of finite abelian groups*, Annals of Math. *99* (1974), 118–153.

[Co] COHEN, M. M., *A course in simple-homotopy theory*, Springer Graduate texts in Math. *10* (1973).

[H-L] HATCHER, A., and LAWSON, T., *Stability theorems for "Concordance implies isotopy" and "h-cobordism implies diffeomorphism"*, Duke Math. J. *43* (1976), 555–560.

[Ha 1] HAUSMANN, J-Cl., *h-cobordism entre variétés homeomorphes*, Comment. Math. Helv. *50* (1975), 9–13.

[Ha 2] ——, *Manifolds without middle dimensional handles, in Preparation.*

[Hu] HUDSON, J. F. P., *Piecewise linear topology*, Benjamin 1969.

[La 1] LAWSON, T., *Inertial h-cobordism with finite fundamental group*, Proc. AMS *44* (1974), 492–496.

[La 2] ——, *Stability Theorems for Inertial h-cobordisms,*

[La 3] ——, *Open book decompositions for odd dimensional manifolds*, Topology *17* (1978), 189–192.

[Ke] KERVAIRE, M., *Le théorème de Barden-Mazur-Stallings*, Comment. Math. Helv. *40* (1965), 31–42.

[Mi 1] MILNOR, J., *Two complexes which are homeomorphic but combinatorially distinct*, Annals of Math. *74* (1961), 575–590.

[Mi 2] ——, *Whitehead Torsion*, Bull. AMS *72* (1966), 358–426.

[Qu] QUINN, F., *Open book decomposition and the bordism of automorphism*, to appear in Topology.

[St] STEIN, M., *Whitehead groups of finite groups*, Bull. AMS *84* (1978), 201–212.

[Wa 1] WALL, C. T. C., *Classification problems in differential topology-IV*, Thickenings, Topology *5* 73–94.

[Wa 2] ——, *Surgery on compact manifolds*, Academic Press, 1970.

[Ni] WILKENKEMPER E. *Manifolds as open books*, Bull. AMS *79* (1973) 45–51.

The Institute for Advanced Study
Princeton, USA

University of Geneva
Switzerland

Received July 1979