Injectivity of local quasi-isometries.

Comment. Math. Helvetici 57 (1982) 202–220 0010–2571/82/002202–19$01.50+0.20/0

Injectivity of local quasi-isometries

F. W. GEHRING[1]

1. Introduction

Suppose that E is a set in $\bar{R}^n$, the one point compactification of euclidean n-space R^n, $n \geq 2$, and suppose that f is a mapping from E into $\bar{R}^n$. We say that f is an *L-quasi-isometry* in E if

$$\frac{1}{L} \leq \frac{|f(x_1)-f(x_2)|}{|x_1-x_2|} \leq L \tag{1}$$

for each pair of points $x_1, x_2 \in E-\{\infty\}$ and if $f(\infty)=\infty$ whenever $\infty \in E$. We say that f is a *local L-quasi-isometry* in E if for each $L'>L$ each $x \in E$ has a neighborhood U such that f is an L'-quasi-isometry in $E \cap U$.

Suppose that f is a local L-quasi-isometry in a domain D in R^n. If $L=1$, then f is an isometry in D and hence injective there. (See, for example, Theorem IV in [11].) Simple examples show that f need not be injective if $L>1$. It was F. John who first noticed that for certain domains D, f will be injective provided L is close enough to 1.

For each domain $D \subset R^n$ we let $L(D)$ denote the supremum of the numbers $L \geq 1$ with the property that each local L-quasi-isometry in D is injective. We say that D is *rigid* if $L(D)>1$.

John established the following interesting result in 1969 (Theorem A in [12]). See also [7] and [8].

THEOREM 1. *If D is an open ball or half space, then $L(D) \geq 2^{1/4}$.*

This result was generalized by John and then extended recently by Martio and Sarvas to a very broad class of domains. We say that $D \subset R^n$ is a *uniform domain* if there exist constants a and b with the following property. Each pair of points

---

[1] This research was supported in part by grants from the U.S. National Science Foundation (Grant MCS 79-01713) and the Finnish Ministry of Education.

$x_1, x_2 \in D$ can be joined by a rectifiable arc α in D so that

$$l(\alpha) \leq a\,|x_1 - x_2| \tag{2}$$

and so that for each $x \in \alpha$

$$\min_{j=1,2} l(\alpha_j) \leq b\,\mathrm{d}\,(x, \partial D), \tag{3}$$

where α_1, α_2 denote the components of $\alpha - \{x\}$. Here $l(\alpha)$ denotes the euclidean length of α and $\mathrm{d}\,(x, \partial D)$ the distance from x to ∂D.

Martio and Sarvas showed that uniform domains are rigid by establishing the following result (Theorem 3.8 in [14]).

THEOREM 2. *If D is a uniform domain, then $L(D) \geq c > 1$ where c depends only on the constants a and b.*

The present paper is concerned with the problem of identifying the domains in R^n which are rigid. In particular, we characterize in Section 2 the finitely connected plane domains which have this property. It turns out that each boundary component of such a domain is either a point or a quasicircle, that is, the image of a circle or a line under a quasiconformal mapping of $\bar{R}^2$. In Section 3 we establish an extension theorem for quasi-isometries. We then apply this result in Section 4 to show that if D is a simply connected rigid domain in R^2 and if f is a local L-quasi-isometry in D with $L < L(D)$, then f is not only injective in D but has an extension as a quasi-isometry to all of R^2.

2. Rigid plane domains

Throughout the remainder of this paper we shall use complex notation to denote points in R^2. For $z_0 \in R^2$ and $0 < r < \infty$ we let $B(z_0, r)$ denote the open disk with center z_0 and radius r. Finally for each domain $D \subset \bar{R}^2$ we let $D^* = \bar{R}^2 - \bar{D}$.

In this section we characterize the finitely connected domains in R^2 which are rigid. We begin with a technical lemma concerning a special class of quasi-isometries.

LEMMA 1. *Suppose that $\phi(t)$ is a real valued function defined in $(0, \infty)$, that*

$$|\phi(t_1) - \phi(t_2)| \leq a \left|\log \frac{t_1}{t_2}\right| \tag{4}$$

for $t_1, t_2 \in (0, \infty)$ *and that*

$$f(z) = \begin{cases} ze^{i\phi(|z|)} & \text{if } 0<|z|<\infty, \\ 0 & \text{if } z=0. \end{cases} \tag{5}$$

Then f *is a* $(1+a)$*-quasi-isometry in* R^2.

Proof. Choose distinct points $z_1, z_2 \in R^2$ with $|z_1| \leq |z_2|$. If $z_1 \neq 0$, then

$$\begin{aligned} |f(z_1)-f(z_2)| &\leq |z_1-z_2| + |z_1|\,|e^{i\phi(|z_1|)} - e^{i\phi(|z_2|)}| \\ &\leq |z_1-z_2| + |z_1|\,|\phi(|z_1|)-\phi(|z_2|)| \\ &\leq |z_1-z_2| + a\,|z_1|\left|\log\frac{|z_1|}{|z_2|}\right| \\ &\leq (1+a)\,|z_1-z_2| \end{aligned}$$

by (4), while

$$|f(z_1)-f(z_2)| = |z_2| \leq (1+a)\,|z_1-z_2|$$

if $z_1=0$. Since f^{-1} is given by (5) with $-\phi$ in place of ϕ, the above argument can be applied to f^{-1} to complete the proof.

We next use Lemma 1 to obtain a geometric property of plane domains D with $L(D)>1$.

LEMMA 2. *Suppose that* D *is a domain in* R^2 *with* $L(D) \geq c > 1$. *Then there exists a constant* b, *depending only on* c, *such that for each* $z_0 \in R^2$ *and* $0<r<\infty$, $D \cap \partial B(z_0, r)$ *lies in component of*

$$G = D \cap (B(z_0, br) - \bar{B}(z_0, r/b)).$$

Proof. Choose $b \in (1, \infty)$ so that

$$1 + \frac{\pi}{\log b} < c, \tag{6}$$

and suppose there exist points $z_1, z_2 \in D \cap \partial B(z_0, r)$ which belong to different components G_1, G_2 of G. By making a change of variable we may assume that

$z_0=0$. Choose $\theta\in[-\pi,\pi]$ so that $z_2=z_1e^{i\theta}$ and let f be as in (5) with

$$\phi(t)=\begin{cases} 0 & \text{if} \quad 0<t\le\dfrac{r}{b} \quad \text{or} \quad br\le t<\infty, \\[2ex] \dfrac{\log\dfrac{bt}{r}}{\log b}\,\theta & \text{if} \quad \dfrac{r}{b}\le t\le r, \\[2ex] \dfrac{\log\dfrac{br}{t}}{\log b}\,\theta & \text{if} \quad r\le t\le br. \end{cases}$$

Then ϕ satisfies (4) with $a=\pi/\log b$ and f is a $(1+a)$-quasi-isometry in R^2 by Lemma 1. Set

$$g(z)=\begin{cases} z & \text{if} \quad z\in D-G_1, \\ f(z) & \text{if} \quad z\in G_1. \end{cases} \tag{7}$$

If U is any open disk in D, then either $U\subset D-G_1$, in which case $g(z)=z$ in U, or $U\subset G_1\cup(D-G)$, in which case $g(z)=f(z)$ in U. Hence g is a local $(1+a)$-quasi-isometry in D. Since $z_2\notin G_1$,

$$g(z_2)=z_2=z_1e^{i\theta}=z_1e^{i\phi(|z_1|)}=g(z_1)$$

and g is not injective in D. Thus $c\le 1+a$. This contradicts (6) and establishes the desired conclusion.

We say that $C\subset\bar{R}^2$ is a *K-quasicircle* if it is the image of a circle or line under a K-quasiconformal mapping $f:\bar{R}^2\to\bar{R}^2$. Similarly $D\subset\bar{R}^2$ is said to be a *K-quasidisk* if ∂D is a K-quasiciricle.

We have next the following information about the boundary of a rigid plane domain.

LEMMA 3. *Suppose that D is a domain in R^2 with $L(D)\ge c>1$. Then each component C of ∂D is either a point or a K-quasicircle where K depends only on c. Moreover if C_1 and C_2 are components of ∂D, then*

$$\min_{j=1,2}\operatorname{dia}(C_j)\le a\,\mathrm{d}(C_1,C_2) \tag{8}$$

where a is a constant which depends only on c.

Here dia (C_j) denotes the diameter of C_j and d (C_1, C_2) the distance between C_1 and C_2.

Proof. Choose $b \in (1, \infty)$ so that (6) holds and suppose that $z_0 \in R^2$, $0<r<\infty$ and $z_1, z_2 \in D \cap \bar{B}(z_0, r)$. Let α be any arc joining z_1 and z_2 in D. If α does not lie in $\bar{B}(z_0, r)$, then $\alpha \cap \bar{B}(z_0, r)$ contains two components α_1, α_2 which join z_1, z_2 to $w_1, w_2 \in \partial B(z_0, r)$, respectively. Lemma 2 implies that w_1 and w_2 can be joined by an arc β in $D \cap \bar{B}(z_0, br)$ and hence $\alpha_1 \cup \beta \cup \alpha_2$ joins z_1 and z_2 in $D \cap \bar{B}(z_0, br)$. A similar argument shows that any pair of points $z_1, z_2 \in D - B(z_0, r)$ can be joined in $D - B(z_0, r/b)$. Hence D is *b-locally connected* and by Lemma 5 in [3], each component C of ∂D is either a point or a K-quasicircle where K depends only on b.

Suppose next that C_1 and C_2 are distinct components of ∂D, choose $z_1 \in C_1$ and $z_2 \in C_2$ so that

$$|z_1 - z_2| = \mathrm{d}\,(C_1, C_2) = 2r$$

and let $z_0 = \frac{1}{2}(z_1 + z_2)$. We shall use Lemma 2 to show that C_1 or C_2 lies in $B(z_0, b^2 r)$ and hence that

$$\min_{j=1,2} \mathrm{dia}\,(C_j) \leq 2b^2 r.$$

This will establish (8) with $a = b^2$.

Suppose that C_1 and C_2 do not lie in $B(z_0, b^2 r)$ and let D_0 denote the component of $\bar{R}^2 - (C_1 \cup C_2)$ which contains D. Then

$$F_1 = \bar{R}^2 - (D_0 \cap B(z_0, b^2 r)), \qquad F_2 = \bar{B}(z_0, r)$$

are continua with

$$F_1 \cap F_2 = (C_1 \cup C_2) \cap \bar{B}(z_0, r) = \{z_1, z_2\}. \tag{9}$$

Hence by Theorem V.11.5 in [15], there exist points w_1, w_2 which lie in different components G_1, G_2 of

$$\bar{R}^2 - (F_1 \cup F_2) = D_0 \cap (B(z_0, b^2 r) - \bar{B}(z_0, r))$$

but which can be joined by an arc α in

$$\bar{R}^2 - F_1 = D_0 \cap B(z_0, b^2 r).$$

Next (9) and Theorem V.16.2 in [15] imply that w_1, w_2 are not separated by $C_1 \cup C_2 \cup \bar{B}(z_0, r)$ and hence can be joined by an arc β in $D_0 - \bar{B}(z_0, r)$. Thus for $j = 1, 2$, $\alpha \cup \beta$ contains a curve which joins $\partial B(z_0, r)$ to $\partial B(z_0, b^2 r)$ in G_j; hence

$$H_j = G_j \cap \partial B(z_0, br) \neq \emptyset.$$

Since each component of H_j is an open arc in D_0 with endpoints in $C_1 \cup C_2$, $D \cap H_j \neq \emptyset$ and we conclude that $D \cap \partial B(z_0, br)$ does not lie in a component of $D \cap (B(z_0, b^2 r) - \bar{B}(z_0, r))$. This contradicts Lemma 2 and thus establishes the desired conclusion.

Finally we have the following relations between quasidisks and rigid plane domains.

THEOREM 3. *If D is a K-quasidisk in R^2, then $L(D) \geq c > 1$ where c depends only on K. Conversely if D is a simply connected proper subdomain of R^2 with $L(D) \geq c > 1$, then D is a K-quasidisk where K depends only on c.*

Proof. If D is a K-quasidisk in R^2, then by Corollary 2.33 in [14], D is a uniform domain where the constants a and b in (2) and (3) depend only on K. (For an alternative proof see Theorem III.2.3 in [4].) Hence $L(D) \geq c > 1$ where $c = c(K)$ by Theorem 2. The converse is a consequence of Lemma 3.

THEOREM 4. *A finitely connected domain D in R^2 is rigid if and only if each component of ∂D is either a point or a quasicircle.*

Proof. If D is bounded by a finite number of points or quasicircles, then D is uniform by Theorem 5 in [16] and Theorem 5 in [6]; hence D is rigid by Theorem 2. The converse follows from Lemma 3.

The problem of characterizing rigid plane domains D is more difficult when D is infinitely connected. For example, if $\mathscr{C}$ denotes the collection of boundary components of a rigid domain D in R^2, then

$$\sup_{C, C' \in \mathscr{C}} \frac{\min(\operatorname{dia}(C), \operatorname{dia}(C'))}{\operatorname{d}(C, C')} < \infty$$

by Lemma 3. Hence one must take into account not only the shape but the relative size and position of the boundary components when D has infinite connectivity.

We conclude this section by exhibiting a plane domain D which is rigid but not uniform; thus the converse of Theorem 2 does not hold. The existence of such a domain is an immediate consequence of the following result.

THEOREM 5. *If D is a rigid domain in R^2 and if E is a discrete subset of D, then $D-E$ is a rigid domain.*

Proof. Suppose that U is an open disk with center at z_0 and let $U_0 = U - \{z_0\}$. Then since $L(D)$ is invariant under similarity mappings, Theorem 4 implies that $L(U_0)$ is an absolute constant c which exceeds 1.

Suppose next that f is a local L-quasi-isometry in $D-E$ with $L < \min(L(D), c)$. Given $z_0 \in E$ we can choose an open disk U centered at z_0 such that

$$U_0 = U - \{z_0\} \subset D - E.$$

If $z_1, z_2 \in U_0$, then for each $\varepsilon > 0$ we can find an arc α joining z_1 and z_2 in U_0 with

$$l(\alpha) \leq (1+\varepsilon)\,|z_1 - z_2|.$$

Since f is a local L-quasi-isometry in U_0,

$$|f(z_1) - f(z_2)| \leq l(f(\alpha)) \leq L l(\alpha) \leq L(1+\varepsilon)\,|z_1 - z_2|,$$

and letting $\varepsilon \to 0$ yields

$$|f(z_1) - f(z_2)| \leq L\,|z_1 - z_2|. \tag{10}$$

Then (10) implies that f has a continuous extension in U which satisfies (10) for $z_1, z_2 \in U$. Next since $L < c$, f is injective in U_0, and it follows that f is injective and hence a homeomorphism in U. Choose an open disk V about $f(z_0)$ with $V \subset f(U)$. Then $g = (f \mid U)^{-1}$ is a local L-quasi-isometry in $V_0 = V - \{f(z_0)\}$, and the above argument applied to g shows that f is an L-quasi-isometry in $g(V)$. Thus f has an extension to D which is a local L-quasi-isometry in a neighborhood of each point of E and hence in D. Then since $L < L(D)$, f is injective in D. Hence f is injective in $D-E$,

$$L(D-E) \geq \min(L(D), c) > 1$$

and $D-E$ is a rigid domain.

Now let B denote the unit disk and let

$$E = \left\{ z = \left(1 - \frac{1}{j}\right) \exp\left(\frac{2\pi i k}{j^2}\right) : k = 1, 2, \ldots, j^2, j = 2, 3, \ldots \right\}.$$

Then $D=B-E$ is rigid by Theorem 5. On the other hand if $z_1=0$ and if $z_2\in D$ with $|z_2|\geq 1-1/2j$ and $j\geq 2$, then each rectifiable arc α joining z_1 and z_2 in D must contain a point z with $|z|=1-1/j$ and

$$\min_{j=1,2} l(\alpha_j)\geq \frac{j}{2\pi}\,\mathrm{d}\,(z,\partial D),$$

where α_1, α_2 denote the components of $\alpha-\{z\}$. Hence there exists no constant b for which D satisfies condition (3) and D is not a uniform domain.

3. Extension of quasi-isometries

We establish here some extension theorems for plane quasi-isometries. Our arguments are based on a reflection principle for quasidisks due to Ahlfors [1] and estimates for the hyperbolic distance.

If D is a simply connected proper subdomain of R^2, then the *hyperbolic metric* with curvature -1 in D is given by

$$\rho_D(z)=\frac{|g'(z)|}{\mathrm{Im}\,(g(z))},$$

where g is any conformal mapping of D onto the upper half plane H. From standard distortion theorems it follows that

$$\tfrac{1}{2}\leq |g'(z)|\frac{\mathrm{d}\,(z,\partial D)}{\mathrm{Im}\,(g(z))}\leq 2, \tag{11}$$

where $\mathrm{d}\,(z,\partial D)$ denotes the distance from z to ∂D, and hence that

$$\frac{1}{2\mathrm{d}\,(z,\partial D)}\leq \rho_D(z)\leq \frac{2}{\mathrm{d}\,(z,\partial D)}. \tag{12}$$

(See, for example, p. 22 in [17].) Next the *hyperbolic distance* between points $z_1, z_2\in D$ is given by

$$h_D(z_1,z_2)=\inf_\alpha \int_\alpha \rho_D(z)\,|dz|,$$

where the infimum is taken over all rectifiable arcs α joining z_1 and z_2 in D. From

(12) and Lemma 2.1 in [5] it follows that

$$h_D(z_1, z_2) \geq \frac{1}{2} \left| \log \frac{d(z_1, \partial D)}{d(z_2, \partial D)} \right| \tag{13}$$

for $z_1, z_2 \in D$. Next if D is a K-quasidisk, then by (12), Corollary 2.33 in [14] and Theorem 1 in [6],

$$h_D(z_1, z_2) \leq c \log \left(\frac{|z_1 - z_2|}{d(z_1, \partial D)} + 1\right)\left(\frac{|z_1 - z_2|}{d(z_2, \partial D)} + 1\right) + d \cdot \tag{14}$$

for $z_1, z_2 \in D$, where c and d are constants which depend only on K. (Cf. pp. 42–44 in [13].)

We begin with a result on a special class of quasi-isometries.

LEMMA 4. *If D is a Jordan domain in R^2 and if $z_1, z_2 \in D$ with $h_D(z_1, z_2) \leq a$, then there exists an L-quasi-isometry $f: \bar{D} \to \bar{D}$ such that f is the identity on ∂D, $f(z_1) = z_2$ and L depends only on a.*

Proof. Choose a conformal mapping $g: D \to H$ normalized so that $g(z_1) = i$ and $g(z_2) = bi$ where $b > 1$. Then

$$\log b = h_D(z_1, z_2) \leq a$$

and g extends to a homeomorphism which maps $\bar{D}$ onto $\bar{H}$. Set

$$h(w) = \begin{cases} u + ibv & \text{if} \quad w = u + iv \in \bar{H} - \{\infty\}, \\ \infty & \text{if} \quad w = \infty. \end{cases}$$

Then h is continuously differentiable with

$$\left.\begin{array}{l} h_H(h(w), w) = \log b \leq a, \\ \dfrac{1}{b} \dfrac{|dw|}{\operatorname{Im}(w)} \leq \dfrac{|dh(w)|}{\operatorname{Im}(h(w))} \leq \dfrac{|dw|}{\operatorname{Im}(w)} \end{array}\right\} \tag{15}$$

in H and $f = g^{-1} \circ h \circ g$ is a homeomorphism of $\bar{D}$ onto $\bar{D}$ which is the identity on ∂D and maps z_1 onto z_2.

Fix $z \in D$ and set $w = g(z)$. Then

$$\frac{|df(z)|}{|dz|} = \frac{|dh(w)|}{|dw|} \frac{|g'(z)|}{|g'(f(z))|}, \tag{16}$$

while we obtain

$$\left.\begin{aligned} \tfrac{1}{2} \le |g'(z)| \frac{\mathrm{d}(z, \partial D)}{\mathrm{Im}(w)} \le 2 \\ \tfrac{1}{2} \le |g'(f(z))| \frac{\mathrm{d}(f(z), \partial D)}{\mathrm{Im}(h(w))} \le 2 \end{aligned}\right\} \tag{17}$$

from (11). Next by (13) and (15),

$$\frac{1}{2}\left|\log \frac{\mathrm{d}(f(z), \partial D)}{\mathrm{d}(z, \partial D)}\right| \le h_D(f(z), z) = h_H(h(w), w) \le a$$

whence

$$e^{-2a} \le \frac{\mathrm{d}(f(z), \partial D)}{\mathrm{d}(z, \partial D)} \le e^{2a}. \tag{18}$$

Combining (15), (16), (17) and (18) yields

$$\frac{1}{L} \le \frac{|df(z)|}{|dz|} \le L$$

where $L = 4e^{3a}$, and hence f is a local L-quasi-isometry in D. (Cf. p. 395 in [10].) The desired conclusion is now a consequence of the following elementary result.

LEMMA 5. *Suppose that D_1 and D_2 are domains in R^2, that $f: \bar{D}_1 \to \bar{D}_2$ is a homeomorphism and that f is an L_1-quasi-isometry in ∂D_1 and a local L_2-quasi-isometry in D_1. Then f is an L-quasi-isometry in $\bar{D}_1$ where* $L = \max(L_1, L_2)$.

Proof. Fix $z_1, z_2 \in D_1$ and let α be the open segment joining these points in R^2. If $\alpha \subset D_1$, then

$$|f(z_1) - f(z_2)| \le l(f(\alpha)) \le L_2 l(\alpha) \le L\,|z_1 - z_2|.$$

Otherwise for $j=1,2$ let α_j denote the component of $\alpha \cap D_1$ which has z_j as an endpoint and let w_j denote the other endpoint of α_j. Then $w_j \in \partial D_1$, $\alpha_j \subset D_1$ and

$$\begin{aligned}|f(z_1)-f(z_2)| &\leq |f(z_1)-f(w_1)|+|f(w_1)-f(w_2)|+|f(w_2)-f(z_2)| \\ &\leq L\,|z_1-w_1|+L_1\,|w_1-w_2|+L\,|w_2-z_2| \\ &\leq L\,|z_1-z_2|.\end{aligned}$$

Applying this argument to f^{-1} shows that f is an L-quasi-isometry in D_1, and hence in $\bar{D}_1$ by continuity.

Lemma 5 shows that a bijective local quasi-isometry between two domains is a quasi-isometry if the induced boundary correspondence is a quasi-isometry. We can also draw this conclusion without knowledge of the boundary correspondence when the two domains have sufficiently regular boundaries.

LEMMA 6. *Suppose that D_1 and D_2 are K_1- and K_2-quasidisks in R^2 and that $f: D_1 \to D_2$ is a bijective local L_1-quasi-isometry. Then f extends to an L-quasi-isometry of $\bar{D}_1$ onto $\bar{D}_2$ where L depends only on K_1, K_2 and L_1.*

Proof. Fix $z_1, z_2 \in D_1$. By Corollary 2.33 in [14], there exists a rectifiable arc α joining z_1 and z_2 in D_1 such that

$$l(\alpha) \leq a_1\,|z_1-z_2|,$$

where a_1 depends only on K_1. Thus

$$|f(z_1)-f(z_2)| \leq l(f(\alpha)) \leq L_1 l(\alpha) \leq L_1 a_1\,|z_1-z_2|.$$

Next since f is injective, f^{-1} is a local L_1-quasi-isometry in D_2 and arguing as above yields

$$|z_1-z_2| \leq L_1 a_2\,|f(z_1)-f(z_2)|,$$

where a_2 depends only on K_2. Hence f is an L-quasi-isometry in D_1 where $L = \max(L_1 a_1, L_1 a_2)$, and we can extend f to $\bar{D}_1$ by continuity.

We will require the following version of Lemma 4 for the case where D is a quasidisk.

LEMMA 7. *Suppose that D is a K-quasidisk in R^2, that $z_1, z_2 \in D$ and that*

$$\frac{1}{b} \leq \frac{|z_1-z|}{|z_2-z|} \leq b \tag{19}$$

for all $z \in \partial D - \{\infty\}$ *where* b *is a constant. Then there exists an* L*-quasi-isometry* $f: \bar{D} \to \bar{D}$ *such that* f *is the identity on* ∂D, $f(z_1) = z_2$ *and* L *depends only on* K *and* b.

Proof. For $j = 1, 2$ choose $w_j \in \partial D - \{\infty\}$ so that

$$|z_j - w_j| = \mathrm{d}\,(z_j, \partial D).$$

Then by (19),

$$|z_1 - z_2| \leq |z_1 - w_j| + |z_2 - w_j| \leq (b+1)\,\mathrm{d}\,(z_j, \partial D)$$

and hence

$$h_D(z_1, z_2) \leq 2c \log (b+2) + d = a$$

by (14), where c and d depend only on K. The desired conclusion now follows directly from Lemma 4.

We derive now an extension of Ahlfors' reflection principle for quasidisks. (See, for example, Lemma 3 on p. 80 in [2].)

THEOREM 6. *Suppose that* D_1 *is a* K_1*-quasidisk with* $\infty \in \partial D_1$, *that* D_2 *is a Jordan domain in* R^2 *with* $\infty \in \partial D_2$ *and that* $\phi: \partial D_1 \to \partial D_2$ *is an* L_1*-quasi-isometry. Then there exists an* L*-quasi-isometry* $f: \bar{D}_1 \to \bar{D}_2$ *such that* $f = \phi$ *on* ∂D_1 *and* L *depends only on* K_1 *and* L_1. *Suppose further that* $z_1 \in D_1$, $z_2 \in D_2$ *and*

$$\frac{1}{b} \leq \frac{|z_1 - z|}{|z_2 - \phi(z)|} \leq b \tag{20}$$

for all $z \in \partial D_1 - \{\infty\}$ *where* b *is a constant. Then we can choose* f *so that, in addition,* $f(z_1) = z_2$ *and* L *depends only on* K_1, L_1 *and* b.

If we choose $D_2 = D_1^*$ and $\phi(z) = z$, then the first part of Theorem 6 yields the above mentioned result of Ahlfors.

Proof. For $j = 1, 2$ let g_j map D_j conformally onto the upper half plane H. Then g_j extends to a homeomorphism of $\bar{D}_j$ onto $\bar{H}$ and by performing an additional Möbius transformation we may assume that $g_j(\infty) = \infty$. Hence $\psi(x) = g_2 \circ \phi \circ g_1^{-1}(x)$ is a homeomorphism of ∂H onto itself with $\psi(\infty) = \infty$.

Choose $-\infty < x < \infty$ and $t > 0$, let $\alpha_1' = (x, x+t)$ and $\beta_1' = (-\infty, x-t)$, and let $\alpha_1, \alpha_2, \alpha_2'$ and $\beta_1, \beta_2, \beta_2'$ denote the images of α_1' and β_1' under g_1^{-1}, $\phi \circ g_1^{-1}$, ψ respectively. If Γ_1 is the family of arcs joining α_1 to β_1 in D_1, then the extremal length $\lambda(\Gamma_1)$ of Γ_1 is equal to 1. Moreover since D_1 is a K_1-quasidisk,

$$|z_1-z_2|\le c_1\,|z_1-z_3| \tag{21}$$

for each ordered triple of points $z_1, z_2, z_3\in\partial D_1-\{\infty\}$ where c_1 is a constant which depends only on K_1. In particular if we let $z_1=g_1^{-1}(x)$ and $w_1=g_1^{-1}(x+t)$, then the argument on pp. 82–83 in [2] shows that

$$\alpha_1\subset\bar{B}(w_1, r), \qquad r=c_1\,|z_1-w_1|$$

and that

$$\mathrm{d}\,(\alpha_1, \beta_1)\ge s=c_1^{-5}e^{-2\pi}r.$$

Since ϕ is an L_1-quasi-isometry,

$$\alpha_2\subset\bar{B}(w_2, L_1r), \qquad \mathrm{d}\,(\alpha_2, \beta_2)\ge\frac{s}{L_1}$$

where $w_2=\phi(w_1)$, and arguing again as on p. 83 in [2] we see that

$$\lambda(\Gamma_2)\ge\frac{1}{\pi}\left(\frac{s}{L_1^2r+s}\right)^2,$$

where Γ_2 is the family of arcs joining α_2 to β_2 in D_2. This implies that

$$\frac{\psi(x+t)-\psi(x)}{\psi(x)-\psi(x-t)}\le c_2, \tag{22}$$

where c_2 is a constant which depends only on K_1 and L_1. From (22) and the above argument with $\alpha_1'=(x-t, x)$ and $\beta_1'=(x+t, \infty)$ we conclude that

$$\frac{1}{c_2}\le\frac{\psi(x+t)-\psi(x)}{\psi(x)-\psi(x-t)}\le c_2$$

for all such x and t. Set

$$h(z)=\frac{1}{2}\int_0^1(\psi(x+ty)+\psi(x-ty))dt+\frac{i}{2}\int_0^1|\psi(x+ty)-\psi(x-ty)|\,dt$$

for $z=x+iy\in\bar{H}-\{\infty\}$ and $h(\infty)=\infty$. Then h maps $\bar{H}$ homeomorphically onto $\bar{H}$ and h is continuously differentiable and K-quasiconformal in H with

$$\frac{1}{c_3}\frac{|dz|}{\operatorname{Im}(z)}\leq\frac{|dh(z)|}{\operatorname{Im}(h(z))}\leq c_3\frac{|dz|}{\operatorname{Im}(z)},$$

where K and c_3 depend only on c_2, and hence on K_1 and L_1. (See pp. 69–74 in [2] for the case where $\psi(x)$ is increasing in x.) Thus $f_1=g_2^{-1}\circ h\circ g_1$ is a homeomorphism of $\bar{D}_1$ onto $\bar{D}_2$, $f=\phi$ on ∂D_1, f_1 is K-quasiconformal in D_1 and

$$\frac{|df_1(z)|}{|dz|}=\frac{|dh(w)|}{|dw|}\frac{|g_1'(z)|}{|g_2'(f_1(z))|}$$

for $z\in D_1$, $w=g_1(z)$. From (11) applied to g_1 and g_2 we obtain

$$\tfrac{1}{2}\leq|g_1'(z)|\frac{\mathrm{d}(z,\partial D_1)}{\operatorname{Im}(w)}\leq 2,$$

$$\tfrac{1}{2}\leq|g_2'(f_1(z))|\frac{\mathrm{d}(f_1(z),\partial D_2)}{\operatorname{Im}(h(w))}\leq 2.$$

Thus

$$\frac{1}{4c_3}\frac{\mathrm{d}(f_1(z),\partial D_2)}{\mathrm{d}(z,\partial D_1)}\leq\frac{|df_1(z)|}{|dz|}\leq 4c_3\frac{\mathrm{d}(f_1(z),\partial D_2)}{\mathrm{d}(z,\partial D_1)} \tag{23}$$

and it remains to bound the ratio on the left and right sides of (23).

If w_1, w_2, w_3 is an ordered triple of points in $\partial D_2-\{\infty\}$, then

$$|w_1-w_2|\leq c_1L_1^2|w_1-w_3|$$

by (21) and D_2 is a K_2-quasidisk where K_2 depends only on K_1 and L_1. Hence f_1 can be extended by quasiconformal reflection in ∂D_1 and ∂D_2 to yield a K_3-quasiconformal mapping of $\bar{R}^2$ onto itself with $K_3=KK_1^2K_2^2$. Fix $z_1\in D_1$ and $z_2\in\partial D_1-\{\infty\}$, and choose $z_3\in\partial D_1$ so that $|z_3-z_2|=|z_1-z_2|$. Since f_1 is K_3-quasiconformal in $\bar{R}^2$ with $f_1(\infty)=\infty$,

$$|f_1(z_1)-f_1(z_2)|\leq c\,|f_1(z_3)-f_1(z_2)|\leq c_4\,|z_3-z_2|=c_4\,|z_1-z_2|$$

where c and $c_4 = cL_1$ depend only on K_1 and L_1. We thus obtain

$$\frac{1}{c_4}|z_1 - z_2| \le |f_1(z_1) - f_1(z_2)| \le c_4 |z_1 - z_2| \tag{24}$$

for all $z_1 \in D_1$ and $z_2 \in \partial D_1 - \{\infty\}$. In particular, (24) implies that

$$\frac{1}{c_4} d(z, \partial D_1) \le \mathrm{d}(f_1(z), \partial D_2) \le c_4 \, \mathrm{d}(z, \partial D_1)$$

for all $z \in D_1$, and we conclude from (23) that f_1 is a local L_2-quasi-isometry in D_1 with $L_2 = 4c_3c_4$. Lemma 5 then implies that f_1 is an L_3-quasi-isometry in $\bar{D}_1$ where $L_3 = \max(L_1, L_2)$, and choosing $f = f_1$ completes the proof of the first part of Theorem 6.

Finally suppose that $z_1 \in D_1$, $z_2 \in D_2$ and that (20) holds for all $z \in \partial D_1 - \{\infty\}$. If $w \in \partial D_2 - \{\infty\}$, then $z = f_1^{-1}(w) \in \partial D_1 - \{\infty\}$ and

$$\frac{|f_1(z_1) - w|}{|z_2 - w|} = \frac{|f_1(z_1) - f_1(z)|}{|z_1 - z|} \frac{|z_1 - z|}{|z_2 - \phi(z)|}$$

lies between $(bc_4)^{-1}$ and bc_4 by (20) and (24). By Lemma 7 there exists an L_4-quasi-isometry $f_2 : \bar{D}_2 \to \bar{D}_2$ such that f_2 is the identity on ∂D_2, $f_2(f_1(z_1)) = z_2$ and L_4 depends only on K_2 and b. Thus $f = f_2 \circ f_1$ has all the properties required in the second part of Theorem 6.

Finally we require the following result which shows that a certain class of quasi-isometries is invariant under conjugation by inversion.

LEMMA 8. *Suppose that f is an L-quasi-isometry in $E \subset \bar{R}^2$, that*

$$\frac{1}{L} \le \frac{|f(z)|}{|z|} \le L \tag{25}$$

for $z \in E - \{0, \infty\}$ and that $f(0) = 0$ if $0 \in E$. Then $g = T \circ f \circ T^{-1}$ is an L^3-quasi-isometry in $T(E)$ where $T(z) = 1/z$.

Proof. Choose distinct points $w_1, w_2 \in T(E) - \{\infty\}$ and let $z_j = 1/w_j$. If $w_1, w_2 \neq 0$, then $z_1, z_2 \in E - \{0, \infty\}$,

$$\frac{|g(w_1) - g(w_2)|}{|w_1 - w_2|} = \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|} \frac{|z_1|}{|f(z_1)|} \frac{|z_2|}{|f(z_2)|}$$

and hence

$$\frac{1}{L^3} \leq \frac{|g(w_1)-g(w_2)|}{|w_1-w_2|} \leq L^3 \tag{26}$$

by (25). If $w_1=0$, then $g(w_1)=0$,

$$\frac{|g(w_1)-g(w_2)|}{|w_1-w_2|} = \frac{|z_2|}{|f(z_2)|}$$

and again (26) holds. Finally if $\infty \in T(E)$ then $g(\infty)=\infty$ and thus g is an L^3-quasi-isometry in $T(E)$.

We now obtain the main result of this section from combining Lemma 8 and Theorem 6.

THEOREM 7. *Suppose that D_1 is a K_1-quasidisk in R^2, that D_2 is a Jordan domain in R^2 and that $\phi: \partial D_1 \to \partial D_2$ is an L_1-quasi-isometry. Then there exist L-quasi-isometries $f: \bar{D}_1 \to \bar{D}_2$ and $f^*: \bar{D}_1^* \to \bar{D}_2^*$ such that $f=f^*=\phi$ on ∂D_1 and L depends only on K_1 and L_1.*

Proof. Suppose that $\infty \in \partial D_1$. Then D_1 and D_1^* are K_1-quasidisks with $\partial D_1 = \partial D_1^*$ and the existence of f and f^* is an immediate consequence of Theorem 6.

Suppose next that $\infty \notin \partial D_1$. Then $\infty \in D_1^*$ and $\infty \in D_2^*$. By making a preliminary change of variables we may assume that $0 \in \partial D_1$ and that $\phi(0)=0$. For $j=1,2$ let G_j denote the image of D_j under $T(z)=1/z$ and set $\psi = T \circ \phi \circ T^{-1}$. Then G_1 is a K_1-quasidisk with $\infty \in \partial G_1$,

$$\frac{1}{L_1} \leq \frac{|\phi(z)|}{|z|} \leq L_1 \tag{27}$$

for $z \in \partial D_1 - \{0\}$ and hence $\psi: \partial G_1 \to \partial G_2$ is an L_1^3-quasi-isometry by Lemma 8. Theorem 6 then yields L_2-quasi-isometries $g: \bar{G}_1 \to \bar{G}_2$ and $g^*: \bar{G}_1^* \to \bar{G}_2^*$ such that $g=g^*=\psi$ on ∂G_1 and L_2 depends only on K_1 and L_1. In addition, since $0 \in G_1^*$, $0 \in G_2^*$ and

$$\frac{1}{L_1} \leq \frac{|0-z|}{|0-\psi(z)|} \leq L_1$$

for $z \in \partial G_1 - \{\infty\}$ by (27), we can choose g^* so that $g^*(0)=0$. Fix $z_1 \in \bar{G}_1 - \{\infty\}$ and let z_0 be a point where the segment joining 0 to z_1 meets ∂G_1. Then

$$\begin{aligned}|g(z_1)| &\le |g(z_1)-g(z_0)|+|g(z_0)| \le L_2\,|z_1-z_0|+|\psi(z_0)| \\ &\le L_2\,|z_1-z_0|+L_1\,|z_0| \le L_2\,|z_1|\end{aligned}$$

since $L_2 \ge L_1$. Thus by symmetry

$$\frac{1}{L_2} \le \frac{|g(z)|}{|z|} \le L_2$$

for $z \in \bar{G}_1 - \{\infty\}$. Next

$$\frac{1}{L_2} \le \frac{|g^*(z)|}{|z|} \le L_2$$

for $z \in \bar{G}_1^* - \{0, \infty\}$ since g^* is an L_2-quasi-isometry in $\bar{G}_1^*$. Thus $f = T^{-1} \circ g \circ T$ and $f^* = T^{-1} \circ g^* \circ T$ have the required properties by Lemma 8.

COROLLARY 1. *Suppose that D_1 and D_2 are K_1- and K_2-quasidisks in R^2 and that $f: D_1 \to D_2$ is a bijective local L_1-quasi-isometry. Then there exists an L-quasi-isometry $g: \bar{R}^2 \to \bar{R}^2$ such that $g = f$ in D_1 and L depends only on K_1, K_2 and L_1.*

Proof. By Lemma 6, f extends to an L_2-quasi-isometry of $\bar{D}_1$ onto $\bar{D}_2$ where L_2 depends only on K_1, K_2 and L_1. Next Theorem 7 with $\phi = f \mid \partial D_1$ yields an L_2^*-quasi-isometry $f^*: \bar{D}_1^* \to \bar{D}_2^*$ such that $f^* = f$ on ∂D_1 and L_2^* depends only on K_1, K_2 and L_1. Then

$$g = \begin{cases} f & \text{in } \bar{D}_1 \\ f^* & \text{in } \bar{D}_1^*, \end{cases} \tag{28}$$

is the desired extension of f.

COROLLARY 2. *Suppose that C_1 is a K_1-quasicircle and that ϕ is an L_1-quasi-isometry in C_1. Then there exists an L-quasi-isometry $g: \bar{R}^2 \to \bar{R}^2$ such that $g = \phi$ on C_1 and L depends only on K_1 and L_1.*

Proof. Let $C_2 = \phi(C_1)$ and for $j = 1, 2$ let D_j be a component of $\bar{R}^2 - C_j$ chosen so that $D_j \subset R^2$. If f and f^* are the L-quasi-isometries given by Theorem 7, then g defined in (28) is the required extension.

Corollary 2 extends recent results of Jerison and Kenig [9] and of Tukia [18] who consider the cases where C_1 is a line and a circle, respectively.

4. An application

If f is a local L-quasi-isometry in a plane domain D with $L<L(D)$, then f is injective. The following result shows that one can say more whenever D is simply connected.

THEOREM 8. *Suppose that D_1 is a simply connected proper subdomain of R^2 and that f is a local L_1-quasi-isometry in D_1 with $L<L(D_1)$. Then there exists an L-quasi-isometry $g: R^2 \to R^2$ such that $g=f$ in D_1 and L depends only on $L(D_1)$ and L_1.*

Proof. Let $D_2=f(D_1)$ and let g denote any local L_2-quasi-isometry in D_2 with $L_2<L(D_1)/L_1$. Then $g\circ f$ is a local L_1L_2-quasi-isometry in D_1, $g\circ f$ is injective in D_1 since $L_1L_2<L(D_1)$ and hence g is injective in D_2. Thus

$$L(D_2) \geq \frac{L(D_1)}{L_1} > 1. \tag{29}$$

Since f is injective in D_1, f is an L_1^2-quasiconformal mapping of D_1 and hence D_2 is a simply connected proper subdomain of R^2. Then by Theorem 3 and (29) D_1 and D_2 are K_1- and K_2-quasidisks, where K_1 and K_2 depend only on $L(D_1)$ and $L(D_1)/L_1$ respectively, and the existence of g follows from Corollary 1.

Theorem 8 can be interpreted physically if we think of D_1 as a homogeneous elastic body and f as the distortion of D_1 due to a force field. In this case $L(D_1)$ measures the maximum permissible strain in D_1 before D_1 buckles. Theorem 8 asserts that if the strain in D_1 is less than $L(D_1)$, then the shape of D_1 is not substantially changed under the force field.

REFERENCES

[1] L. V. AHLFORS, *Quasiconformal reflections*, Acta Math. *109* (1963) 291–301.

[2] L. V. AHLFORS, *Lectures on quasiconformal mappings*, Van Nostrand Mathematical Studies 10, Princeton 1966.

[3] F. W. GEHRING, *Univalent functions and the Schwarzian derivative*, Comment. Math. Helv. *52* (1977) 561–572.

[4] F. W. GEHRING, *Characteristic properties of quasidisks*, Les Presses de l'Université de Montréal 1982.

[5] F. W. GEHRING and B. P. PALKA, *Quasiconformally homogeneous domains*, J. d'Analyse Math. *30* (1976) 172–199.
[6] F. W. GEHRING and B. G. OSGOOD, *Uniform domains and the quasi-hyperbolic metric*, J. d'Analyse Math. *36* (1979) 50–74.
[7] J. GEVIRTZ, *On the univalence of quasi-isometric mappings*, Rev. Colombiana Mat. *7* (1973) 125–132.
[8] J. GEVIRTZ, *A sharp condition for univalence in euclidean spaces*, Proc. Amer. Math. Soc. *57* (1976) 261–265.
[9] D. S. JERISON and C. E. KENIG, *Hardy spaces, A_∞, and singular integrals on chord-arc domains*, Math. Scand. (to appear).
[10] F. JOHN, *Rotation and strain*, Comm. Pure Appl. Math. *14* (1961) 391–413.
[11] F. JOHN, *On quasi-isometric mappings, I*, Comm. Pure Appl. Math. *21* (1968) 77–110.
[12] F. JOHN, *On quasi-isometric mappings, II*, Comm. Pure Appl. Math. *22* (1969) 265–278.
[13] P. W. JONES, *Extension theorems for BMO*, Indiana Univ. Math. J. *29* (1980) 41–66.
[14] O. MARTIO and J. SARVAS, *Injectivity theorems in plane and space*, Ann. Acad. Sci. Fenn. *4* (1978–79) 383–401.
[15] M. H. A. NEWMAN, *Elements of the topology of plane sets of points*, Camb. Univ. Press 1954.
[16] B. G. OSGOOD, *Univalence criteria in multiply-connected domains*, Trans. Amer. Math. Soc. *260* (1980) 459–473.
[17] C. POMMERENKE, *Univalent functions*, Vandenhoeck and Ruprecht, Göttingen 1975.
[18] P. TUKIA, *The planar Schönflies theorem for Lipschitz maps*, Ann. Acad. Sci. Fenn. *5* (1980) 49–72.

University of Helsinki
Helsinki, Finland

University of Michigan
Ann Arbor, Michigan

Received January 14, 1982