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On the placement problem of Reeb components

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Let $\mathcal{F}$ be a codimension-one smooth foliation on a three-sphere, S^3. By Novikov's theorem [10], there exists a Reeb component in $\mathcal{F}$ and the entire set of Reeb components of $\mathcal{F}$ is "knotted" in S^3 in the sense of [10]. The placement problem of Reeb components in S^3 is not only interesting in itself but also related to linked closed orbits of a transverse vector field to $\mathcal{F}$. In fact, Novikov proved that a vector field transverse to a Reeb foliation on $S^1 \times D^2$ has a periodic trajectory which is isotopic to the circle $S^1 \times 0$ traversed once (see [10], Lemma 9.1). Therefore one can consider a Reeb component as a "stable closed orbit" of any transverse vector field and in the case of studying linked closed orbits of a transverse vector field, the placement problem of Reeb components in S^3 is significant.

In the present paper, we study the placement problem of Reeb components in S^3 and have certain fundamental results about the structure of codimension-one smooth foliations on S^3. In §2, we will prove a decomposition theorem with respect to a codimension-one smooth foliation on S^3 which also asserts that the decomposition is represented by a directed linear graph. The associated graph with a foliation must satisfy some conditions and conversely there exists a smooth foliation on S^3 whose associated graph is the given graph satisfying the conditions. We will call (the cores of) all Reeb components in a codimension-one smooth foliation on S^3 a *Reeb link*. Then it follows that a fibred link is a Reeb link by the standard technique, winding (the ends of) the fibres. So it is natural to consider what conditions on the given foliation on S^3 imply the Reeb link is fibred (we will use a term "spinnable" instead of "fibred" in this paper below, see §1). §3 is devoted to the study in the case of "vertexwise fibred" (precisely, see §1). In §4, we construct some examples which we need in order to realize a smooth foliation according to the given graph. Furthermore in §4 we construct Reeb knots (with smooth foliations) which are "vertexwise fibred" but are not spinnable (fibred).

§1. Statement of results

We work mostly in the smooth category and all foliations we consider will be smooth (C^∞), of codimension-one unless otherwise stated.

Let $\mathcal{F}$ be a foliation on S^3. We denote the entire set of Reeb components of $\mathcal{F}$ by $R = \bigcup_{i=1}^n R_i$, where R_i is a solid torus in S^3 and the restricted foliation $\mathcal{F}|_{R_i}$ is a Reeb foliation on the solid torus.

DEFINITION. A link L in S^3 is a *Reeb link* if it is isotopic to a link which consists of cores of all Reeb components of a foliation on S^3, where a core of a solid torus is a circle $S^1 \times 0$ in some fixed framing $S^1 \times D^2$ of the solid torus.

A Reeb link is non-trivial, that is, no component can be spanned by a disc, not intersecting the others (see Novikov [10], Theorem 9.2) and furthermore it is non-splittable (see Proposition 2.1 below and Roussarie [14]).

DEFINITION. A compact 3-manifold with boundary $M = S^3 - \text{Int } R$ is called *the Reeb link exterior* with respect to $\mathcal{F}$.

Our main aim in this paper is to describe a certain structure of the Reeb link exterior with respect to $\mathcal{F}$. It is well known that every compact leaf in $\mathcal{F}$ is diffeomorphic to a torus. We consider to separate the foliated manifold $(M, \mathcal{F}|_M)$ along compact leaves and have the following graphical representation theorem. Before stating the theorem, we make terminological preliminaries. Let L be a link in S^3 and $N(L)$ a tubular neighbourhood of L in S^3. Then $S^3 - \text{Int } N(L)$ is called an *exterior of a link* or a *link exterior*. Let $(S^1 \times D^2) \# \cdots \# (S^1 \times D^2) \to S^3$ be an embedding of k-times interior-connected sum of solid tori into a three-sphere. Then we call this embedded submanifold a *solid toral sum* in S^3. Moreover let L be a link in a solid toral sum (in S^3) and $N(L)$ a tubular neighbourhood in the solid toral sum. Then the solid toral sum minus $\text{Int } N(L)$ is said to be an *exterior of a link in a solid toral sum* or a *link exterior in a solid toral sum*. Finally, a *foliated I-product* over a manifold T is a foliated manifold $(T \times I, \mathcal{F})$ such that all leaves of $\mathcal{F}$ are transverse to $\{x\} \times I$ for all $x \in T$, where $I = [0, 1]$.

THEOREM 1. *Let $\mathcal{F}$ be a foliation on S^3. Then there exists a directed linear graph (i.e. a one-dimensional complex with directions on edges) $\Gamma_{\mathcal{F}}$ corresponding to $\mathcal{F}$ which represents a structure of $\mathcal{F}$ as follows:*

(1) *vertices; there are three types of vertices* $*$, $\bullet$ *and* $\bigcirc$ *which represent Reeb components, foliated I-products over tori (which may have infinitely many compact leaves) and link exteriors or link exteriors in solid toral sums with foliations whose*

interior leaves are all non-compact, respectively. $*, \bullet$ *and* $\bigcirc$ *are referred to as star, black and white vertices respectively.*

(2) *edges; an edge represents a compact leaf of* $\mathcal{F}$ *along which we cut* S^3 *so as to have submanifolds represented by vertices which are connected by the edge.*

(3) *directions; let* v_1 *and* v_2 *be vertices of one edge in* $\Gamma_{\mathcal{F}}$. *Then a direction from* v_1 *to* v_2, $\overrightarrow{v_1 v_2}$ *means the toral compact leaf corresponding to the edge bounds a solid torus in* S^3 *containing the submanifold represented by the vertex* v_2.

Remark 1. By Alexander's solid torus theorem (see [12]), every torus in S^3 bounds a solid torus on at least one side. Therefore every compact leaf in $\mathcal{F}$ bounds a solid torus so that every edge in $\Gamma_{\mathcal{F}}$ is directed. Moreover cylindrical components which are investigated in [2] do not occur in foliations on S^3. Note that an edge in $\Gamma_{\mathcal{F}}$ is directed in both directions if and only if the corresponding toral compact leaf is unknotted in S^3.

Remark 2. A foliation restricted to the interior of a submanifold which is represented by a white (or a star) vertex is a connected component of $\mathcal{F}$ in the sense of Novikov [10].

Let v be a vertex in a directed linear graph Γ. Suppose that Γ is a tree, that is, Γ is contractible. Suppose that there are $(l + m + n)$-edges which have v as a common vertex in Γ and that l-edges of them are flowing out from v, m-edges of them are flowing into v and n-edges of them have both directions. Then we will call the non-negative integer triple (l, m, n) the *index* of the vertex v.

The following is a characterization of the graphs corresponding to foliations on S^3 by Theorem 1:

THEOREM 2. *Let* $\mathcal{F}$ *be a foliation on* S^3 *and* $\Gamma_{\mathcal{F}}$ *be the graph corresponding to* $\mathcal{F}$ *by Theorem* 1. *Then* $\Gamma_{\mathcal{F}}$ *satisfies the following conditions* (i)-(vi);

   (i) *$\Gamma_{\mathcal{F}}$ is a finite tree,*

   (ii) *there exists a star vertex,*

   (iii) *there does not exist a vertex whose index is* $(0, m_1, 0)$ *nor* $(0, m_2, 1)$ *where* $m_1 \geq 2$ *and* $m_2 \geq 1$,

   (iv) *an index is* $(0, 1, 0)$ *or* $(0, 0, 1)$ *if and only if the vertex is star,*

   (v) *the index of the black vertex is* $(1, 1, 0)$ *or* $(0, 0, 2)$,

   (vi) *black vertices are not connected to each other by one edge.*

Conversely, if a directed linear graph Γ *which satisfies the conditions* (i)-(vi) *is given, then there exists a foliation* $\mathcal{G}$ *on* S^3 *such that* $\Gamma_{\mathcal{G}} = \Gamma$.

By a construction in realization of a foliation from a given graph, we have the following corollary:

COROLLARY. *A composite knot of Reeb knots is a Reeb knot.*

Next we consider the question when a Reeb link is fibred or more generally, "vertexwise fibred". First note that we may assume that the foliation has only finitely many compact leaves so far as we discuss the placement problem of Reeb components. In fact, we can replace each foliated I-product over a torus by a torus in the given foliation by adjusting the holonomy of the boundary leaves of the foliated I-product over a torus into the flat holonomy (i.e. all diffeomorphisms which generate the holonomy are infinitely tangent to the identy at 0), if necessary.

Let $\Gamma_{\mathcal{F}}$ be the graph associated with a given foliation $\mathcal{F}$. We assume there is no black vertex in $\Gamma_{\mathcal{F}}$. The Reeb link of $\mathcal{F}$ is *vertexwise fibred* if each restricted foliation of $\mathcal{F}$ to the interior of the submanifold represented by a vertex is a bundle foliation, that is, there is a fibre bundle structure on the interior of the submanifold over S^1 and the foliation is the one whose leaves are fibres of the fibre bundle.

In order to describe asymptotic behaviour of non-compact leaves in a foliation, we need the concept of ends of an open connected manifold.

DEFINITION. Let F be an open connected manifold. A family ε of non-empty connected open subsets of F is called an *end* of F if ε satisfies the following conditions (i)–(iv);

(i) $\mathrm{Cl}_F(U) - U$ is compact for all $U \in \varepsilon$, where $\mathrm{Cl}_F(\ )$ means the closure with respect to the topology of F,

(ii) if $U, U' \in \varepsilon$, then there is $U'' \in \varepsilon$ such that $U'' \subset U \cap U'$,

(iii) $\bigcap_{U \in \varepsilon} \mathrm{Cl}_F(U) = \phi$, and

(iv) ε is a maximal family satisfying (i), (ii) and (iii).

An end ε of F is called *isolated* if there is $U \in \varepsilon$ such that if an end ε' of F contains U then $\varepsilon' = \varepsilon$.

For the following definitions, let $\mathcal{F}$ be a transversely orientable foliation on an arbitrary closed orientable manifold M.

DEFINITION. Let ε be an end of a non-compact leaf F of $\mathcal{F}$. $L_\varepsilon(F) = \bigcap_{U \in \varepsilon} \mathrm{Cl}_M(U)$ and called the ε-*limit set* of F.

ε-limit sets of non-compact leaves have been studied by many authors. For properties of ε-limit sets, we refer to [1], [6], [8], [9], [11] and [15] for example.

In [8], Nishimori defines a notion which prescribes circumstances of ends of leaves in a foliation as follows:

DEFINITION. Let ε be an end of a leaf F of $\mathcal{F}$. Then ε is a *tame end of depth* 1 if the following conditions (i)–(iii) are satisfied;

(i) ε is isolated,

(ii) $L_\varepsilon(F) \cap F = \phi$, and

(iii) ε approaches $L_\varepsilon(F)$ from one side, that is, for all $x \in L_\varepsilon(F)$ there exist $\delta > 0$ and $U \in \varepsilon$ such that $\varphi(\{x\} \times]-\delta, 0[) \cap U = \phi$ for some transverse flow $\varphi : M \times \mathbf{R} \to M$.

For more details, see Nishimori [8] and [9].

Next we define a notion of *depth* of leaves (see Nishimori [8]). Let F_1 and F_2 be leaves of $\mathcal{F}$. We say $F_1 \leq F_2$ if and only if $F_1 \subset \mathrm{Cl}_M(F_2)$. Moreover we write $F_1 < F_2$ if and only if $F_1 \leq F_2$ and $F_1 \neq F_2$. For a leaf F of $\mathcal{F}$, we denote by $d(F)$ the supremum of integers k such that there exists a sequence of leaves $F_1, \ldots, F_k$ of $\mathcal{F}$ satisfying $F_1 < \cdots < F_k = F$. Let $d(\mathcal{F})$ be the supremum of $d(F)$ where F runs through $\mathcal{F}$. We call $d(F)$ and $d(\mathcal{F})$ the *depth* of F and $\mathcal{F}$ respectively.

Now we can state our result. Recall that a leaf is said to be *proper* if it is not asymptotic to itself.

THEOREM 3. *Let $\mathcal{F}$ be a foliation on S^3. Then the following four conditions are equivalent:*

(i) $d(\mathcal{F}) = 2$,

(ii) *all leaves of $\mathcal{F}$ are proper and all ends of non-compact leaves are isolated,*

(iii) *all ends of non-compact leaves are tame ends of depth* 1, *and*

(iv) *for the associated graph $\Gamma_\mathcal{F}$,* (a) *each restricted foliation of $\mathcal{F}$ to the interior of the submanifold represented by a white vertex is a bundle foliation, and* (b) *for each non-compact leaf F of a foliated I-product over a torus represented by a black vertex, there exist compact leaves F_1 and F_2 such that the restricted foliation to the component of the foliated I-product over a torus minus $F_1 \cup F_2$ which contains F is a bundle foliation.*

For foliations which have only finitely many compact leaves, we have the following corollary:

COROLLARY 1. *Let $\mathcal{F}$ be a foliation on S^3 which has only finitely many compact leaves. Then the following are equivalent:*

(i) $d(\mathcal{F}) = 2$,

(ii) *all leaves of $\mathcal{F}$ are proper and all ends of non-compact leaves are isolated,*

(iii) *all ends of non-compact leaves are tame ends of depth* 1, *and*

(iv) *the Reeb link of $\mathcal{F}$ is vertexwise fibred.*

As noted above, since the link type of the Reeb link does not change when we replace each foliated I-product over a torus by a torus, we note that the conditions (i)–(iii) of Theorem 3 imply that the Reeb link is vertexwise fibred.

Since $d(\mathcal{F}) \neq 1$ for a foliation on S^3, by Nishimori's result (Theorem 1 in [8]) and Theorem 3 we have the following:

COROLLARY 2. *Let $\mathcal{F}$ be a foliation on S^3. Then each condition of Theorem 3 is equivalent to the one that $(\mathcal{F}, \leqq)$ is a partially ordered set and $\mathcal{F}$ is almost without holonomy.*

Let L be a link in S^3. We say L is a *spinnable* if there exists a fibration $\pi : S^3 - L \to S^1$ and a tubular neighbourhood of each component K_i of L framed as $S^1 \times D^2$ with $K_i \approx S^1 \times 0$ such that the restriction of π to $S^1 \times (D^2 - 0)$ is the map into S^1 given by $(x, y) \mapsto y/|y|$. Ordinarily, a spinnable link is referred as a fibred link (see [12]). However, in order to avoid confusion with vertexwise fibred, we call it as above. Let L be a spinnable link in S^3. Then one can construct a foliation on S^3 whose Reeb link is L by winding ends of all fibres to the boundary of a smaller tubular neighbourhood of each component K_i of L (see [16]). We also call this foliation or the restriction of this foliation to the Reeb link exterior *spinnable*. Note that the associated graph with this foliation is as Figure 1.

As a detailed part of Theorem 3, we have the following:

PROPOSITION 1. *Let $\mathcal{F}$ be a foliation on S^3 which satisfies the condition of Theorem 3. Let $\Gamma_{\mathcal{F}}$ be the graph associated with $\mathcal{F}$. Then the restriction of $\mathcal{F}$ to the interior of the submanifold represented by a white vertex whose index $(1, 0, 0)$ is spinnable. In particular, if the associated graph is $\bigcirc \longrightarrow *$, then the Reeb knot is spinnable with respect to $\mathcal{F}$.*

When we forget a foliation on the Reeb link exterior and observe only a link type of the Reeb link, we have the following proposition in contrast with
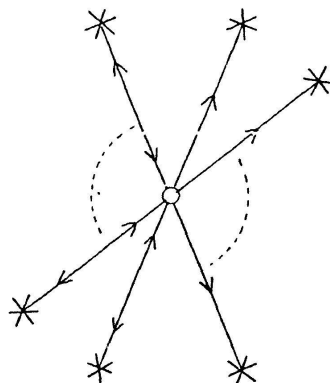

Fig. 1

Proposition 1:

PROPOSITION 2. *There exists a vertexwise-fibred Reeb knot which is not a spinnable knot. In fact, the twisted double of any Reeb knot is a Reeb knot.*

For the definition of a twisted double of a knot, see the proof of Proposition 2.

§2. A graphical representation of a foliation on S^3

In this section, we prove Theorem 1 and the first half of Theorem 2. First we state the following:

PROPOSITION 2.1. *The Reeb link exterior of any Reeb link is irreducible, that is, each embedded 2-sphere bounds a 3-ball in the Reeb link exterior. In other words, each Reeb link is non-splittable.*

Proposition 2.1 follows directly from Novikov's theorem [10] and the sphere theorem (see [3], [12]). Roussarie [14] proved a more general result. For more details, see [14].

Recall that $\mathcal{F}$ is a foliation on S^3 and $R = \bigcup_{i=1}^n R_i$ is the entrie set of Reeb components of $\mathcal{F}$. Let M denote the Reeb link exterior with respect to $\mathcal{F}$. A foliated I-product over a torus in $\mathcal{F}$ is meant to be a $T^2 \times I$ in S^3 with the induced foliation which is a foliated I-product over a torus. We say a foliated I-product over a torus in $\mathcal{F}$ is maximal if it is maximal with respect to the relation of inclusion. Then we can take all maximal foliated I-products over tori in $\mathcal{F}$; $C_1, \ldots, C_m$. Note that there are only finitely many maximal foliated I-products over tori in $\mathcal{F}$. Also note that there are only finitely many compact leaves which are not contained in $\bigcup_{i=1}^m C_i$. We denote the compact leaves by $S_1, \ldots, S_l$.

Let M_i be the closure of a connected component of $M - ((\bigcup_{i=1}^m C_i) \cup (\bigcup_{i=1}^l S_i))$ in M. Then $M_i \cap M_j$ is a torus in S^3 if $i \neq j$ and $M_i \cap M_j \neq \phi$.

Lemma 2.2. *M_i is a link exterior or a link exterior in a solid toral sum in S^3.*

Proof. Obviously, the boundary ∂M_i is a union of tori. Let $\partial M_i = \bigcup_{a=1}^s U_a$, where U_a is a torus in S^3. By Alexander's solid torus theorem, there exists a solid torus V_a in S^3 such that $\partial V_a = U_a$ for each U_a. We can assume that $M_i \cap \text{Int } V_a = \phi$ for an unknotted solid torus V_a in S^3, taking the other solid torus for $U_a = \partial V_a$ in S^3, if necessary.

First, if $M_i \cap \text{Int } V_a = \phi$ for any $a = 1, \ldots, s_i$, then $M_i = S^3 - \bigcup_{a=1}^s \text{Int } V_a$ so that M_i is a link exterior.

Next, suppose that there exists $a \in \{1, \ldots, s_i\}$ such that $M_i \cap \text{Int } V_a \neq \phi$. The proof is divided into two cases as follows: I. $V_b \subset V_a$ for any $b \in \{1, \ldots, s_i\}$, and II. otherwise case, that is, there is $b \in \{1, \ldots, s_i\}$ such that $V_b \not\subset V_a$. We need two sublemmas.

SUBLEMMA 1. *Let V_1 and V_2 be solid tori in S^3 such that $V_1 \cap V_2 \neq \phi$, $\partial V_1 \cap \partial V_2 = \phi$, $V_1 \not\subset V_2$ and $V_2 \not\subset V_1$. Then $V_1 \cup V_2 = S^3$.*

SUBLEMMA 2. *Let V_1 and V_2 be knotted solid tori in S^3 such that $V_1 \cap V_2 \neq \phi$, $\partial V_1 \cap \partial V_2 = \Phi$, $V_1 \not\subset V_2$ and $V_2 \not\subset V_1$. Then $V_1 \cap V_2$ is a solid toral sum in S^3, that is, there is an embedding $(S^1 \times D^2) \mathbin{\#} (S^1 \times D^2) \to S^3$ whose image is $V_1 \cap V_2$.*

These sublemmas can be easily proved. So we omit the proof of Sublemma 1 and we only sketch the proof of sublemma 2.

Sketch of the proof of Sublemma 2. By the fact that V_i's are knotted and irreducible and by general position argument, we assert that there exists a meridian disc of V_1 which does not intersect ∂V_2 (changing the suffixes of V_i's, if necessary). Then there exists a 3-ball B^3 in $\text{Int } V_1$ such that $\partial V_2 \subset \text{Int } B^3$, $\partial B^3 \subset \text{Int } V_2$ and $\partial V_1 \subset S^3 - \text{Int } B^3 \subset V_2$. This implies $V_1 \cap V_2$ is diffeomorphic to $(S^1 \times D^2) \mathbin{\#} (S^1 \times D^2)$. This proves Sublemma 2.

Case I. First we assert that $V_b \cap V_c = \phi$ if $b \neq a$ and $c \neq a$. In fact, assume $V_b \cap V_c \neq \phi$, then $V_b \cup V_c \subset V_a$ so that $V_b \subset V_c$ or $V_c \subset V_b$ by Sublemma 1. We can assume $V_b \subset V_c$. Then ∂V_c separates ∂V_a and ∂V_b. This contradicts that ∂V_a, ∂V_b and ∂V_c are boundary components of M_i. Therefore $V_b \cap V_c = \phi$ for any $b, c \in \{1, \ldots, s_i\} - \{a\}$. This implies $M_i \cap \text{Int } V_b = \phi$ for any $b \neq a$, and consequently M_i is an exterior of a link in a solid torus V_a.

Case II. $\partial V_b \subset V_a$ since $M_i \subset V_a$. Therefore $V_a \cap V_b \neq \phi$ and also $M_i \cap \text{Int } V_b \neq \phi$. Note that $V_a \not\subset V_b$ and that V_a and V_b are knotted respectively by the definition of V_i's. So, by Sublemma 2 $V_a \cap V_b$ is diffeomorphic to $(S^1 \times D^2) \mathbin{\#} (S^1 \times D^2)$. Let $V_{b_1}, \ldots, V_{b_n}$ be the solid tori which are not contained in V_a. Then V_{b_i} does not contain V_{b_j} nor V_a for any $i, j = 1, \ldots, n$, and $i \neq j$. Therefore, for each pair of the solid tori $V_a, V_{b_1}, \ldots, V_{b_n}$, the same situation occurs. One can prove that $V_a \cap V_{b_1} \cap \cdots \cap V_{b_n}$ is diffeomorphic to $(S^1 \times D^2) \mathbin{\#} \cdots \mathbin{\#} (S^1 \times D^2)$, an $(n+1)$-times connected sum of solid tori, by the

same consideration as Sublemma 2. Let $V_{c_1}, \ldots, V_{c_m}$ be the solid tori which are contained in V_a. Then $M_i = V_a \cap V_{b_1} \cap \cdots \cap V_{b_n} - V_{c_1} \cup \cdots \cup V_{c_m}$. This completes the proof of Lemma 2.2.

Proof of Theorem 1. As the observation just before Lemma 2.2, we may take maximal foliated I-products over tori $C_1, \ldots, C_m$. We assign to $C_1, \ldots, C_m$ black vertices. By the maximality of the foliated I-products over tori, this assignment is well-defined, that is, non-negative integer m and submanifolds C_i's depend only on $\mathscr{F}$. Then submanifolds M_i's which are defined just before Lemma 2.2 are well-defined and by definition interior leaves in $\mathscr{F}|_{M_i}$ are all non-compact. We assign to M_i's white vertices and to Reeb components star vertices. Finally, connect those vertices each other by an edge according that the submanifolds which are represented by the vertices intersect on a torus in S^3. Direct all edges according to Alexander's solid torus theorem, then we have the desired directed linear graph $\Gamma_{\mathscr{F}}$. Now, by Lemma 2.2 and our construction of the graph, the assertions of the theorem is obvious.

Next we prove the first half of Theorem 2. We will prove the latter half and Corollary to Theorem 2 in §4.

Proof of Theorem 2. *Part* 1. We prove the graph $\Gamma_{\mathscr{F}}$ satisfies the conditions. By the definition, $\Gamma_{\mathscr{F}}$ is a finite connected graph. Since each torus in S^3 separates S^3 there is no loop in $\Gamma_{\mathscr{F}}$. This shows (i). (ii) follows from Novikov's theorem [10]. (v) and (vi) are obvious. Next we prove (iii). Let $\lambda_1, \ldots, \lambda_m$ be loops in solid torus $S^1 \times D^2$ defined by $\lambda_i(t) = (i/m, (\exp(2\pi it))/2)$, where $S^1 = \mathbf{R}/\mathbf{Z}$, $D^2 = \{z \in \mathbf{C} \mid |z| \leq 1\}$ and $\lambda_i : [0, 1] \to S^1 \times D^2$. Let $X_m = S^1 \times D^2 - \bigcup_{i=1}^m \mathrm{Int}\, N(\lambda_i)$, where $N(\lambda_i)$ is a small tubular neighbourhood in $S^1 \times D^2$. We consider $S^1 \times D^2$ as X_0. Then a submanifold of S^3 represented by a vertex whose index is $(0, m_1, 0)$, where $m_1 \geq 2$, is diffeomorphic to $X_{m_1 - 1}$. For, as in the proof of Lemma 2.2, in particular Sublemma 2, a submanifold whose index is $(0, m_1, 0)$ is a solid toral sum in S^3 and an $(m + 1)$-times connected sum of solid tori is diffeomorphic to X_m. Note that the vertex is not a black vertex. Any boundary component $\partial_i X_m$ of X_m is compressible in X_m, that is, natural homomorphism $\pi_1(\partial_i X_m) \to \pi_1(X_m)$ induced by the inclusion is not monic. Therefore, by Novikov's theorem [10], any foliation on X_m with boundary components as leaves has a compact leaf in $\mathrm{Int}\, X_m$ (see [10]). This contradicts our construction. A submanifold whose index is $(0, m_2, 1)$ is diffeomorphic to $X_{m_2 - 1} - \mathrm{Int}\, N(\lambda)$, where λ is a loop in $\mathrm{Int}\, X_{m_2 - 1}$ and $N(\lambda)$ is a small tubular neighbourhood of λ. Consider the foliation on $X_{m_2 - 1} - \mathrm{Int}\, N(\lambda)$ induced from $\mathscr{F}$. Since each interior leaf of the foliation is not compact, each boundary component of $X_{m_2 - 1} - \mathrm{Int}\, N(\lambda)$ must be incompressible by Novikov's theorem. In other words, λ must pass through each meridian disc of all boundary components in $X_{m_2 - 1}$. Therefore the solid torus corresponding to $N(\lambda)$ in S^3 must

be knotted by non-cancellation theorem on companionship (see Theorem 4.D.9 in [12]). This contradicts the definition of the index. So there is no vertex whose index is $(0, m_1, 0)$ nor $(0, m_2, 1)$, where $m_1 \geq 2$ and $m_2 \geq 1$, in $\Gamma_{\mathscr{F}}$. This shows (iii). The index of a star vertex is obviously $(0, 1, 0)$ or $(0, 0, 1)$. Conversely, if the index of a vertex is $(0, 1, 0)$ or $(0, 0, 1)$, then the corresponding submanifold is a solid torus in S^3. By our construction (and the condition (v)), interior leaves of the restricted foliation to the solid torus are all non-compact. Then it is a Reeb foliation by Novikov's result [10]. Now we have proved the first half of Theorem 2.

§3. Reeb links which are vertexwise fibred

Let F be a non-compact leaf in an arbitrary foliated manifold $(M, \mathscr{F})$.

DEFINITION. A point $y \in M$ is a *limit point* of F if there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in F such that $\{x_n\}$ has no accumulating point in F with respect to the topology of F as an abstract manifold but converges to y with respect to the topology of M. The set of all limit points of F is called the *limit set* of F and is denoted by $L(F)$.

In order to prove Theorem 3, we need some lemmas about limit sets and ε-limit sets. Let M be a compact manifold and F a non-compact leaf of a foliation $\mathscr{F}$ on M. Let ε be an end of F. A subset of M is called *saturated* if it is a union of leaves of $\mathscr{F}$.

LEMMA 3.1. (1) $L_\varepsilon(F)$ *and* $L(F)$ *are non-empty compact saturated subsets of* M. (2) $L_\varepsilon(F)$ *is connected.* (3) $L(F)$ *is a union of all* ε-*limit sets of all ends of* F; $L(F) = \bigcup_{a \in A} L_{\varepsilon_a}(F)$, *where* $\{\varepsilon_a\}_{a \in A}$ *is the set of all ends of* F.

LEMMA 3.2. F *is a proper leaf if and only if* $L(F) \cap F = \phi$.

LEMMA 3.3. (1) F *is a proper leaf if and only if* $L(F) = \mathrm{Cl}_M(F) - F$. (2) F *is a non-proper leaf if and only if* $L(F) = \mathrm{Cl}_M(F)$.

LEMMA 3.4. *Let* F *be a proper leaf and let* ε *be an end of* F. *Assume* $L_\varepsilon(F)$ *be a union of proper leaves, then* $L_\varepsilon(F)$ *contains a compact leaf.*

It is easy to prove Lemma 3.1, 3.2 and 3.3, and we omit the proof.

Proof of Lemma 3.4. Let $\mathcal{K}$ be the set of all non-empty compact saturated sets in $L_\varepsilon(F)$. Note that $\mathcal{K} \neq \phi$ and $\mathcal{K}$ is ordered by inclusion. Then every non-empty linear ordered subset of $\mathcal{K}$ has a lower bound. Therefore there exists a minimal element in $\mathcal{K}$ by Zorn's lemma. Let K denote a minimal element. If K contains a non-compact leaf G, then, since G is proper, $\mathrm{Cl}_M(G) - G$ is a closed saturated subset by Lemma 3.1 and 3.3, and $\mathrm{Cl}_M(G) - G \subsetneqq K$. This contradicts the minimality of K. Therefore K contains no non-compact leaf and thus, by the minimality, K is just a compact leaf.

Now we prove Theorem 3.

Proof of Theorem 3. First, note that if all leaves of $\mathcal{F}$ are proper every ε-limit set contains a compact leaf by Lemma 3.4. The equivalence between (ii) and (iii) follows immediately from Lemma 3.1, 3.2 and the fact that each compact leaf in S^3 separates S^3.

The rest will be proved as (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii). First assume (ii). Let M_i be a submanifold represented by a white vertex. Let $\partial M_i = S_1 \cup \cdots \cup S_{l_i}$ where S_j is a toral boundary component. Since all leaves of $\mathcal{F}|_{M_i}$ are proper and all interior leaves of $\mathcal{F}|_{M_i}$ are non-compact, we can take a *staircase* around each S_j by Nishimori's theorem [7]. Precisely, let $f : [0, \varepsilon[\to [0, \delta]$ be a contracting diffeomorphism which preserves 0 and $X_f = S^1 \times I \times [0, \varepsilon[/\sim$ a quotient manifold where $(x, 0, t) \sim (x, 1, f(t))$, and let $\mathcal{F}_f$ be a foliation on X_f induced from the product foliation $\{S^1 \times I \times t\}_{t \in [0, \varepsilon[}$ on $S^1 \times I \times [0, \varepsilon[$. Then there exists an embedding $h : X_f \to M_i$ such that $h(S^1 \times I \times 0/\sim) = S_j$ and $h^*\mathcal{F} = \mathcal{F}_f$. It follows from this description that there exists a torus in M_i which is transverse to $\mathcal{F}|_{M_i}$ and is parallel to S_j, that is, there exists an embedding $\varphi : T^2 \times I \to M_i$ such that $\varphi(T^2 \times 0) = S_j$ and $\varphi(T^2 \times 1)$ is the torus. In fact, one can construct it in a staircase $h(X_f)$ as follows: Let $\varepsilon' < \varepsilon$, and let $\psi : S^1 \times I \to S^1 \times I \times [0, \varepsilon[$ be an embedding defined by $\psi(x, t) = (x, t, (f(\varepsilon') - \varepsilon')t + \varepsilon')$. Then after rounding corners $\psi(S^1 \times I)/\sim$ is a smooth torus in X_f which is transverse to $\mathcal{F}_f$. Therefore we have tori in M_i which are transverse to $\mathcal{F}$ and are parallel to respective boundary components. Let $T_1, \ldots, T_{l_i}$ be the tori. Let M_i' be the closure in M_i of the connected component of $M_i - \bigcup_{j=1}^{l_i} T_j$ which does not contain ∂M_i. Then $\partial M_i' = \bigcup_{j=1}^{l_i} T_j$ and $\partial M_i'$ is transverse to the restricted foliation $\mathcal{F}|_{M_i'}$. Let F' be any leaf of $\mathcal{F}|_{M_i'}$ and let F be the leaf of $\mathcal{F}|_{M_i}$ such that $F' \subset F$. Then F is a non-compact proper leaf and $\bigcup_\varepsilon L_\varepsilon(F) = L(F) = \mathrm{Cl}_{M_i}(F) - F$. By Lemma 3.4, $L_\varepsilon(F)$ contains a boundary component of M_i for each end ε of F. Note that $L_\varepsilon(F)$ consists of just one leaf if and only if ε is a tame end of depth 1 by a result of Nishimori [9]. This is the case. All ends of F are isolated by assumption so that they are cut off by cutting M_i along transversal tori T_j's. Therefore F' is compact. Consequently all leaves of $\mathcal{F}|_{M_i'}$ are

compact and one can construct a fibration $\pi'_i : M'_i \to S^1$ such that $\mathscr{F}|_{M'_i} = \{\pi'^{-1}_i(\theta) \mid \theta \in S^1\}$. It is clear that π'_i extends into Int M_i so that $\mathscr{F}|_{\mathrm{Int}\, M_i}$ is a bundle foliation. This implies (a) of (iv). One can see that $\mathscr{F}$ satisfies the condition (b) by a similar observation.

Next assume (iv). For each non-compact leaf F of $\mathscr{F}$, $\mathrm{Cl}_M(F) - F = L(F) = \bigcup_\varepsilon L_\varepsilon(F)$ and $L(F)$ consists of compact leaves by the conditions (a) and (b). This implies $d(\mathscr{F}) = 2$.

Assume $d(\mathscr{F}) = 2$. Then $\mathscr{F}$ is almost without holonomy, that is, all non-compact leaves have no holonomy (see Theorem 1 in [8]). Moreover each leaf of $\mathscr{F}$ satisfies the assumption of Theorem 6 in [8], so we conclude that all leaves of $\mathscr{F}$ are proper and that each end of a non-compact leaf is isolated. This can be proved directly without difficulty and is an easy case of Nishimori's theorem. This completes the proof.

Proof of Proposition 1. Let M_i be the submanifold represented by a white vertex with the index $(1, 0, 0)$. M_i is a knot exterior. Let T be a transversal torus parallel to the boundary of M_i and let M'_i be as in the proof of Theorem 3. First we prove the following assertion:

Assertion. For any leaf F' of $\mathscr{F}|_{M'_i}$, $\partial F' = F' \cap \partial M'_i$ is connected. In fact, $\partial F'$ is a longitude of $\partial M'_i$.

Proof of Assertion. Let n be the number of connected components of $\partial F'$. Note that $n \neq 0$. By construction of T, $\partial F' = l_1 \cup \cdots \cup l_n$ is a torus link in $\partial M'_i$, that is, l_i's are all parallel in the torus $\partial M'_i$ and each l_i is a generator of $\pi_1(\partial M'_i)$. Moreover each orientation of l_i induced from F' is the same since $\mathscr{F}$ is transversely orientable. Therefore, let $\partial : H_2(M'_i, \partial M'_i) \to H_1(\partial M'_i)$ be the boundary homomorphism, then $\partial[F'] = [\partial F'] = [l_1] + \cdots + [l_n] = n[l]$, where $[\,]$ means the homology class and l is one of l_i's. Moreover $n(i_\#[l]) = i_\#(n[l]) = i_\# \cdot \partial[F'] = 0 \in H_1(M'_i) \cong \mathbf{Z}$ where $i_\# : H_1(\partial M'_i) \to H_1(M'_i)$ is induced by the inclusion. Since $n \neq 0$, $i_\#[l] = 0$ so that l is a longitude of $\partial M'_i$. Next, let $I : H_2(M'_i, \partial M'_i) \otimes H_1(M'_i) \to \mathbf{Z}$ be the intersection form and $\mu \in H_1(M'_i)$ be a generator of $H_1(M'_i)$ represented by a meridian loop. Then we can assume $I([F'], \mu) = n$. There is a section $\sigma : S^1 \to M'_i$ of the fibration π'_i, that is a simple transversal loop in M'_i such that $\sigma(S^1) \cap G'$ is one point for any leaf $G' \in \mathscr{F}|_{M'_i}$. $[\sigma] \neq 0$ in $H_i(M'_i)$, since $(\pi'_i)_\#[\sigma] = (\pi'_i)_\# \cdot \sigma_\#[S^1] = [S^1]$ is the fundamental class. We can assume $I([F'], [\sigma]) = 1$. On the other hand $[\sigma] = m\mu \in H_1(M'_i)$, where $m \neq 0$. Then $1 = I([F'], [\sigma]) = m(I([F'], \mu)) = m \cdot n$, therefore $m = n = 1$ so that $\partial F'$ is connected.

It was also proved in the proof of Assertion that a meridian loop can be taken as another generator of $\pi_1(\partial M'_i)$. Specify these longitude and meridian and attach

a solid torus to M_i' by the orientation-reversing diffeomorphism $S^1 \times \partial D^2 \to \partial M_i'$ which takes $S^1 \times \{1\}$ and $\{0\} \times \partial D^2$ respectively to the specified longitude and meridian of $\partial M_i'$. Then the resulting manifold is S^3 and π_i' extends to S^3 except the core of the solid torus by natural projections $S^{-1} \times (D^2 - 0) \to S^1$. This implies Proposition 1.

§4. Examples and realization of a foliation

Let V be a codimension-zero compact submanifold in S^3 such that $\partial V \neq \phi$ and each component of ∂V is a torus. Let (l, m, n) be the index of V, that is, ∂V consists of $(l + m + n)$-tori in S^3 and l-components bound solid tori outside of V, m-components bound solid tori on the side of V and n-components bound solid tori on both sides. In this section, we will construct a foliation on V with boundary components as leaves which has no compact leaves in interior, where V has an index which is not $(0, m_1, 0)$ nor $(0, m_2, 1)$ where $m_1 \geq 2$ and $m_2 \geq 1$. Furthermore we will prove the rest of Theorem 2 using this construction. Also we will prove Corollary to Theorem 2 and Proposition 2.

EXAMPLE A. In the case that $l \cdot m \cdot n = 0$.

(1) In the case that the index is $(l, 0, 0)$, where $l \geq 1$: (a) Since a $(2l, 3l)$-type torus link is spinnable (see [12], [5]), on a $(2l, 3l)$-type torus link exterior V we have a spinnable foliation. (b) Let V' be a trefoil (i.e. $(2, 3)$-type torus knot) exterior and $\mathcal{F}'$ a spinnable foliation on V' by the standard fibration of a trefoil (see [12]). Take loops $a_1, \ldots, a_{l-1}$ transverse to $\mathcal{F}'$ which are isotopic to meridian loops in V' and let $N(a_i)$ be a small tubular neighbourhood of a_i in V'. Let b_i be a torus knot in $\partial N(a_i)$ and $N(b_i)$ a small tubular neighbourhood (see Figure 2).
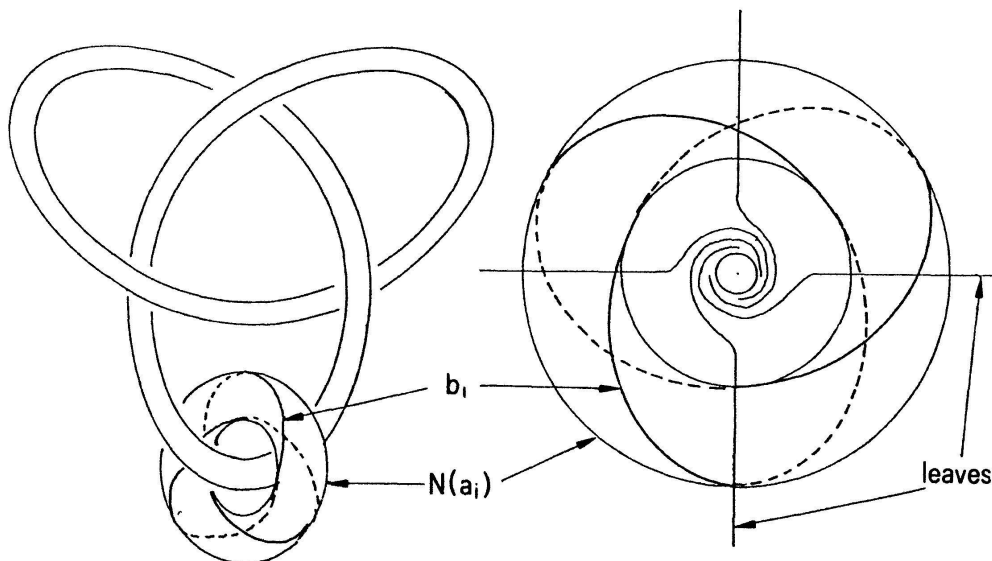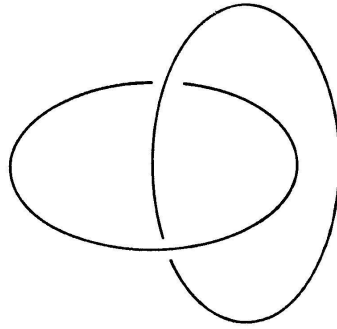

b_i

N(a_i)

leaves

Fig. 2

Fig. 3

Then set $V = V' - \bigcup_{i=1}^{l-1} \mathrm{Int}\, N(b_i)$ and turbulize $\mathscr{F}'$ along b_i's so as to have $\partial N(b_i)$, $i = 1, \ldots, l-1$, as leaves. For the definition of turbulization of a foliation along a transversal loop, we refer to [14] for example. Let $\mathscr{F}$ be the restricted foliation to V. Then $(V, \mathscr{F})$ is a desired foliated submanifold.

(2) In the case that the index is $(0, 0, n)$, where $n \geq 1$: (a) Let V be an $(n, 2n)$-type torus link exterior and $\mathscr{F}$ a foliation on V which is a spinnable foliation with boundary components as leaves. (b) Assume $n \geq 2$. Let V' be an exterior of the link indicated in Figure 3. V' is diffeomorphic to $T^2 \times I$. Let $\mathscr{F}'$ be a foliation on V' by cylinders, that is, for example, let $\mathscr{F}_R$ be a Reeb foliation on an annulus, then $\mathscr{F}' = S^1 \times \mathscr{F}_R = \{S^1 \times L \mid L \in \mathscr{F}_R\}$. Let $a_1, \ldots, a_{n-2}$ be loops defined by $a_i(t) = ((i-1)/(n-2), t, \frac{1}{2}) \in (\mathbf{R}/\mathbf{Z}) \times (\mathbf{R}/\mathbf{Z}) \times I$, and $N(a_i)$ a small tubular neighbourhood for each $i = 1, \ldots, n-2$. Then $V = V' - \bigcup_{i=1}^{n-2} \mathrm{Int}\, N(a_i)$ and turbulize $\mathscr{F}'$ so as to define a foliation on V with boundary components as leaves (see Figure 4).

(3) In the case that the index is $(0, 1, 0)$: In this case V is a knotted solid torus and $\mathscr{F}$ must be a Reeb foliation.

(4) In the case that the index is $(l, 0, n)$, where $l \geq 1$ and $n \geq 1$: Let V' be a standard solid torus in S^3 and $\mathscr{F}'$ a Reeb foliation on V'. Let V' be framed as

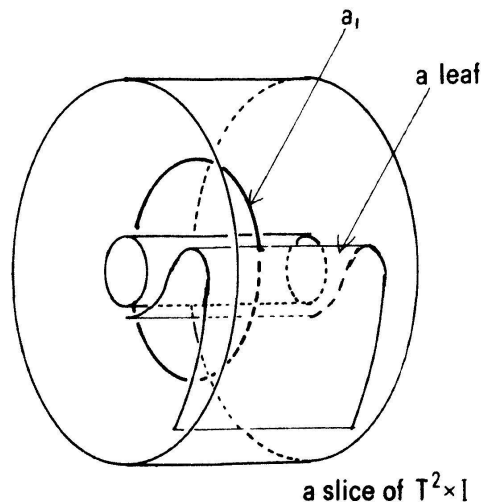

a slice of $T^2 \times I$

Fig. 4

$S^1 \times D^2$ and $a_1, \ldots, a_{n-1}$ loops in V' isotopic to the core of V' which are contained in the sub-solid torus $V'(\frac{1}{3}) \subset V'$, where $V'(\frac{1}{3}) = S^1 \times D^2(\frac{1}{3})$ with respect to the framing and $D^2(\frac{1}{3}) = \{x \in D^2 \mid |x| \leq \frac{1}{3}\}$. Let τ be a $(2l, 3l)$-type torus link in $\partial V'(\frac{2}{3})$ where $V'(\frac{2}{3}) = S^1 \times D^2(\frac{2}{3})$. Note that we can assume a_i and τ to be transverse to $\mathscr{F}'$. Remove small open tubular neighbourhoods of a_i's and τ, and turbulize $\mathscr{F}'$ so that we have a desired foliation.

(5)* In the case that the index is $(0, m, n)$, where $m \geq 1$ and $n \geq 2$: Let $T^3 = \mathbf{R}^3/\mathbf{Z}^3$ be a 3-torus. Let T_x, T_y and T_z be tubular neighbourhoods of x-axis, y-axis and z-axis in general position, respectively. It is known ([4], [13]) that $T^3 - \text{Int}(T_x \cup T_y \cup T_z)$ is diffeomorphic to an exterior of Borromean rings (see Figure 5). Furthermore this diffeomorphism takes the curves α_x, α_y and α_z (as in
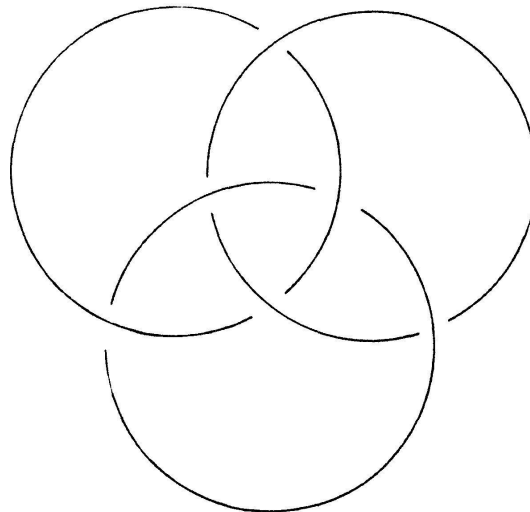

Fig. 5

Figure 6) to the longitudes of the rings, and takes a circle that is parallel to an axis and lie on one of the tubes to a meridian of one of the rings. On the other hand, $T^3 - \text{Int } T_z$ is an S^1-bundle over punctured torus. By Proposition 7.1 of [18], for an irrational rotation ρ, there exist orientation-preserving diffeomorphisms f and g of S^1 (in fact they are in $SL(2, \mathbf{R})$ acting on S^1) such that $\rho = f \cdot g \cdot f^{-1} \cdot g^{-1}$. Consider the foliated S^1-bundle over punctured torus which is determined by the total holonomy diffeomorphisms f and g. Then we have a foliation on $T^3 - \text{Int } T_z$ which is transverse to x-axis, y-axis and the boundary ∂T_z and induces on ∂T_z the irrational linear foliation by ρ. We have transversal loops $a_1, \ldots, a_{m-1}$ which are isotopic to α_z in $T^3 - \text{Int}(T_x \cup T_y \cup T_z)$ (see Figure 7) since the foliation induces an irrational linear foliation on ∂T_z. Furthermore loops $b_1, \ldots, b_{n-2}$ which are parallel to x-axis (Figure 7) are also transverse to the foliation.
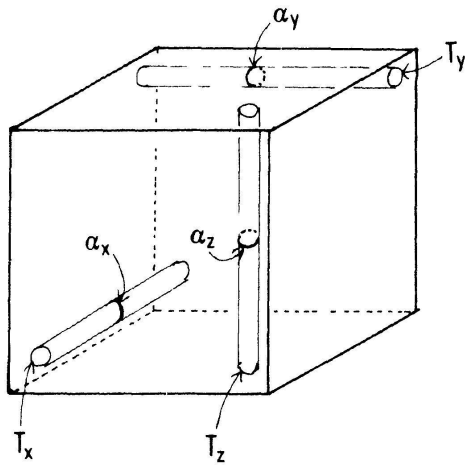
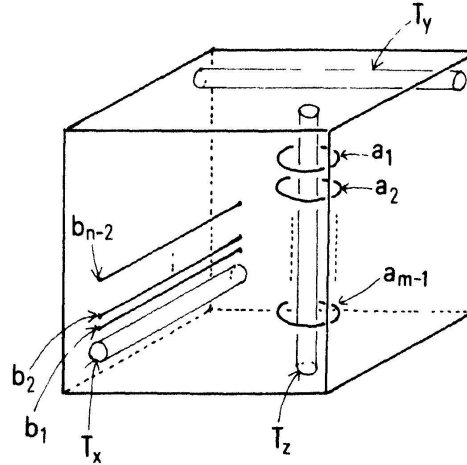Fig. 6   Fig. 7

Let $N(a_i)$ and $N(b_j)$ be small tubular neighbourhoods and let $M = T^3 - \text{Int}\,(T_x \cup T_y \cup T_z \cup (\bigcup_{i=1}^{m-1} N(a_i)) \cup (\bigcup_{j=1}^{n-2} N(b_j)))$. M is diffeomorphic to an exterior of the link indicated in Figure 8. By turbulizing the foliation along x-axis, y-axis, a_i's and b_j's, we can make $\partial T_x, \partial T_y, \partial N(a_i)$ and $\partial N(b_j)$, where $i = 1, \ldots, m-1$ and $j = 1, \ldots, n-2$, leaves of the new foliation. One can wind the irrational linear foliation on ∂T_z in order to make ∂T_z a leaf. Finally we have a foliation $\mathscr{F}$ on M which has the boundary components as leaves and whose interior leaves are all non-compact. We have a foliated submanifold which has the index $(0, m, n)$ by embedding $(M, \mathscr{F})$ into S^3 as follows: Let $L = \bigcup_{i=1}^{m+n} K_i$ be the link indicated in Figure 8. Recall that M is an exterior of L. Note that each
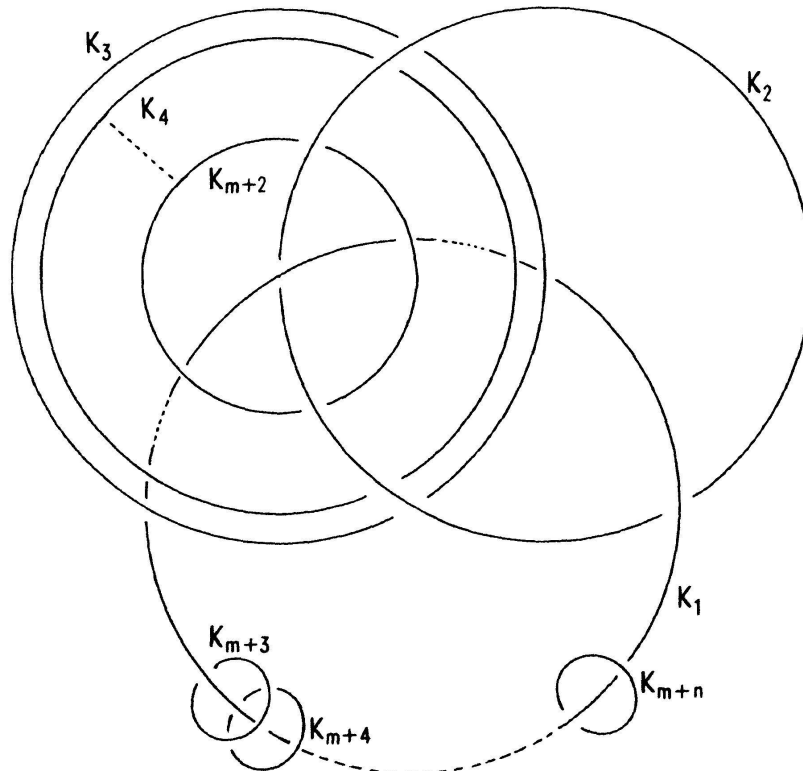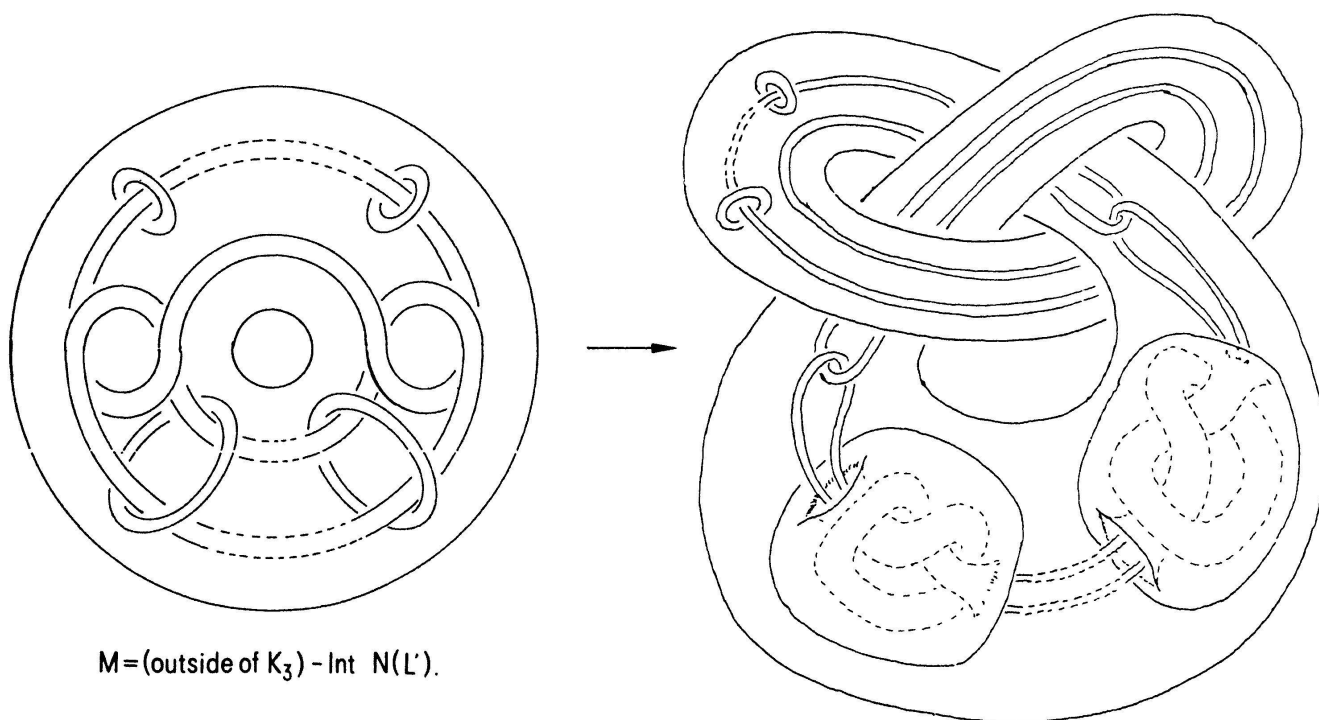

Fig. 8

M = (outside of K_3) - Int N(L').

Fig. 9

component of L is a trivial knot in S^3. Consider M as an exterior of the link L' $= L - K_3$ in a solid torus which is an exterior of the component K_3 (see Figures 8 and 9). Now we construct an embedding. Let U_i be a slice of M which contains $\partial N(K_{3+i})$ and a part of $\partial N(K_2)$ and $\partial N(K_3)$, and does not contain any other boundary components (see Figure 10). We denote by P a two-punctured disc and by A an annulus. Then U_i is diffeomorphic to $(P \times I) \bigcup_\alpha (A \times I)$ where $\alpha : \partial A \times I \to \partial P \times I$ is an embedding which takes $\partial A \times \{\frac{1}{2}\}$ to $\partial_1 P \times \{\frac{1}{3}, \frac{2}{3}\}$, where $\partial_1 P$ is a component of ∂P (see Figure 11). Note that the meridian of the solid torus is the longitude of K_3. First we embed $P \times I$ so as to make a local knotted $P \times I$ just between two components of $\alpha(\partial A \times I)$. Then we attach $A \times I$ so as to cover the knotted part (see Figure 12). Now we have an embedding $U_i \to S^3$. We join these embeddings for $1 \le i \le m - 1$ and embed the other slice of M such that the boundary torus of the solid torus, that is, $\partial N(K_3)$ is embedded to be the boundary of a knotted solid torus in S^3 (see Figure 9). Then we have a desired embedding $M \to S^3$.

(6) In the case that the index is $(l, m, 0)$, where $l \ge 1$ and $m \ge 1$: We will construct an example of this type in Example B.

EXAMPLE B. In the case that $l \cdot m \cdot n \ne 0$ and that $l \cdot m \ne 0$ and $n = 0$ ((6) of Example A).

We apply the technique in Example A (2) (b) and A (1) (b). We also apply a similar construction of an embedding in Example A (5). Let $(M', \mathcal{F}')$ be the foliated manifold $(V', \mathcal{F}')$ in Example A (2) (b). Precisely, $M' = (\mathbf{R}/\mathbf{Z}) \times (\mathbf{R}/\mathbf{Z}) \times I$
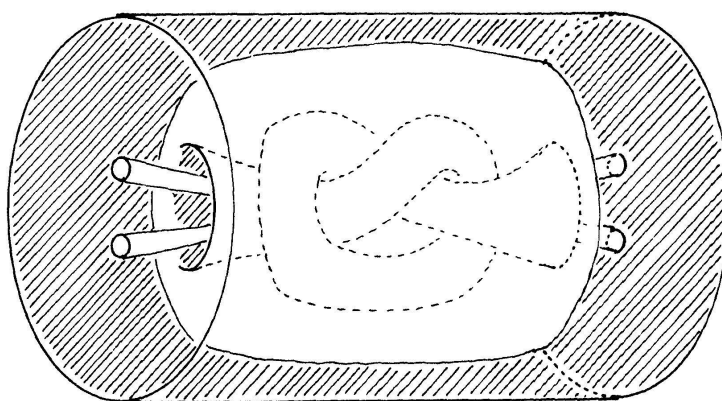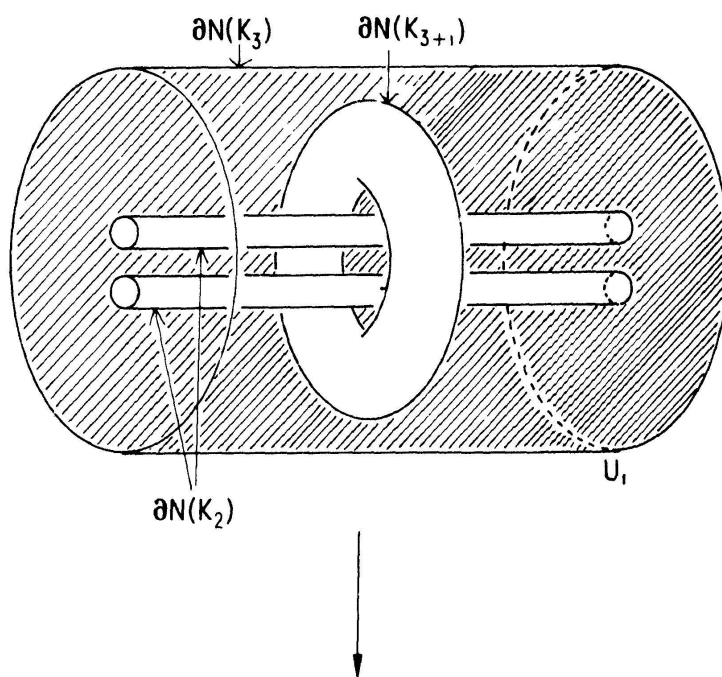
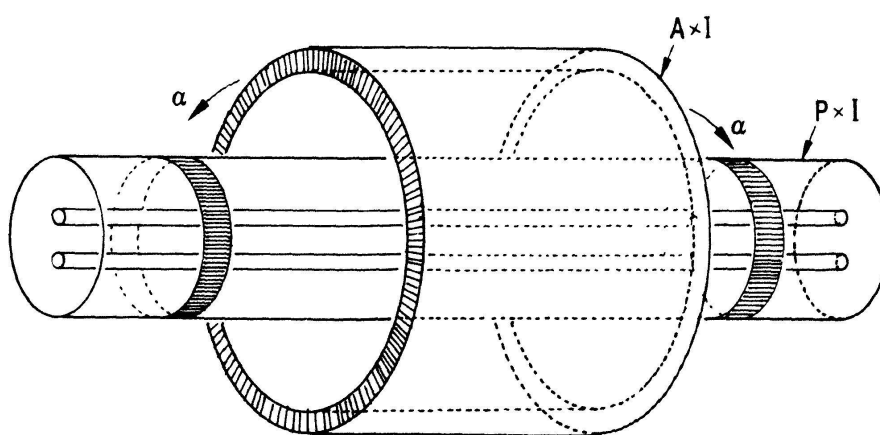∂N(K₃) ∂N(K₃₊₁)

∂N(K₂)

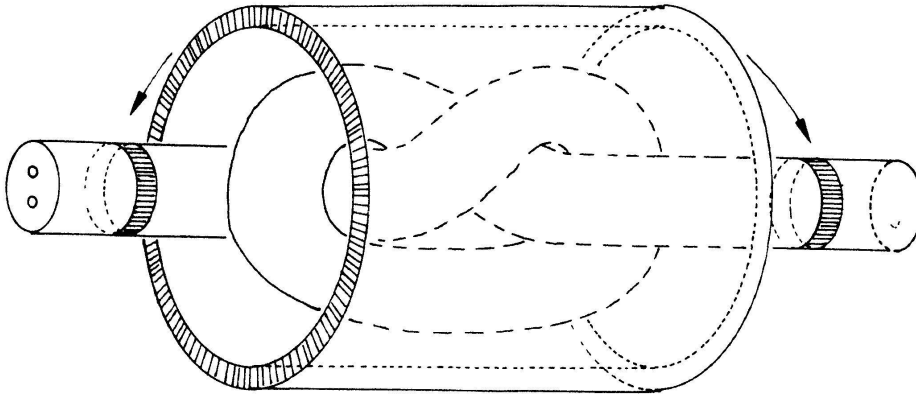U₁

Fig. 10

A×I

α α

P×I

Fig. 11

Fig. 12

and $\mathscr{F}' = S^1 \times \mathscr{F}_R = \{S^1 \times L \mid L \in \mathscr{F}_R\}$ where $\mathscr{F}_R$ is a Reeb foliation on an annulus. Let $b_1, \ldots, b_{l+m+n-2}$ be loops defined by $b_i(t) = ((i-1)/(l+m+n-2), t, \frac{1}{2}) \in (\mathbf{R}/\mathbf{Z}) \times (\mathbf{R}/\mathbf{Z}) \times I$ and $N(b_1), \ldots, N(b_{l+m+n-2})$ small tubular neighbourhoods. Let $c_1, \ldots, c_{l-1}$ be torus knots in $\partial N(b_{m+n}), \ldots, \partial N(b_{l+m+n-2})$ respectively and $N(c_1), \ldots, N(c_{l-1})$ small tubular neighbourhoods. Set $M = M' - \text{Int}$ $((\bigcup_{i=1}^{m+n-1} N(b_i)) \cup (\bigcup_{i=1}^{l-1} N(c_i)))$ and $\mathscr{F}$ a foliation which is obtained by turbulizing $\mathscr{F}'$ along $b_1, \ldots, b_{m+n-1}$ and $c_1, \ldots, c_{l-1}$, and has $\partial N(b_1), \ldots, \partial N(b_{m+n-1})$ and $\partial N(c_1), \ldots, \partial N(c_{l-1})$ as leaves. Then we may embed $(M, \mathscr{F})$ into S^3 whose image is a desired foliated submanifold. The construction of an embedding is essentially the same as Example A (5). Let $U_i = ([2i-3)/2(l+m+n-2), (2i-1)/2(l+m+n-2)] \times (\mathbf{R}/\mathbf{Z}) \times I) \cap M$. Then, for $n + 1 \leq i \leq m + n - 1$, embed U_i into S^3 as the same way in Example A (5) except that we replace the two-punctured disc with a punctured disc. For the other U_i's, embed as it is. We join these embedding so that the torus which is the image of a connected component of $\partial M' \subset \partial M$ is the boundary of a knotted solid torus (see Figure 13). In the above construction, set $n = 0$, then we have an example of the type $(l, m, 0)$, where $l \geq 1$ and $m \geq 1$.

First we prove Corollary to Theorem 2.

Proof of Corollary to Theorem 2. Let $(M_1, \mathscr{F}_1)$ and $(M_2, \mathscr{F}_2)$ be Reeb knot exteriors. We can assume that each boundary leaf has the flat holonomy by adjusting the given holonomy if necessary. In Example B, set $l = 1, m = 2$ and $n = 0$. That is, let $X = T^2 \times I - \text{Int } N(\lambda)$, where $\lambda : [0, 1] \to (\mathbf{R}^2/\mathbf{Z}^2) \times I$ is defined by $\lambda(t) = (0, t, \frac{1}{2})$ and $N(\lambda)$ is a small tubular neighbourhood of λ, and let $\mathscr{F}$ be a foliation on X constructed from $(T^2 \times I, \mathscr{F}')$ in Example A (2) (b) by turbulizing along λ. Note that, in Example B, each boundary component corresponding to the second component of the index can be embedded to be the boundary of any knotted solid torus in S^3. Then there exists an embedding of X into S^3 which patches M_1 and M_2 together so as to construct S^3 by adding one Reeb component (compare Figure 13). We have a foliation on S^3 which consists of $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}$ and a
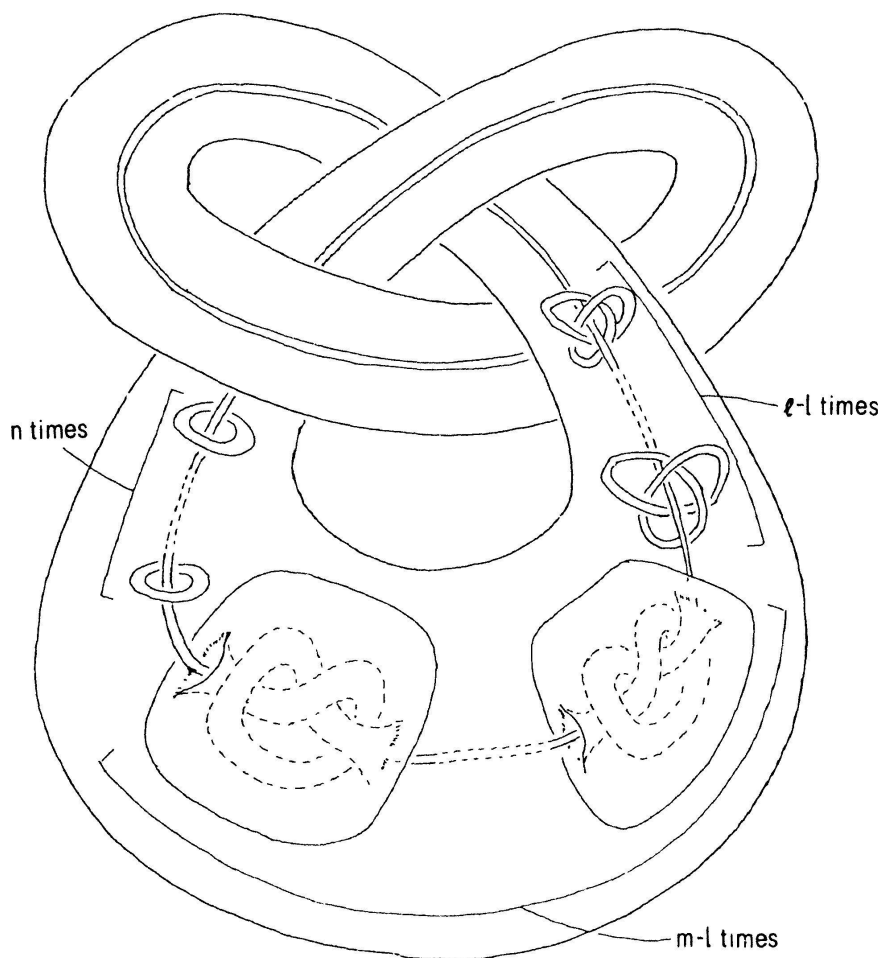
Fig. 13

Reeb component so that the resulting Reeb knot is the composite knot of the given two Reeb knots.

Proof of Theorem 2. *Part* 2. Let Γ be a graph satisfying conditions (i)–(vi). We can assume Γ has no black vertex. For, we can replace a compact leaf with a foliated I-product over a torus at will. Then there exists a vertex whose index is $(l, 0, n)$. In fact, since Γ is a directed finite tree, there must exist a "source" in Γ. Let $v_1^0, \ldots, v_{s_0}^0$ be all such vertices. We can construct foliated submanifolds of S^3 corresponding to $v_1^0, \ldots, v_{s_0}^0$. Precisely, we can construct foliated manifolds $(M_j^0, \mathscr{F}_j^0)$ and embeddings $f_j^0 : M_j^0 \to S^3$ such that each $f_j^0(M_j^0)$ has the index of v_j^0 by Example A (1) (a), (2) (a) and (4). Note that all knotted boundary-tori in $f_j^0(M_j^0)$'s are boundaries of tubular neighbourhoods of trefoils. If v_i^0 is connected to v_j^0 by one edge, then they are connected to each other by an edge with both direction. This means the boundary components of $f_i^0(M_i^0)$ and $f_j^0(M_j^0)$ which are corresponding to the edge are unknotted. Therefore we can glue $f_i^0(M_i^0)$ and $f_j^0(M_j^0)$ on the boundary components by deforming f_i^0 and f_j^0 by isotopy. Next, let $v_1^1, \ldots, v_{s_1}^1$ be vertices which are connected to v_j^0's by one edge. Then the second components of the indices of these vertices are positive and the vertices are

connected to v_j^0's by the edges corresponding to the second components of the indices. Let $(M_j^1, \mathcal{F}_j^1)$ and $f_j^1 : M_j^1 \to S^3$ be a foliated manifold and an embedding respectively corresponding to v_j^1 as Example A (3), (5) and Example B ($1 \leq j \leq s_1$). In this stage, we can construct the embedding f_j^1 so that all boundary components of $f_j^1(M_j^1)$ which are corresponding to edges with only one-sided directions are boundaries of tubular neighbourhoods of trefoils in S^3 (compare Example A and B). Therefore, by isotopy, we can glue $f_j^1(M_j^1)$ to some of $f_i^0(M_i^0)$'s according to Γ. By the same reason, there is no problem for gluing at the vertex which is a "confluence", that is, the second component of the index of the vertex is greater than one. This construction works well inductively. Consequently we have a foliation on S^3 whose graph is the given one. This completes the proof of Theorem 2.

In order to prove Proposition 2, we need the following (see [12], Chapter 10-H, Corollary 9):

LEMMA 4.1. *Let K be a spinnable knot in S^3. Then the Alexander polynomial of K, $\Delta(K: t)$ is monic, that is, the leading coefficient of $\Delta(K; t)$ is ± 1.*

Proof of Proposition 2. Let K_0 be any knot in S^3 and let K be a doubled knot with the companion K_0 with twisting number n. Precise construction of K is as follows: Let $L = K_1 \cup K_2$ be Whitehead's link (see Figure 14) and $N(K_1)$ a tubular neighbourhood of K_1 in S^3. Then $S^3 - \text{Int } N(K_1)$ is a solid torus with a knot K_2. Let $h_n : S^3 - \text{Int } N(K_1) \to N(K_0)$ be the diffeomorphism such that $h_n(\text{longitude}) = \text{longitude} + n(\text{meridian})$ and set $K = h_n(K_2)$. K is also called a twisted double of K_0 with a certain twisting number. For the details, we refer to [12]. Then, as is well known fact, $\Delta(K; t) = n \cdot t^2 - (2n \pm 1) \cdot t + n$, where $\Delta(K; t)$ is the Alexander polynomial of K (see [12], [17]). On the other hand, Whitehead's link is spinnable (see [12]). Therefore a twisted double of any Reeb knot is a Reeb knot. Set the twisting number $n \neq \pm 1$, then we have Proposition 2.
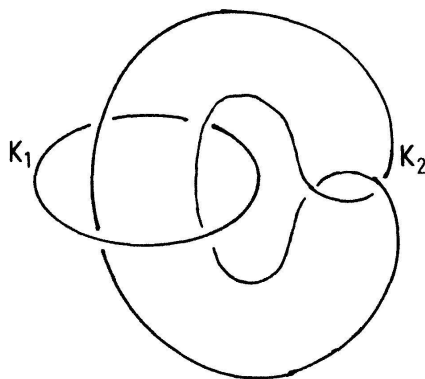

Fig. 14

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