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Smooth solutions of the heat and wave equations

By Stephen Smale

Section 1

The motivation for this work was to try to give proofs for the existence of C^∞ solutions of the heat and wave equations on bounded domains by Fourier methods. I wanted to show that the Fourier series (i.e., eigenfunction expansion) of solutions would converge not just in L^2, but smoothly to smooth solutions. In contrast to more abstract methods, eigenfunction methods bring the existence theory closer to the practice of physics, and also to ordinary differential equations and numerical methods as well.

In fact I found that by the addition of an extra term generated by the boundary of the domain, one could obtain this smooth convergence. For this proof one needs no significant estimates beyond those needed for the elliptic theory. And in general, our proof below gives sharp results by simple conceptual arguments.

The difficulty with Fourier expansions can be seen in the problem: $(\partial u/\partial t)-(\partial^2 u/\partial x^2)=f$ satisfying $u(0,x)=v(x)$, $u(t,0)=u(t,1)=0$. Here the data $f:R^+\times[0,1]\to R$ and $v:[0,1]\to R$ are given, and $u:R^+\times[0,1]\to R$ is to be found. If $f(t,x)=\sum_{n\in Z^+} a_n(t)\sin n\pi x$ is a Fourier expansion which converges in $C^2[0,1]$, then $f''(t,0)=f''(t,1)=0$. This is a special condition on f.

We state now our problem in general for the heat equation. Let Ω be a closed bounded set of $\mathbf{R}^n$ with smooth (i.e., C^∞) boundary $\partial\Omega$ and let $R^+=[0,\infty)$. Let $L=-\Delta$, Δ the usual Laplacian, or more generally any self-adjoint real elliptic (smooth) operator on $C^\infty(\Omega)$ with no eigenvalue equal to 0 (see Section 2). Suppose the following C^∞ data are given: $f:R^+\times\Omega\to R$, initial condition $u_0:\Omega\to R$ and Dirichlet boundary data $g:R^+\times\Omega\to R$ with $g(0,x)=0$. We seek

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a solution, a C^∞ function $u: R^+ \times \Omega \to R$ such that

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + Lu = f \quad \text{on} \quad R^+ \times \Omega, \\
&u(0,x) = u_0(x), \quad \text{all} \quad x \in \Omega \quad \text{and} \\
&u(t,x) = g(t,x) \quad \text{all} \quad x \in \partial\Omega.
\end{aligned} \tag{1}
$$

We may incorporate the data into f. More precisely let $v = u - u_0 - g$ and $h = f - (\partial g/\partial t) - Lg - Lu_0$. The main problem becomes: Find $C^\infty v: R^+ \times \Omega \to R$ such that $(\partial v/\partial t) + Lv = h$, $v(0,x) = 0$, $x \in \Omega$ and $v(t,x) = 0$, $x \in \partial\Omega$. Thus we may take $u_0 \equiv 0$, $g \equiv 0$ in (1) and ask:

Given $f: R^+ \times \Omega \to R$, C^∞, when is there a C^∞ function $u: R^+ \times \Omega \to R$ such that $\dfrac{\partial u}{\partial t} + Lu = f$ on $R^+ \times \Omega$, $u(t,x) = 0$ if $t = 0$ or $x \in \partial\Omega$? (2)

For the answer define a sequence of polynomials in 2 variables by:

$$P_k(L,T) = \sum_{i=0}^{k} (-1)^i L^{k-i} T^i \quad \text{for} \quad k = 0, 1, 2, \ldots.$$

Thus

$$P_0 = 1, \qquad P_1 = L - T, \qquad P_2 = L^2 - LT + T^2, \quad \text{etc.}$$

Main theorem

A necessary and sufficient condition for the solution of (2) is that $P_k(L,T)f]_{x \in \partial\Omega}^{t=0} = 0$, all k where $T = (\partial/\partial t)$. Similarly, for the wave equation. A NASC for the existence of a C^∞ function $u: R \times \Omega \to R$ satisfying $(\partial^2 u/\partial t^2) + Lu = f$ on $R \times \Omega$ with $u(0,x) = (\partial/\partial t)u(0,x) = 0$ all x and $u(t,x) = 0$, all $x \in \partial\Omega$ is that

$$P_k(L,T)f\Big|_{\substack{t=0\\ x\in\partial}} = 0 \quad \text{and} \quad P_k(L,T)f'\Big|_{\substack{t=0\\ x\in\partial}} = 0 \quad \text{for all} \quad k = 0, 1, \ldots$$

where $T = (\partial^2/\partial t^2)$. Here f' denotes $(\partial f/\partial t)$.

The condition of f in this theorem is a kind of compatibility condition which can be translated to non-trivial initial data via the previously defined function h. While the necessity of the condition comes out of the proof, one can test directly for the necessity as follows. Suppose u is a solution, given f as in the first part of the main theorem. Then $P_k(L, T)(L+T)u = P_k(L, T)f$ so

$$(L^{k+1} \pm T^{k+1})u = P_k(L, T)f. \tag{3}$$

But by the boundary conditions, if $x \in \partial\Omega$, then $T^{k+1}u(t, x) = 0$ all t. Similarly if $t = 0$, $L^{k+1}u(t, x) = 0$ all x. Thus if both $t = 0$ and $x \in \partial\Omega$, the left hand side of (3) vanishes and so does $P_k(L, T)f$. The same argument works for the second part noting first that $(T+L)u' = f'$. Thus only the sufficiency has to be proved.

One can reasonably ask: to what extent is our main theorem a new result in partial differential equations (apart from the methodology introduced here)? I have not seen it explicitly in the literature and the mathematicians in partial differential equations I've talked to were unaware of it. However, it overlaps and is close to, e.g., the work of Solonnikov in [5] and Rauch–Massey [9]. On the other hand Solonnikov doesn't discuss the wave equation and has a different generalization of the classic heat equation so that his compatibility conditions don't come out so neatly; they are only given by a recurrence relation.

Rauch–Massey treat only the hyperbolic case, first order hyperbolic systems explicitly, and again these conditions are given by a recurrence relation. Also they suppose $t \geqslant 0$ in contrast to our treatment (in the hyperbolic case) where $R \times \Omega$ has no corners. They state that their methods can be applied to hyperbolic equations of higher order than one.

In texts where heat and wave equations on bounded domains are treated, e.g., Friedman [2], [3], Lions [8], Treves [10], the results presented are not so sharp and the proofs seem more complicated. Treves does use eigenfunction expansions, but only to obtain weaker solutions.

Also some of the PDE literature is not very clear as to what are natural initial value problems for the heat and wave equation on bounded domains. For example, in the well-known paper of Lax and Milgram [7], p. 182, it is stated: "if the initial function U_0 is sufficiently differentiable, $u(t)$ approaches u_0 as t tends to zero not only in the L_2 sense but pointwise." But later, p. 184, "... if u_0 is sufficiently smooth, i.e., belongs to the domain of A^m..." The domain of A^m is basically one of our H^s_*. And u_0 can be even $C\infty$ and not in H^s_*.

Section 2 is devoted to the elliptic theory and section 3 gives the proof of the main theorem. Extensions and generalizations of the main theorem are discussed in section 4.

Finally I wish to acknowledge brief but useful discussions with Vic Guillemin, Dick Palais and Bob Seeley among others.

Section 2

Our methods depend heavily on the Sobolev spaces $H^s(\Omega)=H^s$, $s=0,1,2,\ldots$ With $\Omega\subset R^n$ as in section 1, recall that H^s consists of all real valued functions on Ω with (generalized) derivatives up through order s in $L^2(\Omega)$. A complete norm on H^s is given by

$$|u|^2_{H^s}=\sum_{0\leq|\alpha|\leq s}\int_\Omega |D_\alpha u|^2$$

where α is a multi-index, $\alpha=(\alpha_1,\ldots,\alpha_n)$, α_i is a non-negative integer, $\sum\alpha_i=|\alpha|$, and $D_\alpha u=(\partial^{\alpha_1}/\partial x_1)\cdot\cdots\cdots(\partial^{\alpha_n}/\partial x_n)$. The norm is induced by a inner product and H^0 coincides with $L^2(\Omega)$.

The Sobolev imbedding theorem asserts that $H^{s+k}\subset C^s(\Omega)$ if $k>n/2$ (where $n=\dim\Omega$) and the inclusion is continuous for all $s\geq 0$. Here $C^s(\Omega)$ is the Banach space of C^s functions on Ω, natural norm. See e.g. [3] or [10] for this and other background on Sobolev spaces. The Rellich theorem states that the inclusion $H^s\to H^{s-1}$ is compact.

Let H^1_0 be the closure of C^∞_0 in H^1 where C^∞_0 is the subset of $C^\infty(\Omega)$ of functions which are zero on $\partial\Omega$.

Let $J:H^m\to H^1$ be the natural inclusion and $H^m\cap H^1_0=J^{-1}(H^1_0)$. Since H^1_0 is a closed linear subspace of H^1, and J a continuous linear map, $H^m\cap H^1_0$ is a closed linear subspace of H^m. This space $H^m\cap H^1_0$ is the set of all functions in H^m which are essentially zero on $\partial\Omega$. It is a natural space for the Dirichlet boundary conditions for second order elliptic operators that we will consider, with m independent of the order of the operator or the dimension of Ω.

These elliptic operators are linear maps $L:C^\infty(\Omega)\to C^\infty(\Omega)$ of the form

$$(Lu)(x)=\sum_{0\leq|\alpha|\leq k} a_\alpha(x)D_\alpha u(x)$$

where α is a multi-index, k is the order and $a_\alpha:\Omega\to R$ are C^∞ functions (all functions are real valued here). We will assume $k=2$, for our notation. Our standing hypotheses on L are

L is elliptic. (1)

For each $x \in \Omega$ the polynomial $\sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha \neq 0$ all $\xi \in R^n$, if $\xi \neq 0$, where $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$.

L is self-adjoint. (2)

I.e., $(Lu, v) = (u, Lv)$ for all $u, v \in C_0^\infty$ where (u, v) denotes the L^2 inner product.

$L : C_0^\infty \to C^\infty(\Omega)$ is injective. (no "eigenvalue" is zero) (3)

Condition (3) just make things go more simply. If L satisfies (1) and (2) it can be "translated" to satisfy (3).

As we remarked above, we use second order notation for L throughout. This comes into the boundary conditions in particular. But the proofs go over immediately to arbitrary order. Thus we suppose L is second order and so

$$(Lu)(x) = \sum_{1 \leq i,j \leq n} a_{ij}(x)\frac{\partial^2 y}{\partial x_i \partial x_j} + \sum_{k=1}^{n} b_k(x)\frac{\partial u}{\partial x_k} + c(x)u(x)$$

where $(a_{ij}(x))$ is a negative definite matrix for each x, negative definite rather than positive definite by our convention.

The map $L : C_0^\infty \to C^\infty(\Omega)$ extends naturally to $L : H^m \cap H_0^1 \to H^{m-2}$.

Fundamental theorem of elliptic theory

For each $m = 2, 3, \dots$ $L : H^m \cap H_0^1 \to H^{m-2}$ is an isomorphism. That is, L has a bounded linear (2-sided) inverse $G : H^{m-2} \to H^m \cap H_0^1$.

This could be considered as a regularity theorem, including boundary regularity. For a proof see e.g. [3].

The maps, L, G and inclusions J described above make sense with various domains; sometimes will use them without specifying this domain if the context makes it clear.

A second theorem from the elliptic theory is that providing L with eigenfunctions.

Eigenfunction theorem

Suppose given an elliptic self-adjoint operation $L : C_0^\infty \to C^\infty$ as above. Then there exist a non-decreasing sequence of real numbers $\lambda_1, \lambda_2, \dots$ called eigenvalues, *with $\lambda_i \to \infty$ as $i \to \infty$, and a sequence of elements ϕ_i of C_0^∞ called eigenfunctions so that $L\phi_i = \lambda_i \phi_i$. Furthermore the ϕ_i constitute a Hilbert basis for $L^2(\Omega) = H^0$.*

We sketch how the proof of the eigenfunction theorem follows from the Fundamental theorem. Consider

$$\begin{tikzcd}
H^2 \cap H_0^1 \arrow[rr, "G_0"] \arrow[dr, "J"'] & & H^2 \cap H_0^1 \arrow[dl, shift left, "L"] \\
& H^o \arrow[ur, shift left, "G"] &
\end{tikzcd}$$

Then $G_0 = GJ$ is compact using Rellich and self-adjoint relative to a Hilbert structure on $H^2 \cap H_0^1$ induced from that on H^0 via L. Apply the spectral theorem for compact self-adjoint operators (the simplest spectral theorem, see e.g. [6]) to G_0 to obtain real μ_i, $\psi_i \in H^2 \cap H_0^1$ with $G_0\psi_i = \mu_i \psi_i$. Take $\lambda_i = 1/\mu_i$ indexed so that the λ_i are non-decreasing, and $\phi_i = \lambda_i \psi_i$. The ψ_i are a Hilbert basis for $H^2 \cap H_0^1$ and $L\psi_i = \lambda\psi_i = \phi_i$ a basis for H^0. Finally, the repeated use of the Fundamental theorem applied to $L\phi_i = \lambda_i\phi_i$ implies that $\phi_i \in H^m \cap H_0^1$ every m and thus $\phi_i \in C^\infty$ by the Sobolev theorem.

Define H_*^m, $m = 0, 1, \ldots$ as the closure of the subspace of H^m spanned by the eigenfunctions ϕ_i. For example it follows from the above that $H_*^0 = H^0$, $H_*^1 = H_0^1$, $H_*^2 = H^2 \cap H_0^1$ and that $H_*^m \subset H^m \cap H_0$ for $m \geq 1$. But H_*^3 is a proper subspace of $H^3 \cap H_0^1$ since H_0^1 is a proper subspace of H^1 and $L: H^3 \cap H_0^1 \to H^1$ is an isomorphism. In fact $H_*^3 = L^{-1}(H_0^1)$. Since in general H_*^m is not all of $H^m \cap H_0^1$, the simple expansion by eigenfunctions is not sufficient to give smooth solutions for the heat and wave equation.

It follows from the eigenfunction theorem that (the restriction) $L: H_*^m \to H_*^{m-2}$ is an isomorphism with inverse $G: H_*^{m-2} \to H_*^m$, $m \geq 2$. Actually one may define H_*^m without the use of eigenfunctions by

$$H_*^{2m+2} = G^m(H_*^2) = L^{-m}(H_*^2), \quad H_*^{2m+1} = G^m(H_*^1), \quad m \geq 0.$$

Consider the composition

$$\begin{cases} G_0: H^m \cap H_0^1 \to H^m \cap H_0^1, \\ H^m \cap H_0^1 \xrightarrow{J} H^{m-2} \xrightarrow{G} H^m \cap H_0^1 \end{cases}$$

PROPOSITION. *The image of* G_0^s *is contained in* H_*^m *for* $s \geq [m-1/2]$, *the largest integer in* $(m-1/2)$. *This is false for* $s < [m-1/2]$.

The proposition is a kind of spectral theorem for the operator $G_0 : H^m \cap H_0^1 \to H^m \cap H_0^1$. It implies for example if $m > 2$, there is no Hilbert structure on $H^m \cap H_0^1$ so that G_0 is self-adjoint. On the other hand modulo H_*^m, G_0 is nilpotent, and on H_*^m, G_0 has the spectral theory defined by $G_0\phi_i = (1/\lambda_i)\phi_i$.

The proof of the proposition can perhaps best be seen by studying the following diagram for m even (m odd goes similarly):

$$\begin{tikzcd}
 & & H^6 & \cdots \arrow[l] \\
 & H^4 & H^6 \cap H_0^1 \arrow[l, "L"'] \arrow[u, phantom, sloped, "\subset"] \arrow[dl, "J"] & \cdots \arrow[l] \\
H^2 & H^4 \cap H_0^1 \arrow[l, "L"'] \arrow[u, phantom, sloped, "\subset"] \arrow[dl, "J"] & F \arrow[l] \arrow[u, phantom, sloped, "\subset"] & \cdots \arrow[l] \\
H^2 \cap H_0^1 = H_*^2 \arrow[u, phantom, sloped, "\subset"] & H_*^4 \arrow[l, "L"] \arrow[u, phantom, sloped, "\subset"] & H_*^6 \arrow[l, "L"] \arrow[u, phantom, sloped, "\subset"] & \cdots \arrow[l]
\end{tikzcd}$$

Here $F = L^{-1}(H^4 \cap H_0^1)$ with $L : H^6 \cap H_0^1 \longrightarrow H^4$ etc. Now $G_0 : H^4 \cap H_0^1 \to H^4 \cap H_0^1$ is of form $G_0 = GJ = L^{-1}J$. So $im(G_0) \subset H_*^4$. Similarly $G_0 : H^6 \cap H_0^1 \to H^6 \cap H_0^1$ is given by $G_0 = GJGJ = G^2J^2$ and $im(G_0^2) \subset im(G^2 H_*^2)$ or H_*^6. Continue in the same way to finish the proof.

Section 3

The goal of this section is to prove the main theorem of section 1. We do that first for the case that the data can be expanded in a Fourier series. More precisely:

PROPOSITION. *Suppose* $t \to w_t$, $t \geq 0$ *is a* C^∞ *curve in* H_*^m *and* $\hat{v} \in H_*^m$. *Let* l *satisfy* $m - 2l \geq 2$, $l > 0$. *Then there is a unique* C^l *curve* $t \to v_t$ *in* H_*^{m-2l} *such that* $J_1(\partial v_t/\partial t) + Lv_t = J_1 J w_t$ *with* $v_0 = J\hat{v}$.

Here $J_1 : H_*^{m-2l} \to H_*^{m-2(l+1)}$ and $J : H_*^m \to H_*^{m-2l}$ are all inclusion maps. One may relax the C^∞ condition on $t \to w_t$ as the proof shows.

COROLLARY. *Under the same hypotheses, there exists a unique C^l curve $t \to v_t$ in H_*^{m-2l} such that $v_0 = J\hat{v}$ and $(I + G_0 T)v_t = Jw_t$.*

Here $I: H_*^{m-2l} \to H_*^{m-2l}$ is the identity and $T = \partial/\partial t$. For the corollary simply apply G to the equation of the proposition.

The main part of the proposition is contained in the following lemma.

LEMMA 1 (of Fourier type). *Under the hypotheses of the proposition, there is a unique C^1 curve v_t in H_*^{m-2} such that $v_0 = J_1\hat{v}$ and*

$$J_1 \frac{\partial}{\partial t} v_t + Lv_t = J_1 w_t, \tag{1}$$

where $J_1: H_*^m \to H_*^{m-2}$ is the inclusion.

Postponing momentarily the proof of the lemma, we see how the proposition is a consequence via a simple induction. Say v_t is a C^k curve in H_*^{m-2k} satisfying (1), J_1 the appropriate inclusion. Apply $J_0: H_*^{m-2k} \to H_*^{m-2k-2}$ to both sides to obtain that v_t is c^{k+1} in H_*^{m-2k-2}.

For the proof of the Lemma, first examine just what convergence in H_*^m means. Say $m = 2k$ (we only use these results for m even; and for m odd, the proofs are similar). Then since $L^k: H_*^m \to H_*^0$ is an isomorphism, $\sum_{i=1}^{\infty} c_i\phi_i$ converges in H_*^m if and only if $\sum c_i\lambda_i^k\phi_i$ converges in $H_*^0 = L^2$ or equivalently $\sum |c_i\lambda_i^k|^2 < \infty$.

Now expand the data of the lemma in a Fourier series, i.e., we may write $\hat{v} = \sum_{i=1}^{\infty} c_i\phi_i$ and $w_t = \sum_{i=1}^{\infty} a_i(t)\phi_i$ in H_*^m. Hence the c_i are constants and the $a_i(t)$ are real valued functions of t. In fact, $a_i(t)$ is C^∞ since it is the projection of a C^∞ function. For $u_t = \sum_{i=1}^{\infty} b_i(t)\phi_i$, the equation of the lemma is

$$\sum b_i'(t)\phi_i + \sum \lambda_i b_i(t)\phi_i = \sum a_i(t)\phi_i$$

or for each i,

$$b_i'(t) + \lambda_i b_i(t) = a_i(t), \qquad b_i(0) = c_i.$$

The unique solution is (see practically any book on ordinary differential equations)

$$b_i(t) = e^{-\lambda_i t}\left[\int_0^t a_i(s)e^{\lambda_i s}\, ds + c_i\right].$$

We claim that the curve u_t defined above in terms of the $b_i(t)$ converges in H^m_* and satisfies the properties in the lemma.

Since for each $t \geq 0, e^{-\lambda_i t} \leq 1$ except for a finite number of i, and $\sum c_i\phi_i$ converges in H^m_*, it follows that $\sum c_i e^{-\lambda_i t}\phi_i$ also converges in H^m_* for each t. The continuity in t of this sum is an easy check which we leave to the reader.

Next we show that $\sum d_i(t)\phi_i$ converges to a continuous function of t in H^m_* where $m = 2k$ and

$$d_i(t) = \int_0^t a_i(s)e^{\lambda_i(s-t)}\, ds.$$

First estimate by Cauchy's inequality,

$$|d_i(t)|^2 = \int_0^t |a_i(s)|^2\, ds \int_0^t e^{2\lambda_i(s-t)}\, ds$$

$$= \int_0^t |a_i(s)|^2\, ds \frac{1}{2\lambda_i}[1 - e^{-2\lambda_i t}]$$

Thus

$$\sum_{\lambda_i \geq 1/2} |d_i(t)|^2 \lambda_i^{2k} \leq \sum_{\lambda_i \geq 1/2} \int_0^t |a_i(s)|^2 \lambda_i^{2k}\, ds \leq K$$

where

$$K = \max_{0 \leq s \leq t} |L^k w_s|^2_{H^0}$$

Thus $\sum d_i(t)\phi_i$ converges in H^m_* and so does $u_t = \sum b_i(t)\phi_i$. The continuity in t is proved similarly. The rest of the proof of lemma 1 follows from the definition of $b_i(t)$ obtaining u_t C^1 in t in H^{m-2}_*.

Now consider the general problem of section 1. Thus $C^\infty f: R^+ \times \Omega \to R$ is given and the problem is to find $C^\infty u: R^+ \times \Omega \to R$ satisfying zero boundary conditions such that

$$Tu + Lu = f \quad \text{on} \quad R^+ \times \Omega \tag{2}$$

where $T = \partial/\partial t$. Let $f_t(x) = f(t, x)$; then it is easily seen that the map $R^+ \to H^k$ given by $t \to f_t$ is a C^∞ curve in H^k any k. Let $J: H^{k+2} \cap H^1_0 \to H^k$ be the inclusion and consider the following version of (2).

$$TJu + Lu_t = f_t, \quad u_t \quad \text{a curve in} \quad H^{k+2} \cap H^1_0, \quad u_0 = 0. \tag{3}$$

Let $m=k+2$ and apply $G:H^k\to H^m\cap H_0^1$ to both sides of (3) to obtain

$$(I+TG_0)u_t=g_t,\qquad u_t\in H^m\cap H_0^1,\qquad u_0=0 \tag{4}$$

where the datum $g_t=Gf_t$ is now a curve in $H^m\cap H_0^1$.

This form suggests trying to invert $I+TG_0$ or to look at:

$$u_t=[I-(G_0T)+(G_0T)^2-\cdots+(-1)^s(G_0T)^s]g_t+v_t \tag{5a}$$

$$(I+TG_0)v_t=(-G_0T)^{s+1}g_t=w_t \tag{5b}$$

For s large enough $w_t\in H_*^m$ by the proposition of section 1. We may apply the above Corollary to solve (5b) for v_t, and with an appropriate boundary condition at $t=0$, put this in (5a) to obtain our desired solution u_t.

Motivated by the above, we proceed more formally.

Set $m=2k=4l$, l some positive integer. The data f define a curve $t\to f_t$ in H^{m-2}. Let $g_t=Gf_t$ be the corresponding curve in $H^m\cap H_0^1$, C^∞ in t. Define $\gamma_t=[I-(G_0T)+\cdots+(-G_0T)^{k-2}]g_t$, which is a C^∞ curve in $H^m\cap H_0^1$ for $0\le t<\infty$.

LEMMA 2. $\gamma_0\in H_*^m$

Proof. Denote by $C_q(f)$ the condition of the theorem $P_q(L,T)f\,|_{t=0}\in C_0^\infty$, $q=0,1,\ldots,$. We will show that if $C_q(f)$ for $q\le k-2$, then $\sum_{i=0}^{k-2}(-TG_0)^ig_t\,|_{t=0}\in H_*^m$. Let $J:H^j\to H^{j-2}$ be the inclusion for various j and suppose f_t is the curve in H^{m-2} defined by the inclusion $C^\infty\to H^{m-2}$. Define $R_q=\sum_{i=0}^{q}(-TJ)^iL^{q-i}f_t\,|_{t=0}$ and note $R_q=(-TJ)^qf_t\,|_{t=0}+LR_{q-1}$. This latter could be used as an inductive definition of R_q starting with $R_{-1}=0$. R_q lies *a priori* in $H^{m-2(q+1)}$, but $C_q(f)$ implies that R_q lies in $H^{m-2(q+1)}\cap H_0^1$. Now suppose $C_q(f)$ is true for $q\le k-2$. Then $R_{k-2}\in H^2\cap H_0^1=H_*^2$, so $G^{k-1}R_{k-2}\in H_*^m$. By the inductive definition of R_q above, the L used in the definition of R_{k-2} have domain some $H^s\cap H_0^1$ so $GL=$ identity. Thus $G^{k-1}R_{k-2}=\sum_0^{k-2}(-TG_0)^ig_t\,|_{t=0}\in H_*^m$.

The curve $w_t=(-G_0T)^{k-1}g_t$ lies in H_*^m by the proposition of section 2. Let $J:H^m\cap H_0^1$ and $J:H_*^m\to H_*^k$ denote the inclusion. Apply the Corollary of the proposition in this section to obtain v_t in H_*^k of class C^l in t such that $(I+G_0T)v_t=Jw_t$, $v_0=-J\gamma_0$. Now define u_t in $H^k\cap H_0^1$ by $u_t=J\gamma_t+v_t$; so u_t is C^l in t, $u_0=J\gamma_0+v_0=0$ and $(I+G_0T)u_t=Jg_t$ in $H^k\cap H_0^1$.

If different l, say l_1, l_2, above are chosen, the corresponding u defined by the above process agree in $H^k\cap H_0^1$ where $k=2l$, $l=\min(l_1,l_2)$ using the uniqueness in the Corollary. Thus we obtain a u_t, which lies in each $H^k\cap H_0^1$. Thus by the Sobolev theorem, u_t is C^∞ and we have proved the first half of the main theorem.

For the second part, the above proof, with a couple of modifications which we state, is applicable.

The first modification is in lemma 1 and its consequences. One obtains a different ordinary differential equation, namely

$$b_i''(t)+\lambda_i b_i(t)=a_i(t), \qquad i=1,2,\dots$$

$$\text{with} \quad b_i(0)=c_i, \qquad b_i'(0)=d_i$$

where $u_t=\sum b_i(t)\phi_i$, is to be found and $\sum c_i\phi_i=u_0$, $\sum d_i\phi_i=u_0'$, $\sum a_i(t)\phi_i=f_t$ are prescribed. This differential equation has as its unique solution, if $\lambda_i>0$

$$b_i(t)=b_i\cos t\sqrt{\lambda_i}+d_i\frac{\sin t\sqrt{\lambda_i}}{\sqrt{\lambda_i}}+\int_0^t a_i(s)\frac{\sin(t-2)\sqrt{\lambda_i}}{\sqrt{\lambda_i}}, \quad \text{all} \quad t.$$

The finite number of equations with $\lambda_i<0$ are handled as easily. Now one proceeds as before, with similar estimates to get convergence of $\sum b_i(t)\phi_i=u_t$.

The other modification relates to Lemma 2; but here just apply that construction of f and f' as well.

Section 4

This section is a series of remarks on extensions and relations to other problems of the above.

Section 3 of this paper could be considered as a theory of separation of variables for boundary value problems in PDE. It works well for problems which are the product of understood problems. Thus the evolution problems above are the product of space and time problems. We give more examples to illustrate this point.

Consider $\Delta u=f$ on the rectangle $\Omega=\Omega_a\times\Omega_b$ where $\Omega_a=[0,a]$, $\Omega_b=[0,b]$. Given $C^\infty f:\Omega\to R$, find a C^∞ solution $u:\Omega\to R$ such that $u=0$ on $\partial\Omega$. Write $(t,x)\in\Omega_a\times\Omega_b$ and $\Delta u=(\partial^2u/\partial t^2)+(\partial^2u/\partial x^2)=Tu-Lu$ and proceed as before to obtain NASC on f for the existence of a solution u. The Fourier lemma and proposition at the beginning of section 3 must be replaced by a simple spectral analysis of T (similar to that of L).

A second example is the wave equation on the same domain, $(T+L)u=f$ on Ω with Dirichlet boundary conditions $u=0$ on $\partial\Omega$(!). This problem has been considered By Fritz John [4], V. Arnold [1] and others. Now the above analysis applies. Besides the compatibility condition on f, one needs in general that the ratio a/b be not rational (or not even close to rational?).

Finally we list some ways in which the main theorem might be extended.

(A) If condition (3) on the elliptic operator is dropped, i.e., some eigenvalues are allowed to be zero, the methods extend easily to yield similar results.

(B) If L is not self-adjoint, one could no doubt replace the Fourier lemma of section 3 by a different existence proof, the rest being the same as before.

(C) The extension to complex coefficients or systems should not require substantial changes.

(D) The operator T in the theorem of section 1 could be replaced by any ordinary linear differential operator with leading coefficient 1. Then the results would have to be modified at the boundary condition $t=0$. Schrödinger's equation on bounded spatial domains thus can be included.

(E) Ω could be a compact manifold with boundary

(F) Perhaps one could obtain C solutions to Navier-Stokes on compact $\Omega \subset R^n$, $\partial\Omega$ smooth, for small time via eigenfunction expansions this way.

(G) In the main theorem of section 1, L is time independent. I am not sure how the extension of this result to the case of time dependent L should go.

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