

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 57 (1982)

**Artikel:** Injectivity of local quasi-isometries.  
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**DOI:** <https://doi.org/10.5169/seals-43883>

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## Injectivity of local quasi-isometries

F. W. GEHRING<sup>(1)</sup>

### 1. Introduction

Suppose that  $E$  is a set in  $\bar{R}^n$ , the one point compactification of euclidean  $n$ -space  $R^n$ ,  $n \geq 2$ , and suppose that  $f$  is a mapping from  $E$  into  $\bar{R}^n$ . We say that  $f$  is an  $L$ -quasi-isometry in  $E$  if

$$\frac{1}{L} \leq \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|} \leq L \quad (1)$$

for each pair of points  $x_1, x_2 \in E - \{\infty\}$  and if  $f(\infty) = \infty$  whenever  $\infty \in E$ . We say that  $f$  is a *local  $L$ -quasi-isometry* in  $E$  if for each  $L' > L$  each  $x \in E$  has a neighborhood  $U$  such that  $f$  is an  $L'$ -quasi-isometry in  $E \cap U$ .

Suppose that  $f$  is a local  $L$ -quasi-isometry in a domain  $D$  in  $R^n$ . If  $L = 1$ , then  $f$  is an isometry in  $D$  and hence injective there. (See, for example, Theorem IV in [11].) Simple examples show that  $f$  need not be injective if  $L > 1$ . It was F. John who first noticed that for certain domains  $D$ ,  $f$  will be injective provided  $L$  is close enough to 1.

For each domain  $D \subset R^n$  we let  $L(D)$  denote the supremum of the numbers  $L \geq 1$  with the property that each local  $L$ -quasi-isometry in  $D$  is injective. We say that  $D$  is *rigid* if  $L(D) > 1$ .

John established the following interesting result in 1969 (Theorem A in [12]). See also [7] and [8].

**THEOREM 1.** *If  $D$  is an open ball or half space, then  $L(D) \geq 2^{1/4}$ .*

This result was generalized by John and then extended recently by Martio and Sarvas to a very broad class of domains. We say that  $D \subset R^n$  is a *uniform domain* if there exist constants  $a$  and  $b$  with the following property. Each pair of points

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<sup>1</sup> This research was supported in part by grants from the U.S. National Science Foundation (Grant MCS 79-01713) and the Finnish Ministry of Education.

$x_1, x_2 \in D$  can be joined by a rectifiable arc  $\alpha$  in  $D$  so that

$$l(\alpha) \leq a |x_1 - x_2| \quad (2)$$

and so that for each  $x \in \alpha$

$$\min_{j=1,2} l(\alpha_j) \leq b d(x, \partial D), \quad (3)$$

where  $\alpha_1, \alpha_2$  denote the components of  $\alpha - \{x\}$ . Here  $l(\alpha)$  denotes the euclidean length of  $\alpha$  and  $d(x, \partial D)$  the distance from  $x$  to  $\partial D$ .

Martio and Sarvas showed that uniform domains are rigid by establishing the following result (Theorem 3.8 in [14]).

**THEOREM 2.** *If  $D$  is a uniform domain, then  $L(D) \geq c > 1$  where  $c$  depends only on the constants  $a$  and  $b$ .*

The present paper is concerned with the problem of identifying the domains in  $R^n$  which are rigid. In particular, we characterize in Section 2 the finitely connected plane domains which have this property. It turns out that each boundary component of such a domain is either a point or a quasicircle, that is, the image of a circle or a line under a quasiconformal mapping of  $\bar{R}^2$ . In Section 3 we establish an extension theorem for quasi-isometries. We then apply this result in Section 4 to show that if  $D$  is a simply connected rigid domain in  $R^2$  and if  $f$  is a local  $L$ -quasi-isometry in  $D$  with  $L < L(D)$ , then  $f$  is not only injective in  $D$  but has an extension as a quasi-isometry to all of  $R^2$ .

## 2. Rigid plane domains

Throughout the remainder of this paper we shall use complex notation to denote points in  $R^2$ . For  $z_0 \in R^2$  and  $0 < r < \infty$  we let  $B(z_0, r)$  denote the open disk with center  $z_0$  and radius  $r$ . Finally for each domain  $D \subset \bar{R}^2$  we let  $D^* = \bar{R}^2 - \bar{D}$ .

In this section we characterize the finitely connected domains in  $R^2$  which are rigid. We begin with a technical lemma concerning a special class of quasi-isometries.

**LEMMA 1.** *Suppose that  $\phi(t)$  is a real valued function defined in  $(0, \infty)$ , that*

$$|\phi(t_1) - \phi(t_2)| \leq a \left| \log \frac{t_1}{t_2} \right| \quad (4)$$

for  $t_1, t_2 \in (0, \infty)$  and that

$$f(z) = \begin{cases} ze^{i\phi(|z|)} & \text{if } 0 < |z| < \infty, \\ 0 & \text{if } z = 0. \end{cases} \quad (5)$$

Then  $f$  is a  $(1+a)$ -quasi-isometry in  $R^2$ .

*Proof.* Choose distinct points  $z_1, z_2 \in R^2$  with  $|z_1| \leq |z_2|$ . If  $z_1 \neq 0$ , then

$$\begin{aligned} |f(z_1) - f(z_2)| &\leq |z_1 - z_2| + |z_1| |e^{i\phi(|z_1|)} - e^{i\phi(|z_2|)}| \\ &\leq |z_1 - z_2| + |z_1| |\phi(|z_1|) - \phi(|z_2|)| \\ &\leq |z_1 - z_2| + a |z_1| \left| \log \frac{|z_1|}{|z_2|} \right| \\ &\leq (1+a) |z_1 - z_2| \end{aligned}$$

by (4), while

$$|f(z_1) - f(z_2)| = |z_2| \leq (1+a) |z_1 - z_2|$$

if  $z_1 = 0$ . Since  $f^{-1}$  is given by (5) with  $-\phi$  in place of  $\phi$ , the above argument can be applied to  $f^{-1}$  to complete the proof.

We next use Lemma 1 to obtain a geometric property of plane domains  $D$  with  $L(D) > 1$ .

**LEMMA 2.** *Suppose that  $D$  is a domain in  $R^2$  with  $L(D) \geq c > 1$ . Then there exists a constant  $b$ , depending only on  $c$ , such that for each  $z_0 \in R^2$  and  $0 < r < \infty$ ,  $D \cap \partial B(z_0, r)$  lies in component of*

$$G = D \cap (B(z_0, br) - \bar{B}(z_0, r/b)).$$

*Proof.* Choose  $b \in (1, \infty)$  so that

$$1 + \frac{\pi}{\log b} < c, \quad (6)$$

and suppose there exist points  $z_1, z_2 \in D \cap \partial B(z_0, r)$  which belong to different components  $G_1, G_2$  of  $G$ . By making a change of variable we may assume that



$z_0 = 0$ . Choose  $\theta \in [-\pi, \pi]$  so that  $z_2 = z_1 e^{i\theta}$  and let  $f$  be as in (5) with

$$\phi(t) = \begin{cases} 0 & \text{if } 0 < t \leq \frac{r}{b} \text{ or } br \leq t < \infty, \\ \frac{\log \frac{bt}{r}}{\log b} \theta & \text{if } \frac{r}{b} \leq t \leq r, \\ \frac{\log \frac{br}{t}}{\log b} \theta & \text{if } r \leq t \leq br. \end{cases}$$

Then  $\phi$  satisfies (4) with  $a = \pi/\log b$  and  $f$  is a  $(1+a)$ -quasi-isometry in  $R^2$  by Lemma 1. Set

$$g(z) = \begin{cases} z & \text{if } z \in D - G_1, \\ f(z) & \text{if } z \in G_1. \end{cases} \quad (7)$$

If  $U$  is any open disk in  $D$ , then either  $U \subset D - G_1$ , in which case  $g(z) = z$  in  $U$ , or  $U \subset G_1 \cup (D - G)$ , in which case  $g(z) = f(z)$  in  $U$ . Hence  $g$  is a local  $(1+a)$ -quasi-isometry in  $D$ . Since  $z_2 \notin G_1$ ,

$$g(z_2) = z_2 = z_1 e^{i\theta} = z_1 e^{i\phi(|z_1|)} = g(z_1)$$

and  $g$  is not injective in  $D$ . Thus  $c \leq 1 + a$ . This contradicts (6) and establishes the desired conclusion.

We say that  $C \subset \bar{R}^2$  is a  $K$ -quasicircle if it is the image of a circle or line under a  $K$ -quasiconformal mapping  $f: \bar{R}^2 \rightarrow \bar{R}^2$ . Similarly  $D \subset \bar{R}^2$  is said to be a  $K$ -quasidisk if  $\partial D$  is a  $K$ -quasicircle.

We have next the following information about the boundary of a rigid plane domain.

**LEMMA 3.** *Suppose that  $D$  is a domain in  $R^2$  with  $L(D) \geq c > 1$ . Then each component  $C$  of  $\partial D$  is either a point or a  $K$ -quasicircle where  $K$  depends only on  $c$ . Moreover if  $C_1$  and  $C_2$  are components of  $\partial D$ , then*

$$\min_{j=1,2} \text{dia}(C_j) \leq a \, d(C_1, C_2) \quad (8)$$

where  $a$  is a constant which depends only on  $c$ .

Here  $\text{dia}(C_j)$  denotes the diameter of  $C_j$  and  $d(C_1, C_2)$  the distance between  $C_1$  and  $C_2$ .

*Proof.* Choose  $b \in (1, \infty)$  so that (6) holds and suppose that  $z_0 \in \mathbb{R}^2$ ,  $0 < r < \infty$  and  $z_1, z_2 \in D \cap \bar{B}(z_0, r)$ . Let  $\alpha$  be any arc joining  $z_1$  and  $z_2$  in  $D$ . If  $\alpha$  does not lie in  $\bar{B}(z_0, r)$ , then  $\alpha \cap \bar{B}(z_0, r)$  contains two components  $\alpha_1, \alpha_2$  which join  $z_1, z_2$  to  $w_1, w_2 \in \partial B(z_0, r)$ , respectively. Lemma 2 implies that  $w_1$  and  $w_2$  can be joined by an arc  $\beta$  in  $D \cap \bar{B}(z_0, br)$  and hence  $\alpha_1 \cup \beta \cup \alpha_2$  joins  $z_1$  and  $z_2$  in  $D \cap \bar{B}(z_0, br)$ . A similar argument shows that any pair of points  $z_1, z_2 \in D - B(z_0, r)$  can be joined in  $D - B(z_0, r/b)$ . Hence  $D$  is  $b$ -locally connected and by Lemma 5 in [3], each component  $C$  of  $\partial D$  is either a point or a  $K$ -quasicircle where  $K$  depends only on  $b$ .

Suppose next that  $C_1$  and  $C_2$  are distinct components of  $\partial D$ , choose  $z_1 \in C_1$  and  $z_2 \in C_2$  so that

$$|z_1 - z_2| = d(C_1, C_2) = 2r$$

and let  $z_0 = \frac{1}{2}(z_1 + z_2)$ . We shall use Lemma 2 to show that  $C_1$  or  $C_2$  lies in  $B(z_0, b^2r)$  and hence that

$$\min_{j=1,2} \text{dia}(C_j) \leq 2b^2r.$$

This will establish (8) with  $a = b^2$ .

Suppose that  $C_1$  and  $C_2$  do not lie in  $B(z_0, b^2r)$  and let  $D_0$  denote the component of  $\bar{\mathbb{R}}^2 - (C_1 \cup C_2)$  which contains  $D$ . Then

$$F_1 = \bar{\mathbb{R}}^2 - (D_0 \cap B(z_0, b^2r)), \quad F_2 = \bar{B}(z_0, r)$$

are continua with

$$F_1 \cap F_2 = (C_1 \cup C_2) \cap \bar{B}(z_0, r) = \{z_1, z_2\}. \quad (9)$$

Hence by Theorem V.11.5 in [15], there exist points  $w_1, w_2$  which lie in different components  $G_1, G_2$  of

$$\bar{\mathbb{R}}^2 - (F_1 \cup F_2) = D_0 \cap (B(z_0, b^2r) - \bar{B}(z_0, r))$$

but which can be joined by an arc  $\alpha$  in

$$\bar{\mathbb{R}}^2 - F_1 = D_0 \cap B(z_0, b^2r).$$

Next (9) and Theorem V.16.2 in [15] imply that  $w_1, w_2$  are not separated by  $C_1 \cup C_2 \cup \bar{B}(z_0, r)$  and hence can be joined by an arc  $\beta$  in  $D_0 - \bar{B}(z_0, r)$ . Thus for  $j = 1, 2$ ,  $\alpha \cup \beta$  contains a curve which joins  $\partial B(z_0, r)$  to  $\partial B(z_0, b^2 r)$  in  $G_j$ ; hence

$$H_j = G_j \cap \partial B(z_0, br) \neq \emptyset.$$

Since each component of  $H_j$  is an open arc in  $D_0$  with endpoints in  $C_1 \cup C_2$ ,  $D \cap H_j \neq \emptyset$  and we conclude that  $D \cap \partial B(z_0, br)$  does not lie in a component of  $D \cap (B(z_0, b^2 r) - \bar{B}(z_0, r))$ . This contradicts Lemma 2 and thus establishes the desired conclusion.

Finally we have the following relations between quasidisks and rigid plane domains.

**THEOREM 3.** *If  $D$  is a  $K$ -quasidisk in  $R^2$ , then  $L(D) \geq c > 1$  where  $c$  depends only on  $K$ . Conversely if  $D$  is a simply connected proper subdomain of  $R^2$  with  $L(D) \geq c > 1$ , then  $D$  is a  $K$ -quasidisk where  $K$  depends only on  $c$ .*

*Proof.* If  $D$  is a  $K$ -quasidisk in  $R^2$ , then by Corollary 2.33 in [14],  $D$  is a uniform domain where the constants  $a$  and  $b$  in (2) and (3) depend only on  $K$ . (For an alternative proof see Theorem III.2.3 in [4].) Hence  $L(D) \geq c > 1$  where  $c = c(K)$  by Theorem 2. The converse is a consequence of Lemma 3.

**THEOREM 4.** *A finitely connected domain  $D$  in  $R^2$  is rigid if and only if each component of  $\partial D$  is either a point or a quasicircle.*

*Proof.* If  $D$  is bounded by a finite number of points or quasicircles, then  $D$  is uniform by Theorem 5 in [16] and Theorem 5 in [6]; hence  $D$  is rigid by Theorem 2. The converse follows from Lemma 3.

The problem of characterizing rigid plane domains  $D$  is more difficult when  $D$  is infinitely connected. For example, if  $\mathcal{C}$  denotes the collection of boundary components of a rigid domain  $D$  in  $R^2$ , then

$$\sup_{C, C' \in \mathcal{C}} \frac{\min(\text{dia}(C), \text{dia}(C'))}{d(C, C')} < \infty$$

by Lemma 3. Hence one must take into account not only the shape but the relative size and position of the boundary components when  $D$  has infinite connectivity.

We conclude this section by exhibiting a plane domain  $D$  which is rigid but not uniform; thus the converse of Theorem 2 does not hold. The existence of such a domain is an immediate consequence of the following result.

**THEOREM 5.** *If  $D$  is a rigid domain in  $R^2$  and if  $E$  is a discrete subset of  $D$ , then  $D - E$  is a rigid domain.*

*Proof.* Suppose that  $U$  is an open disk with center at  $z_0$  and let  $U_0 = U - \{z_0\}$ . Then since  $L(D)$  is invariant under similarity mappings, Theorem 4 implies that  $L(U_0)$  is an absolute constant  $c$  which exceeds 1.

Suppose next that  $f$  is a local  $L$ -quasi-isometry in  $D - E$  with  $L < \min(L(D), c)$ . Given  $z_0 \in E$  we can choose an open disk  $U$  centered at  $z_0$  such that

$$U_0 = U - \{z_0\} \subset D - E.$$

If  $z_1, z_2 \in U_0$ , then for each  $\varepsilon > 0$  we can find an arc  $\alpha$  joining  $z_1$  and  $z_2$  in  $U_0$  with

$$l(\alpha) \leq (1 + \varepsilon) |z_1 - z_2|.$$

Since  $f$  is a local  $L$ -quasi-isometry in  $U_0$ ,

$$|f(z_1) - f(z_2)| \leq l(f(\alpha)) \leq Ll(\alpha) \leq L(1 + \varepsilon) |z_1 - z_2|,$$

and letting  $\varepsilon \rightarrow 0$  yields

$$|f(z_1) - f(z_2)| \leq L |z_1 - z_2|. \quad (10)$$

Then (10) implies that  $f$  has a continuous extension in  $U$  which satisfies (10) for  $z_1, z_2 \in U$ . Next since  $L < c$ ,  $f$  is injective in  $U_0$ , and it follows that  $f$  is injective and hence a homeomorphism in  $U$ . Choose an open disk  $V$  about  $f(z_0)$  with  $V \subset f(U)$ . Then  $g = (f|U)^{-1}$  is a local  $L$ -quasi-isometry in  $V_0 = V - \{f(z_0)\}$ , and the above argument applied to  $g$  shows that  $f$  is an  $L$ -quasi-isometry in  $g(V)$ . Thus  $f$  has an extension to  $D$  which is a local  $L$ -quasi-isometry in a neighborhood of each point of  $E$  and hence in  $D$ . Then since  $L < L(D)$ ,  $f$  is injective in  $D$ . Hence  $f$  is injective in  $D - E$ ,

$$L(D - E) \geq \min(L(D), c) > 1$$

and  $D - E$  is a rigid domain.

Now let  $B$  denote the unit disk and let

$$E = \left\{ z = \left(1 - \frac{1}{j}\right) \exp\left(\frac{2\pi i k}{j^2}\right) : k = 1, 2, \dots, j^2, j = 2, 3, \dots \right\}.$$

Then  $D = B - E$  is rigid by Theorem 5. On the other hand if  $z_1 = 0$  and if  $z_2 \in D$  with  $|z_2| \geq 1 - 1/2j$  and  $j \geq 2$ , then each rectifiable arc  $\alpha$  joining  $z_1$  and  $z_2$  in  $D$  must contain a point  $z$  with  $|z| = 1 - 1/j$  and

$$\min_{j=1,2} l(\alpha_j) \geq \frac{j}{2\pi} d(z, \partial D),$$

where  $\alpha_1, \alpha_2$  denote the components of  $\alpha - \{z\}$ . Hence there exists no constant  $b$  for which  $D$  satisfies condition (3) and  $D$  is not a uniform domain.

### 3. Extension of quasi-isometries

We establish here some extension theorems for plane quasi-isometries. Our arguments are based on a reflection principle for quasidisks due to Ahlfors [1] and estimates for the hyperbolic distance.

If  $D$  is a simply connected proper subdomain of  $R^2$ , then the *hyperbolic metric* with curvature  $-1$  in  $D$  is given by

$$\rho_D(z) = \frac{|g'(z)|}{\operatorname{Im}(g(z))},$$

where  $g$  is any conformal mapping of  $D$  onto the upper half plane  $H$ . From standard distortion theorems it follows that

$$\frac{1}{2} \leq |g'(z)| \frac{d(z, \partial D)}{\operatorname{Im}(g(z))} \leq 2, \quad (11)$$

where  $d(z, \partial D)$  denotes the distance from  $z$  to  $\partial D$ , and hence that

$$\frac{1}{2d(z, \partial D)} \leq \rho_D(z) \leq \frac{2}{d(z, \partial D)}. \quad (12)$$

(See, for example, p. 22 in [17].) Next the *hyperbolic distance* between points  $z_1, z_2 \in D$  is given by

$$h_D(z_1, z_2) = \inf_{\alpha} \int_{\alpha} \rho_D(z) |dz|,$$

where the infimum is taken over all rectifiable arcs  $\alpha$  joining  $z_1$  and  $z_2$  in  $D$ . From

(12) and Lemma 2.1 in [5] it follows that

$$h_D(z_1, z_2) \geq \frac{1}{2} \left| \log \frac{d(z_1, \partial D)}{d(z_2, \partial D)} \right| \quad (13)$$

for  $z_1, z_2 \in D$ . Next if  $D$  is a  $K$ -quasidisk, then by (12), Corollary 2.33 in [14] and Theorem 1 in [6],

$$h_D(z_1, z_2) \leq c \log \left( \frac{|z_1 - z_2|}{d(z_1, \partial D)} + 1 \right) \left( \frac{|z_1 - z_2|}{d(z_2, \partial D)} + 1 \right) + d. \quad (14)$$

for  $z_1, z_2 \in D$ , where  $c$  and  $d$  are constants which depend only on  $K$ . (Cf. pp. 42–44 in [13].)

We begin with a result on a special class of quasi-isometries.

**LEMMA 4.** *If  $D$  is a Jordan domain in  $R^2$  and if  $z_1, z_2 \in D$  with  $h_D(z_1, z_2) \leq a$ , then there exists an  $L$ -quasi-isometry  $f: \bar{D} \rightarrow \bar{D}$  such that  $f$  is the identity on  $\partial D$ ,  $f(z_1) = z_2$  and  $L$  depends only on  $a$ .*

*Proof.* Choose a conformal mapping  $g: D \rightarrow H$  normalized so that  $g(z_1) = i$  and  $g(z_2) = bi$  where  $b > 1$ . Then

$$\log b = h_D(z_1, z_2) \leq a$$

and  $g$  extends to a homeomorphism which maps  $\bar{D}$  onto  $\bar{H}$ . Set

$$h(w) = \begin{cases} u + ibv & \text{if } w = u + iv \in \bar{H} - \{\infty\}, \\ \infty & \text{if } w = \infty. \end{cases}$$

Then  $h$  is continuously differentiable with

$$\left. \begin{aligned} h_H(h(w), w) &= \log b \leq a, \\ \frac{1}{b} \frac{|dw|}{\operatorname{Im}(w)} &\leq \frac{|dh(w)|}{\operatorname{Im}(h(w))} \leq \frac{|dw|}{\operatorname{Im}(w)} \end{aligned} \right\} \quad (15)$$

in  $H$  and  $f = g^{-1} \circ h \circ g$  is a homeomorphism of  $\bar{D}$  onto  $\bar{D}$  which is the identity on  $\partial D$  and maps  $z_1$  onto  $z_2$ .

Fix  $z \in D$  and set  $w = g(z)$ . Then

$$\frac{|df(z)|}{|dz|} = \frac{|dh(w)|}{|dw|} \frac{|g'(z)|}{|g'(f(z))|}, \quad (16)$$

while we obtain

$$\left. \begin{aligned} \frac{1}{2} \leq |g'(z)| \frac{d(z, \partial D)}{\text{Im}(w)} \leq 2 \\ \frac{1}{2} \leq |g'(f(z))| \frac{d(f(z), \partial D)}{\text{Im}(h(w))} \leq 2 \end{aligned} \right\} \quad (17)$$

from (11). Next by (13) and (15),

$$\frac{1}{2} \left| \log \frac{d(f(z), \partial D)}{d(z, \partial D)} \right| \leq h_D(f(z), z) = h_H(h(w), w) \leq a$$

whence

$$e^{-2a} \leq \frac{d(f(z), \partial D)}{d(z, \partial D)} \leq e^{2a}. \quad (18)$$

Combining (15), (16), (17) and (18) yields

$$\frac{1}{L} \leq \frac{|df(z)|}{|dz|} \leq L$$

where  $L = 4e^{3a}$ , and hence  $f$  is a local  $L$ -quasi-isometry in  $D$ . (Cf. p. 395 in [10].) The desired conclusion is now a consequence of the following elementary result.

**LEMMA 5.** *Suppose that  $D_1$  and  $D_2$  are domains in  $R^2$ , that  $f: \bar{D}_1 \rightarrow \bar{D}_2$  is a homeomorphism and that  $f$  is an  $L_1$ -quasi-isometry in  $\partial D_1$  and a local  $L_2$ -quasi-isometry in  $D_1$ . Then  $f$  is an  $L$ -quasi-isometry in  $\bar{D}_1$  where  $L = \max(L_1, L_2)$ .*

*Proof.* Fix  $z_1, z_2 \in D_1$  and let  $\alpha$  be the open segment joining these points in  $R^2$ . If  $\alpha \subset D_1$ , then

$$|f(z_1) - f(z_2)| \leq l(f(\alpha)) \leq L_2 l(\alpha) \leq L |z_1 - z_2|.$$

Otherwise for  $j = 1, 2$  let  $\alpha_j$  denote the component of  $\alpha \cap D_1$  which has  $z_j$  as an endpoint and let  $w_j$  denote the other endpoint of  $\alpha_j$ . Then  $w_j \in \partial D_1$ ,  $\alpha_j \subset D_1$  and

$$\begin{aligned} |f(z_1) - f(z_2)| &\leq |f(z_1) - f(w_1)| + |f(w_1) - f(w_2)| + |f(w_2) - f(z_2)| \\ &\leq L |z_1 - w_1| + L_1 |w_1 - w_2| + L |w_2 - z_2| \\ &\leq L |z_1 - z_2|. \end{aligned}$$

Applying this argument to  $f^{-1}$  shows that  $f$  is an  $L$ -quasi-isometry in  $D_1$ , and hence in  $\bar{D}_1$  by continuity.

Lemma 5 shows that a bijective local quasi-isometry between two domains is a quasi-isometry if the induced boundary correspondence is a quasi-isometry. We can also draw this conclusion without knowledge of the boundary correspondence when the two domains have sufficiently regular boundaries.

**LEMMA 6.** *Suppose that  $D_1$  and  $D_2$  are  $K_1$ - and  $K_2$ -quasidisks in  $R^2$  and that  $f: D_1 \rightarrow D_2$  is a bijective local  $L_1$ -quasi-isometry. Then  $f$  extends to an  $L$ -quasi-isometry of  $\bar{D}_1$  onto  $\bar{D}_2$  where  $L$  depends only on  $K_1$ ,  $K_2$  and  $L_1$ .*

*Proof.* Fix  $z_1, z_2 \in D_1$ . By Corollary 2.33 in [14], there exists a rectifiable arc  $\alpha$  joining  $z_1$  and  $z_2$  in  $D_1$  such that

$$l(\alpha) \leq a_1 |z_1 - z_2|,$$

where  $a_1$  depends only on  $K_1$ . Thus

$$|f(z_1) - f(z_2)| \leq l(f(\alpha)) \leq L_1 l(\alpha) \leq L_1 a_1 |z_1 - z_2|.$$

Next since  $f$  is injective,  $f^{-1}$  is a local  $L_1$ -quasi-isometry in  $D_2$  and arguing as above yields

$$|z_1 - z_2| \leq L_1 a_2 |f(z_1) - f(z_2)|,$$

where  $a_2$  depends only on  $K_2$ . Hence  $f$  is an  $L$ -quasi-isometry in  $D_1$  where  $L = \max(L_1 a_1, L_1 a_2)$ , and we can extend  $f$  to  $\bar{D}_1$  by continuity.

We will require the following version of Lemma 4 for the case where  $D$  is a quasidisk.

**LEMMA 7.** *Suppose that  $D$  is a  $K$ -quasidisk in  $R^2$ , that  $z_1, z_2 \in D$  and that*

$$\frac{1}{b} \leq \frac{|z_1 - z|}{|z_2 - z|} \leq b \tag{19}$$



for all  $z \in \partial D - \{\infty\}$  where  $b$  is a constant. Then there exists an  $L$ -quasi-isometry  $f: \bar{D} \rightarrow \bar{D}$  such that  $f$  is the identity on  $\partial D$ ,  $f(z_1) = z_2$  and  $L$  depends only on  $K$  and  $b$ .

*Proof.* For  $j = 1, 2$  choose  $w_j \in \partial D - \{\infty\}$  so that

$$|z_j - w_j| = d(z_j, \partial D).$$

Then by (19),

$$|z_1 - z_2| \leq |z_1 - w_j| + |z_2 - w_j| \leq (b + 1) d(z_j, \partial D)$$

and hence

$$h_D(z_1, z_2) \leq 2c \log(b + 2) + d = a$$

by (14), where  $c$  and  $d$  depend only on  $K$ . The desired conclusion now follows directly from Lemma 4.

We derive now an extension of Ahlfors' reflection principle for quasidisks. (See, for example, Lemma 3 on p. 80 in [2].)

**THEOREM 6.** Suppose that  $D_1$  is a  $K_1$ -quasidisk with  $\infty \in \partial D_1$ , that  $D_2$  is a Jordan domain in  $R^2$  with  $\infty \in \partial D_2$  and that  $\phi: \partial D_1 \rightarrow \partial D_2$  is an  $L_1$ -quasi-isometry. Then there exists an  $L$ -quasi-isometry  $f: \bar{D}_1 \rightarrow \bar{D}_2$  such that  $f = \phi$  on  $\partial D_1$  and  $L$  depends only on  $K_1$  and  $L_1$ . Suppose further that  $z_1 \in D_1$ ,  $z_2 \in D_2$  and

$$\frac{1}{b} \leq \frac{|z_1 - z|}{|z_2 - \phi(z)|} \leq b \quad (20)$$

for all  $z \in \partial D_1 - \{\infty\}$  where  $b$  is a constant. Then we can choose  $f$  so that, in addition,  $f(z_1) = z_2$  and  $L$  depends only on  $K_1$ ,  $L_1$  and  $b$ .

If we choose  $D_2 = D_1^*$  and  $\phi(z) = z$ , then the first part of Theorem 6 yields the above mentioned result of Ahlfors.

*Proof.* For  $j = 1, 2$  let  $g_j$  map  $D_j$  conformally onto the upper half plane  $H$ . Then  $g_j$  extends to a homeomorphism of  $\bar{D}_j$  onto  $\bar{H}$  and by performing an additional Möbius transformation we may assume that  $g_j(\infty) = \infty$ . Hence  $\psi(x) = g_2 \circ \phi \circ g_1^{-1}(x)$  is a homeomorphism of  $\partial H$  onto itself with  $\psi(\infty) = \infty$ .

Choose  $-\infty < x < \infty$  and  $t > 0$ , let  $\alpha'_1 = (x, x + t)$  and  $\beta'_1 = (-\infty, x - t)$ , and let  $\alpha_1, \alpha_2, \alpha'_2$  and  $\beta_1, \beta_2, \beta'_2$  denote the images of  $\alpha'_1$  and  $\beta'_1$  under  $g_1^{-1}$ ,  $\phi \circ g_1^{-1}$ ,  $\psi$  respectively. If  $\Gamma_1$  is the family of arcs joining  $\alpha_1$  to  $\beta_1$  in  $D_1$ , then the extremal

length  $\lambda(\Gamma_1)$  of  $\Gamma_1$  is equal to 1. Moreover since  $D_1$  is a  $K_1$ -quasidisk,

$$|z_1 - z_2| \leq c_1 |z_1 - z_3| \quad (21)$$

for each ordered triple of points  $z_1, z_2, z_3 \in \partial D_1 - \{\infty\}$  where  $c_1$  is a constant which depends only on  $K_1$ . In particular if we let  $z_1 = g_1^{-1}(x)$  and  $w_1 = g_1^{-1}(x + t)$ , then the argument on pp. 82–83 in [2] shows that

$$\alpha_1 \subset \bar{B}(w_1, r), \quad r = c_1 |z_1 - w_1|$$

and that

$$d(\alpha_1, \beta_1) \geq s = c_1^{-5} e^{-2\pi} r.$$

Since  $\phi$  is an  $L_1$ -quasi-isometry,

$$\alpha_2 \subset \bar{B}(w_2, L_1 r), \quad d(\alpha_2, \beta_2) \geq \frac{s}{L_1}$$

where  $w_2 = \phi(w_1)$ , and arguing again as on p. 83 in [2] we see that

$$\lambda(\Gamma_2) \geq \frac{1}{\pi} \left( \frac{s}{L_1^2 r + s} \right)^2,$$

where  $\Gamma_2$  is the family of arcs joining  $\alpha_2$  to  $\beta_2$  in  $D_2$ . This implies that

$$\frac{\psi(x+t) - \psi(x)}{\psi(x) - \psi(x-t)} \leq c_2, \quad (22)$$

where  $c_2$  is a constant which depends only on  $K_1$  and  $L_1$ . From (22) and the above argument with  $\alpha'_1 = (x-t, x)$  and  $\beta'_1 = (x+t, \infty)$  we conclude that

$$\frac{1}{c_2} \leq \frac{\psi(x+t) - \psi(x)}{\psi(x) - \psi(x-t)} \leq c_2$$

for all such  $x$  and  $t$ . Set

$$h(z) = \frac{1}{2} \int_0^1 (\psi(x+ty) + \psi(x-ty)) dt + \frac{i}{2} \int_0^1 |\psi(x+ty) - \psi(x-ty)| dt$$

for  $z = x + iy \in \bar{H} - \{\infty\}$  and  $h(\infty) = \infty$ . Then  $h$  maps  $\bar{H}$  homeomorphically onto  $\bar{H}$  and  $h$  is continuously differentiable and  $K$ -quasiconformal in  $H$  with

$$\frac{1}{c_3} \frac{|dz|}{\text{Im}(z)} \leq \frac{|dh(z)|}{\text{Im}(h(z))} \leq c_3 \frac{|dz|}{\text{Im}(z)},$$

where  $K$  and  $c_3$  depend only on  $c_2$ , and hence on  $K_1$  and  $L_1$ . (See pp. 69–74 in [2] for the case where  $\psi(x)$  is increasing in  $x$ .) Thus  $f_1 = g_2^{-1} \circ h \circ g_1$  is a homeomorphism of  $\bar{D}_1$  onto  $\bar{D}_2$ ,  $f = \phi$  on  $\partial D_1$ ,  $f_1$  is  $K$ -quasiconformal in  $D_1$  and

$$\frac{|df_1(z)|}{|dz|} = \frac{|dh(w)|}{|dw|} \frac{|g'_1(z)|}{|g'_2(f_1(z))|}$$

for  $z \in D_1$ ,  $w = g_1(z)$ . From (11) applied to  $g_1$  and  $g_2$  we obtain

$$\begin{aligned} \frac{1}{2} &\leq |g'_1(z)| \frac{d(z, \partial D_1)}{\text{Im}(w)} \leq 2, \\ \frac{1}{2} &\leq |g'_2(f_1(z))| \frac{d(f_1(z), \partial D_2)}{\text{Im}(h(w))} \leq 2. \end{aligned}$$

Thus

$$\frac{1}{4c_3} \frac{d(f_1(z), \partial D_2)}{d(z, \partial D_1)} \leq \frac{|df_1(z)|}{|dz|} \leq 4c_3 \frac{d(f_1(z), \partial D_2)}{d(z, \partial D_1)} \quad (23)$$

and it remains to bound the ratio on the left and right sides of (23).

If  $w_1, w_2, w_3$  is an ordered triple of points in  $\partial D_2 - \{\infty\}$ , then

$$|w_1 - w_2| \leq c_1 L_1^2 |w_1 - w_3|$$

by (21) and  $D_2$  is a  $K_2$ -quasidisk where  $K_2$  depends only on  $K_1$  and  $L_1$ . Hence  $f_1$  can be extended by quasiconformal reflection in  $\partial D_1$  and  $\partial D_2$  to yield a  $K_3$ -quasiconformal mapping of  $\bar{R}^2$  onto itself with  $K_3 = KK_1^2 K_2^2$ . Fix  $z_1 \in D_1$  and  $z_2 \in \partial D_1 - \{\infty\}$ , and choose  $z_3 \in \partial D_1$  so that  $|z_3 - z_2| = |z_1 - z_2|$ . Since  $f_1$  is  $K_3$ -quasiconformal in  $\bar{R}^2$  with  $f_1(\infty) = \infty$ ,

$$|f_1(z_1) - f_1(z_2)| \leq c |f_1(z_3) - f_1(z_2)| \leq c_4 |z_3 - z_2| = c_4 |z_1 - z_2|$$

where  $c$  and  $c_4 = cL_1$  depend only on  $K_1$  and  $L_1$ . We thus obtain

$$\frac{1}{c_4} |z_1 - z_2| \leq |f_1(z_1) - f_1(z_2)| \leq c_4 |z_1 - z_2| \quad (24)$$

for all  $z_1 \in D_1$  and  $z_2 \in \partial D_1 - \{\infty\}$ . In particular, (24) implies that

$$\frac{1}{c_4} d(z, \partial D_1) \leq d(f_1(z), \partial D_2) \leq c_4 d(z, \partial D_1)$$

for all  $z \in D_1$ , and we conclude from (23) that  $f_1$  is a local  $L_2$ -quasi-isometry in  $D_1$  with  $L_2 = 4c_3c_4$ . Lemma 5 then implies that  $f_1$  is an  $L_3$ -quasi-isometry in  $\bar{D}_1$  where  $L_3 = \max(L_1, L_2)$ , and choosing  $f = f_1$  completes the proof of the first part of Theorem 6.

Finally suppose that  $z_1 \in D_1$ ,  $z_2 \in D_2$  and that (20) holds for all  $z \in \partial D_1 - \{\infty\}$ . If  $w \in \partial D_2 - \{\infty\}$ , then  $z = f_1^{-1}(w) \in \partial D_1 - \{\infty\}$  and

$$\frac{|f_1(z_1) - w|}{|z_2 - w|} = \frac{|f_1(z_1) - f_1(z)|}{|z_1 - z|} \frac{|z_1 - z|}{|z_2 - \phi(z)|}$$

lies between  $(bc_4)^{-1}$  and  $bc_4$  by (20) and (24). By Lemma 7 there exists an  $L_4$ -quasi-isometry  $f_2: \bar{D}_2 \rightarrow \bar{D}_2$  such that  $f_2$  is the identity on  $\partial D_2$ ,  $f_2(f_1(z_1)) = z_2$  and  $L_4$  depends only on  $K_2$  and  $b$ . Thus  $f = f_2 \circ f_1$  has all the properties required in the second part of Theorem 6.

Finally we require the following result which shows that a certain class of quasi-isometries is invariant under conjugation by inversion.

**LEMMA 8.** *Suppose that  $f$  is an  $L$ -quasi-isometry in  $E \subset \bar{R}^2$ , that*

$$\frac{1}{L} \leq \frac{|f(z)|}{|z|} \leq L \quad (25)$$

*for  $z \in E - \{0, \infty\}$  and that  $f(0) = 0$  if  $0 \in E$ . Then  $g = T \circ f \circ T^{-1}$  is an  $L^3$ -quasi-isometry in  $T(E)$  where  $T(z) = 1/z$ .*

*Proof.* Choose distinct points  $w_1, w_2 \in T(E) - \{\infty\}$  and let  $z_i = 1/w_i$ . If  $w_1, w_2 \neq 0$ , then  $z_1, z_2 \in E - \{0, \infty\}$ ,

$$\frac{|g(w_1) - g(w_2)|}{|w_1 - w_2|} = \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|} \frac{|z_1|}{|f(z_1)|} \frac{|z_2|}{|f(z_2)|}$$

and hence

$$\frac{1}{L^3} \leq \frac{|g(w_1) - g(w_2)|}{|w_1 - w_2|} \leq L^3 \quad (26)$$

by (25). If  $w_1 = 0$ , then  $g(w_1) = 0$ ,

$$\frac{|g(w_1) - g(w_2)|}{|w_1 - w_2|} = \frac{|z_2|}{|f(z_2)|}$$

and again (26) holds. Finally if  $\infty \in T(E)$  then  $g(\infty) = \infty$  and thus  $g$  is an  $L^3$ -quasi-isometry in  $T(E)$ .

We now obtain the main result of this section from combining Lemma 8 and Theorem 6.

**THEOREM 7.** *Suppose that  $D_1$  is a  $K_1$ -quasidisk in  $\mathbb{R}^2$ , that  $D_2$  is a Jordan domain in  $\mathbb{R}^2$  and that  $\phi: \partial D_1 \rightarrow \partial D_2$  is an  $L_1$ -quasi-isometry. Then there exist  $L$ -quasi-isometries  $f: \bar{D}_1 \rightarrow \bar{D}_2$  and  $f^*: \bar{D}_1^* \rightarrow \bar{D}_2^*$  such that  $f = f^* = \phi$  on  $\partial D_1$  and  $L$  depends only on  $K_1$  and  $L_1$ .*

*Proof.* Suppose that  $\infty \in \partial D_1$ . Then  $D_1$  and  $D_1^*$  are  $K_1$ -quasidisks with  $\partial D_1 = \partial D_1^*$  and the existence of  $f$  and  $f^*$  is an immediate consequence of Theorem 6.

Suppose next that  $\infty \notin \partial D_1$ . Then  $\infty \in D_1^*$  and  $\infty \in D_2^*$ . By making a preliminary change of variables we may assume that  $0 \in \partial D_1$  and that  $\phi(0) = 0$ . For  $j = 1, 2$  let  $G_j$  denote the image of  $D_j$  under  $T(z) = 1/z$  and set  $\psi = T \circ \phi \circ T^{-1}$ . Then  $G_1$  is a  $K_1$ -quasidisk with  $\infty \in \partial G_1$ ,

$$\frac{1}{L_1} \leq \frac{|\phi(z)|}{|z|} \leq L_1 \quad (27)$$

for  $z \in \partial D_1 - \{0\}$  and hence  $\psi: \partial G_1 \rightarrow \partial G_2$  is an  $L_1^3$ -quasi-isometry by Lemma 8. Theorem 6 then yields  $L_2$ -quasi-isometries  $g: \bar{G}_1 \rightarrow \bar{G}_2$  and  $g^*: \bar{G}_1^* \rightarrow \bar{G}_2^*$  such that  $g = g^* = \psi$  on  $\partial G_1$  and  $L_2$  depends only on  $K_1$  and  $L_1$ . In addition, since  $0 \in G_1^*$ ,  $0 \in G_2^*$  and

$$\frac{1}{L_1} \leq \frac{|0 - z|}{|0 - \psi(z)|} \leq L_1$$

for  $z \in \partial G_1 - \{\infty\}$  by (27), we can choose  $g^*$  so that  $g^*(0) = 0$ . Fix  $z_1 \in \bar{G}_1 - \{\infty\}$  and

let  $z_0$  be a point where the segment joining 0 to  $z_1$  meets  $\partial G_1$ . Then

$$\begin{aligned} |g(z_1)| &\leq |g(z_1) - g(z_0)| + |g(z_0)| \leq L_2 |z_1 - z_0| + |\psi(z_0)| \\ &\leq L_2 |z_1 - z_0| + L_1 |z_0| \leq L_2 |z_1| \end{aligned}$$

since  $L_2 \geq L_1$ . Thus by symmetry

$$\frac{1}{L_2} \leq \frac{|g(z)|}{|z|} \leq L_2$$

for  $z \in \bar{G}_1 - \{\infty\}$ . Next

$$\frac{1}{L_2} \leq \frac{|g^*(z)|}{|z|} \leq L_2$$

for  $z \in \bar{G}_1^* - \{0, \infty\}$  since  $g^*$  is an  $L_2$ -quasi-isometry in  $\bar{G}_1^*$ . Thus  $f = T^{-1} \circ g \circ T$  and  $f^* = T^{-1} \circ g^* \circ T$  have the required properties by Lemma 8.

**COROLLARY 1.** *Suppose that  $D_1$  and  $D_2$  are  $K_1$ - and  $K_2$ -quasidisks in  $R^2$  and that  $f: D_1 \rightarrow D_2$  is a bijective local  $L_1$ -quasi-isometry. Then there exists an  $L$ -quasi-isometry  $g: \bar{R}^2 \rightarrow \bar{R}^2$  such that  $g = f$  in  $D_1$  and  $L$  depends only on  $K_1$ ,  $K_2$  and  $L_1$ .*

*Proof.* By Lemma 6,  $f$  extends to an  $L_2$ -quasi-isometry of  $\bar{D}_1$  onto  $\bar{D}_2$  where  $L_2$  depends only on  $K_1$ ,  $K_2$  and  $L_1$ . Next Theorem 7 with  $\phi = f|_{\partial D_1}$  yields an  $L_2^*$ -quasi-isometry  $f^*: \bar{D}_1^* \rightarrow \bar{D}_2^*$  such that  $f^* = f$  on  $\partial D_1$  and  $L_2^*$  depends only on  $K_1$ ,  $K_2$  and  $L_1$ . Then

$$g = \begin{cases} f & \text{in } \bar{D}_1 \\ f^* & \text{in } \bar{D}_1^*, \end{cases} \quad (28)$$

is the desired extension of  $f$ .

**COROLLARY 2.** *Suppose that  $C_1$  is a  $K_1$ -quasicircle and that  $\phi$  is an  $L_1$ -quasi-isometry in  $C_1$ . Then there exists an  $L$ -quasi-isometry  $g: \bar{R}^2 \rightarrow \bar{R}^2$  such that  $g = \phi$  on  $C_1$  and  $L$  depends only on  $K_1$  and  $L_1$ .*

*Proof.* Let  $C_2 = \phi(C_1)$  and for  $j = 1, 2$  let  $D_j$  be a component of  $\bar{R}^2 - C_j$  chosen so that  $D_j \subset R^2$ . If  $f$  and  $f^*$  are the  $L$ -quasi-isometries given by Theorem 7, then  $g$  defined in (28) is the required extension.

Corollary 2 extends recent results of Jerison and Kenig [9] and of Tukia [18] who consider the cases where  $C_1$  is a line and a circle, respectively.

#### 4. An application

If  $f$  is a local  $L$ -quasi-isometry in a plane domain  $D$  with  $L < L(D)$ , then  $f$  is injective. The following result shows that one can say more whenever  $D$  is simply connected.

**THEOREM 8.** *Suppose that  $D_1$  is a simply connected proper subdomain of  $\mathbb{R}^2$  and that  $f$  is a local  $L_1$ -quasi-isometry in  $D_1$  with  $L < L(D_1)$ . Then there exists an  $L$ -quasi-isometry  $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $g = f$  in  $D_1$  and  $L$  depends only on  $L(D_1)$  and  $L_1$ .*

*Proof.* Let  $D_2 = f(D_1)$  and let  $g$  denote any local  $L_2$ -quasi-isometry in  $D_2$  with  $L_2 < L(D_1)/L_1$ . Then  $g \circ f$  is a local  $L_1 L_2$ -quasi-isometry in  $D_1$ ,  $g \circ f$  is injective in  $D_1$  since  $L_1 L_2 < L(D_1)$  and hence  $g$  is injective in  $D_2$ . Thus

$$L(D_2) \geq \frac{L(D_1)}{L_1} > 1. \quad (29)$$

Since  $f$  is injective in  $D_1$ ,  $f$  is an  $L_1^2$ -quasiconformal mapping of  $D_1$  and hence  $D_2$  is a simply connected proper subdomain of  $\mathbb{R}^2$ . Then by Theorem 3 and (29)  $D_1$  and  $D_2$  are  $K_1$ - and  $K_2$ -quasidisks, where  $K_1$  and  $K_2$  depend only on  $L(D_1)$  and  $L(D_1)/L_1$  respectively, and the existence of  $g$  follows from Corollary 1.

Theorem 8 can be interpreted physically if we think of  $D_1$  as a homogeneous elastic body and  $f$  as the distortion of  $D_1$  due to a force field. In this case  $L(D_1)$  measures the maximum permissible strain in  $D_1$  before  $D_1$  buckles. Theorem 8 asserts that if the strain in  $D_1$  is less than  $L(D_1)$ , then the shape of  $D_1$  is not substantially changed under the force field.

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Received January 14, 1982