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A $1\frac{1}{2}$-dimensional version of Hopf's Theorem on the number of ends of a group

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1. Introduction

If G is a finitely generated group then the first cohomology group with group ring coefficients $H^1(G; \mathbb{Z}G)$ is known to be free-Abelian. H. Hopf [7] has shown that its $\mathbb{Z}$-rank, rk $H^1(G; \mathbb{Z}G)$, attains only the values $0, 1$ or ∞, and the celebrated structure theorem of Hopf–Stallings [7], [12], classifies these three cases in terms of the group theoretic structure of G.

Of course the cohomology group $H^1(G; \mathbb{Z}G)$ carries much more information than just its Abelian group structure. As the coefficient module $\mathbb{Z}G$ is a bi-module $H^1(G; \mathbb{Z}G)$ inherits the structure of a (right) G-module; and by functoriality one can consider the restriction maps

$$\text{res: } H^1(G; \mathbb{Z}G) \to \prod^{m} H^1(S_i; \mathbb{Z}G) \tag{1.1}$$

where $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ is a finite family of finitely generated subgroups of G. The relative versions of Stalling's structure theorem by Swan [13] and Swarup [14] show that the kernel K of (1.1) is free-Abelian of rank $0, 1$ or ∞, and classify these three cases in terms of the structure of the pair $(G, \mathcal{S})$.

In this paper we consider the cokernel $C(G, \mathcal{S})$ of the restriction map (1.1), under the assumption that G is *accessible*. (For a discussion of accessibility refer to [4], but we recall that every finitely generated torsion-free group is accessible by Gruško's Theorem and that it is unknown whether finitely generated non-accessible groups exist). We observe that Heinz Müller's result [9] on the freeness of the cokernel of the restriction map carries readily over to the case of a finite family of subgroups, so that *$C(G, \mathcal{S})$ is always free-Abelian in our situation.* Our main result asserts that *the rank m of $C(G, \mathcal{S})$ is equal to* $0, 1$ or ∞ *except in the very special situation when G contains an infinite cyclic subgroup of finite index, in which case m can attain every value* $0 \le m < \infty$. Then we *classify the three cases $m = 0, 1, \infty$ in terms of the structure of* $(G, \mathcal{S})$. The fact that, in view of the long exact cohomology sequence for the pair $(G, \mathcal{S})$, the cokernel of (1.1) "lies between $H^1(G; \mathbb{Z}G)$ and $H^2(G; \mathbb{Z}G)$" justifies our title.

2. The results

2.1. Our main result is

THEOREM A. *Let G be a finitely generated accessible group and $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ a finite non-empty family of finitely generated infinite subgroups of G, and let* $\operatorname{rk} C(G, \mathcal{S})$ *denote the rank of the (free-Abelian) cokernel of the restriction map* (1.1). *If G contains an infinite cyclic subgroup of finite index then*

$$\operatorname{rk} C(G, \mathcal{S}) = \sum_{i=1}^{m} |G : S_i| - 1;$$

otherwise $\operatorname{rk} C(G, \mathcal{S})$ *is equal to* 0 *or* 1, *or* ∞.

Note that finite groups in the family $\mathcal{S}$ have no influence whatsoever on the cokernel of (1.1) and so we lose no generality by assuming that all groups in $\mathcal{S}$ are infinite.

Next we classify the three cases $\operatorname{rk} C(G, \mathcal{S}) = 0, 1, \infty$ by exhibiting necessary and sufficient conditions for $\operatorname{rk} C(G, \mathcal{S})$ to be 0 or 1, respectively, The case $\operatorname{rk} C(G, \mathcal{S}) = 0$ is then, of course, given by exclusion.

2.2. $\operatorname{rk} C(G, \mathcal{S}) = 1$. In order to state the result when $C(G, \mathcal{S})$ is infinite cyclic we introduce the following notation. Let $(G, \mathcal{S})$ be a pair consisting of a group G and a family $\mathcal{S} = \{S_i \mid i \in I\}$ of subgroups (possibly with repetitions!), and let $F \leq G$ be an auxiliary subgroup. For each index $i \in I$ we choose a system X_i of double coset representatives of $F \backslash G / S_i$ and consider the family

$$\mathcal{S}' = \{F \cap x_i S_i x_i^{-1} \mid x_i \in X_i, i \in I\}.$$

Up to cojugacy within F, $\mathcal{S}'$ is independent of the choice of X_i, $i \in I$. We call $(F, \mathcal{S}')$ the *full subpair of* $(G, \mathcal{S})$ *given by* $F \leq G$.

We define the group pair $(G, \mathcal{S})$ to be a *virtual Poincaré duality pair* if G contains a subgroup of finite index $F \leq G$ such that the full subpair of $(G, \mathcal{S})$ given by F is a Poincaré duality pair in the sense of [2]. Note that F is necessarily torsion-free and that the definition of a virtual Poincaré duality pair is independant of the patricular choice of F by [2], Theorem 7.6.

THEOREM B.[(1)] *Let $(G, \mathcal{S})$ be as in Theorem A. Then* $\operatorname{rk} C(G, \mathcal{S}) = 1$ *if and only if $(G, \mathcal{S})$ is a virtual Poincaré duality pair of dimension* 2.

Thus in view of [2] Theorem 9.3 we have $\operatorname{rk} C(G, \mathcal{S}) = 1$ if and only if G

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[1] Eckmann and Müller have recently obtained a different proof of Theorem B and a direct description of all virtual Poincaré duality pairs of dimension 2. See "Plane motion groups and virtual Poincaré duality of dimension 2". Preprint, Forschungsinstitut für Mathematik 1981, ETH, Zürich.

contains a free subgroup of finite index, each S_i contains an infinite cyclic subgroup of finite index, and the relative cohomology group $H^2(G, \mathcal{S}; \mathbb{Z}G)$ is $\cong \mathbb{Z}$.

It was shown by Eckmann and Müller [5] that the 2-dimensional Poincaré duality pairs are geometric, that is, given by the fundamental group and the peripheral subgroup system of a compact surface-with-boundary. This yields the

COROLLARY.[(1)] *Let* $(G, \mathcal{S})$ *be as in Theorem A and assume G is torsion-free. Then* $\operatorname{rk} C(G, S) = 1$ *if and only if G is a free group having a basis* $\{t_1, t_2, \ldots, t_{m-1}, x_1, \ldots, x_n\}$, *such that the subgroups* $S_i \in \mathcal{S}$ *are conjugate to the infinite cyclic subgroups* $gp(t_1), \ldots, gp(t_{m-1}), gp(t_1 \cdots t_{m-1} r)$, *where*

$$r = [x_1, x_2][x_3, x_4] \cdots [x_{n-1}, x_n], \; n \text{ even} \geq 0$$

if $C(G, \mathcal{S})$ *has trivial G-action, and*

$$r = x_1^2 x_2^2 \cdots x_n^2, \qquad n \geq 0$$

2.3. $\operatorname{rk} C(G, \mathcal{S}) = 0$. In order to exhibit the structure of $(G, \mathcal{S})$ when the restriction map (1.1) is surjective we have to consider simultanous decompositions of G and the subgroups S_i as fundamental groups of graphs of groups. In order to handle the family $\mathcal{S}$ it is convenient to consider graphs of groups $(\mathfrak{G}, X)$ where the underlying graph X is not necessarily connected and define its "fundamental group" $\pi_1(\mathfrak{G}, X)$ to be the family of fundamental groups of the connected components.

In more detail: Let $X(i)$, $i \in I$, denote the connected components of the (oriented) graph X, with vertices $V(X(i))$ and (positive) edges $E(X(i))$, and let $\mathfrak{G}(i)$ be the corresponding system of vertex groups G_v, $v \in V(X(i))$ and edge groups $G_e \leq G_{o(e)}$, $G_{\bar{e}} \leq G_{t(e)}$, $e \in E(X(i))$. Then $\pi_1(\mathfrak{G}, X)$ stands for the family of groups $G(i) = \pi_1(\mathfrak{G}(i), X(i))$, $i \in I$. Recall that $G(i)$ is generated by the vertex groups G_v, $v \in V(X(i))$ and stable letters p_e, $e \in E(X(i))$, subject to the following defining relations.

$$p_e^{-1} G_e p_e = G_{\bar{e}}, \qquad e \in E(X(i))$$

$p_e = 1$ for all edges e in a maximal tree of $X(i)$.

So let $G = \pi_1(\mathfrak{G}, X)$, with X connected, and $\mathcal{S} = \pi_1(\mathfrak{S}, Y)$ with Y arbitrary, and let $V(X)$, $V(Y)$ be the set of vertices and $E(X)$, $E(Y)$ the set of (positive) edges of X resp. Y.

DEFINITION. We say that the decompositions of G and $\mathcal{S}$ are compatible (via an orientation preserving graph map $f: Y \to X$) if there are elements $c_v \in G$,

$v \in V(X)$, such that the following holds

$$c_v^{-1} S_v c_v \leq G_{f(v)} \quad \text{for every vertex} \quad v \in V(Y) \tag{2.1}$$

$$c_{o(e)} p_{f(e)} = p_e c_{t(e)} \quad \text{for every edge} \quad e \in E(Y), \tag{2.2}$$

where p_e and $p_{f(e)}$ stand for the stable letters corresponding to the (positive) edges e resp. $f(e)$.

Note that if G and $\mathcal{S}$ have compatible decompositions via f, $G = \pi_1(\mathfrak{G}, X)$, $\mathcal{S} = \pi_1(\mathfrak{S}, Y)$, then the same holds for any family $\mathcal{S}' = \{S_i' \mid i \in I\}$ with $S_i' = g_i^{-1} S_i g_i$, $g_i \in G$. Indeed, let $(\mathfrak{S}_i, Y_i)$, $i \in I$, be the connected components of $(\mathfrak{S}, Y)$. Then conjugating each $\mathcal{S}_i = \pi_1(\mathfrak{S}_i, Y_i) \leq G$ by g_i yields a decomposition $\mathcal{S}' = \pi_1(\mathfrak{S}', Y)$ satisfying (2.1), and (2.2), where for each $v \in V(Y_i)$ and $e \in E(Y_i)$ c_v is to be replaced by $g_i^{-1} c_v$ and p_e by $g_i^{-1} p_e g_i$.

Now we are in a position to state

THEOREM C. *Let $(G, \mathcal{S})$ be as in Theorem A. Then $C(G, \mathcal{S}) = 0$ if and only if G and $\mathcal{S}$ have compatible decompositions $G = \pi_1(\mathfrak{G}, X)$, $\mathcal{S} = \pi_1(\mathfrak{S}, Y)$ given by a graph map $f: Y \to X$ which is bejective on the edges, such that the following holds:*

(i) *all edge groups of G are finite and coincide with the corresponding (conjugate) edge groups of $\mathcal{S}$*

(ii) *all vertex groups of $\mathcal{S}$ have ≤ 1 end.*

As a special case Theorem 3 contains a splitting result which is related to those of Swan [13], Lemma 7.1, and Wall [15].

COROLLARY. *Let G be a torsion-free finitely generated group and $\mathcal{S}$ a finite family of finitely generated free subgroups of G. Then $C(G, \mathcal{S}) = 0$ if and only if G is the free product $G = S_1' * \cdots * S_m' * K$ where $S_i' \leq G$ is a subgroup conjugate to S_i $1 \leq i \leq m$, and $K \leq G$ is an auxiliary subgroup.*

Proof. If res is surjective G and $\mathcal{S}$ have decompositions $G = \pi_1(\mathfrak{G}, X)$, $\mathcal{S} = \pi_1(\mathfrak{S}, Y)$ satisfying the properties (i), (ii) of Proposition 7.2. Hence all edge groups are trivial and all vertex groups S_v of $\mathcal{S}$ have ≤ 1 end. Since S_v is free this means that $S_v = 1$, and $\mathcal{S} = \pi_1(\mathfrak{S}, Y)$ is the family of fundamental groups (in the topological sense) of the connected components Y_i of Y. Since $X = f(Y_i)$ the fundamental group of X is free product of $\pi_1(f(Y_i))$ and an auxiliary group K_1, and clearly $G \cong \pi_1(X) * K_2$ where K_2 is the tree product along a maximal tree of X. Finally $\pi_1(f(Y_i)) \cong \pi_1(Y_i) * K_{3i}$ because f identifies certain vertices; note that one has to choose base points and use conjugation to adapt the elements $c_v \in G$ so that the last isomorphism involves conjugation.

3. Two preliminary lemmas

3.1. Let G be a group and K a commutative ring with nontrivial unity. Recall that a KG-module M is said to be of type $(FP)_n$, where n is an integer ≥ 0 or $n = \infty$, if M has a projective resolution which is finitely generated in all dimensions $\leq n$. If M is of type $(FP)_\infty$ and of finite projective dimension then M is said to be of type (FP). If the trivial G-module K is of type $(FP)_n$ (resp. of type (FP)) then we say that the group G is of type $(FP)_n$ over K (resp. of type (FP) over K).

LEMMA 3.1 (Stallings [12]). *Let K be a field and assume that G has no K-torsion. Let V be a non-trivial KG-module of finite K-dimension. then we have*

(a) *The KG-module V is of type $(FP)_n$ if and only if the group G is of type $(FP)_n$ over K.*

(b) *The projective dimension of the KG-module V is equal to the chomology dimension* $\mathrm{cd}_K G$ *of G over K.*

Proof. Let $\mathbf{P} \to K$ be a projective resolution of the KG-module K. Then $\mathbf{P} \otimes_K V$ is a projective resolution of V. And if $\mathbf{P}$ is finitely generated (resp. of finite length) so is $\mathbf{P} \otimes_K V$.

Conversely: Assume first that V is of type $(FP)_n$. By induction one may assume that $P_0, P_1, \ldots, P_{n-1}$ are finitely generated, hence so are $P_i \otimes_K V$, $i = 1, 2, \ldots, n-1$.

Let $R = \ker(P_{n-1} \to P_{n-2})$. Since V is of type $(FP)_n$, $R \otimes_K V$ is finitely generated over KG; hence so is R, and therefore G is of type $(FP)_n$ over K.

Now assume V is of projective dimension $\leq n$. Then $R \otimes_K V$ is a projective KG-module. Let F be a free KG-module and $f: F \to R$ an epimorphism. There is a KG-homomorphism $g: R \otimes_K V \to F \otimes_K V$ which splits $f \otimes 1$. Stallings defines to such a map g the "transfer trace" $g_V^*: R \to V$ as follows: for every $r \in R$ and a fixed basis $\{v_1, v_2, \ldots, v_n\}$ of V one has

$$g(r \otimes v_i) = \sum_{i=1}^{n} g_{ij}(r) \otimes v_j$$

and we can put

$$g_V^*(r) = \sum_{i=1}^{n} g_{ii}(r)$$

It is easy to check that $g_V^*: R \to F$ is a KG-homomorphism which does not depend upon the choice of the basis $\{v_1, v_2, \ldots, v_n\}$, and that the composite map $f \cdot g_V^*: F \to R$ is multiplication by $n = \dim_K V$. Since G has no K-torsion $\frac{1}{n} g_V^*: R \to F$ splits f, and R is projective.

3.2. There is an immediate Corollary which improves Lemma 3.2(b) provided the cohomology dimension $\mathrm{cd}_K G$ is known to be finite.

COROLLARY 3.2. *Let K be a field, G a group of finite cohomology dimension over K, and M a KG-module containing a non-trivial submodule $V \le M$ of finite K-dimension. Then $\mathrm{cd}_K G$ is equal to the projective dimension of M.*

Proof. Let A be a KG-module such that $\mathrm{Ext}^m_{KG}(V, A) \neq 0$, where $m = \mathrm{cd}_K G$. Since the projective dimension of any KG-module is $\le m$ we obtain from the long exact Ext-sequence

$$\mathrm{Ext}^m_{KG}(M, A) \to \mathrm{Ext}^m_{KG}(V, A) \to \underbrace{\mathrm{Ext}^{m+1}_{KG}(M/V, A)}_{=0}$$

that $\mathrm{Ext}^m_{KG}(M, A) \neq 0$. Hence the projective dimension of M is $\ge m$ and hence $= m = \mathrm{cd}_K G$.

4. Resolutions of end groups by permutation modules

4.1. Let G be an infinite finitely generated accessible group and $\mathcal{S} = \{S_i \mid i \in I\}$ a finite family of finitely generated subgroups of G. In this section we deduce a finite resolution of the relative cohomology group $H^1(G, \mathcal{S}; \mathbb{Z}G)$ regarded as a right G-module. For definitions and notation concerning the cohomology of a pair $(G, \mathcal{S})$ we refer to [2]. Thus we consider the short exact sequence

$$\Delta_{G/\mathcal{S}} \rightarrowtail \mathbb{Z}G/\mathcal{S} \xrightarrow{\varepsilon} \mathbb{Z} \tag{4.1}$$

where $\mathbb{Z}(G/\mathcal{S})$ is an abbreviation for the direct sum of all permutation modules $\mathbb{Z}G/S_i$, $i \in I$, and ε is the obvious augmentation. Then

$$H^k(G, \mathcal{S}; ZG) = \begin{cases} H^k(G; \mathbb{Z}G), & \text{if } \mathcal{S} = \varnothing \\ \mathrm{Ext}^{k-1}_G(\Delta_{G/\mathcal{S}}, \mathbb{Z}G) & \text{if } \mathcal{S} \neq \varnothing \end{cases}$$

Note that $H^0(G, \mathcal{S}; A) = 0$ for $\mathcal{S} \neq \varnothing$; and replacing the subgroups $S_i \in \mathcal{S}$ by conjugates leads to an isomorphic relative group. Finally, we shall use the abbreviation $H^n(\mathcal{S};\mathbb{Z}G)$ for the direct product of the groups $H^n(S_i;\mathbb{Z}G)$, $i\in I$.

4.2. Let I_{fin} (resp. I_{inf}) denote the set of all $i\in I$ with S_i finite (resp. infinite), and put

$$\mathcal{S}_{\text{fin}}=\{S_i \mid i\in I_{\text{fin}}\},\qquad \mathcal{S}_{\text{inf}}=\{S_i \mid i\in I_{\text{inf}}\}.$$

From $\mathbb{Z}G/\mathcal{S}=\mathbb{Z}G/\mathcal{S}_{\text{fin}}\oplus\mathbb{Z}G/\mathcal{S}_{\text{inf}}$ one easily obtains a short exact sequence of left G-modules.

$$\mathbb{Z}G/\mathcal{S}_{\text{fin}} \rightarrowtail \Delta_{G/\mathcal{S}} \twoheadrightarrow \Delta_{G/\mathcal{S}_{\text{inf}}}$$

and the corresponding Ext-Sequence yields the short exact sequence of right G-modules.

$$0\to H^0(\mathcal{S}_{\text{fin}};\mathbb{Z}G)\to H^1(G,\mathcal{S};\mathbb{Z}G)\to H^1(G,\mathcal{S}_{\text{inf}};\mathbb{Z}G)\to 0. \tag{4.2}$$

Now, $H^0(\mathcal{S}_{\text{fin}};\mathbb{Z}G)$ is the direct product of the (right) permutation modules $\mathbb{Z}(S_i\backslash G)$, $i\in I_{\text{fin}}$.

4.3. It remains to consider the cohomology group $H^1(G,\mathcal{S}_{\text{inf}};\mathbb{Z}G)$, which – by the long exact sequence for the pair $(G,\mathcal{S}_{\text{inf}})$ – is isomorphic to the kernel of the restriction map $H^1(G;\mathbb{Z}G)\to H^1(\mathcal{S}_{\text{inf}};\mathbb{Z}G)$. If the kernel is $=0$ then, by Swarup's relative version of Stalling's Structure Theorem [14] one can replace the groups in $\mathcal{S}_{\text{inf}}$ by suitable conjugates in such a way that G can be written as the fundamental group of a graph of groups $(\mathfrak{G},X)$ with finite edge groups and with every group of $\mathcal{S}_{\text{inf}}$ contained in one of the vertex groups. Let V be the set of vertices and E the set of positive edges of X. $\mathcal{S}_{\text{inf}}$ can be written as a disjoint union of families $\mathcal{S}_v$ of subgroups of the edge groups, G_v, $v\in V$. If $H^1(G_v,\mathcal{S}_v;\mathbb{Z}G)\neq 0$ for some $v\in V$ one can repeat the decomposition procedure. But as G is accessible the decomposition stops after a finite number of steps. Hence we can assume that $H^1(G_v,\mathcal{S}_v;\mathbb{Z}G)=0$ for all $v\in V$.

The relative Mayer–Vietoris sequence (cf. [2], Theorems 3.2 and 3.3, which can be generalized to arbitrary graphs of groups) now yields a short exact sequence of right G-modules.

$$0\to\prod_V H^0(G_v,\mathcal{S}_v;\mathbb{Z}G)\to\prod_E H^0(G_e;\mathbb{Z}G)\to H^1(G,\mathcal{S}_{\text{inf}};\mathbb{Z}G)\to 0. \tag{4.3}$$

Of course $H^0(G_v,\mathcal{S}_v;\mathbb{Z}G)=0$ if either $\mathcal{S}_v\neq\varnothing$ or G_v is infinite. If G_v is finite then $\mathcal{S}_v=\varnothing$ and $H^0(G_v,\mathcal{S}_v;\mathbb{Z}G)\cong\mathbb{Z}(G_v\backslash G)$, and similarly for $H^0(G_e;\mathbb{Z}G)$. Thus

(4.3) can be written as

$$0 \to \prod_{V_{\mathrm{fin}}} \mathbb{Z}(G_v \backslash G) \to \prod_{E} \mathbb{Z}(G_e \backslash G) \to H^1(G, \mathcal{S}_{\mathrm{inf}}; \mathbb{Z}G) \to 0, \tag{4.4}$$

where $V_{\mathrm{fin}} \subseteq V$ is the set of all vertices v with G_v finite.

4.4. From the short exact sequence (4.2) and (4.4) we deduce three things. Firstly, the permutation modules $\mathbb{Z}(U\backslash G)$ for a finite subgroup $U \leq G$ are of type $(FP)_\infty$. Since the index sets V_{fin}, E and I_{fin} are finite it follows that the G-module $H^1(G, \mathcal{S}; \mathbb{Z}G)$ is of type $(FP)_\infty$. Secondly, when tensored with $\mathbb{Q}$, a permutation module $\mathbb{Z}(U\backslash G)$, U finite, becomes a projective $\mathbb{Q}G$-module. Hence using (4.4) and (4.2) one can construct a finite projective resolution of $H^1(G, \mathcal{S}; \mathbb{Q}G)$. This yields a bound for the projective dimension and the Euler characteristic of this $\mathbb{Q}G$-module. Using the notation of [3] (in fact extending it slightly) we write $\chi(M)$ for the Hattori–Stallings-rank of a $\mathbb{Q}G$-module of type (FP) – recall that $\chi(M)$ is a finite $\mathbb{Q}$-linear combination of conjugacy classes in G – and $\mu(M) \in \mathbb{Q}$ for its coefficient of $1 \in G$.

We summarize:

THEOREM 4.1. *Let G be a finitely generated infinite accessible group and $\mathcal{S} = \{S_i \mid i \in I\}$ a finite family of finitely generated subgroups of G. Then the right G-module $H^1(G, \mathcal{S}; \mathbb{Z}G)$ is of type $(FP)_\infty$. The $\mathbb{Q}G$-module $H^1(G, \mathcal{S}; \mathbb{Q}G)$ is of type (FP) and of projective dimension ≤ 2; and its Euler characteristic is given by*

$$\mu(H^1(G, \mathcal{S}; \mathbb{Q}G)) = \sum_E \frac{1}{|G_e|} - \sum_{V_{\mathrm{fin}}} \frac{1}{|G_v|} + \sum_{I_{\mathrm{fin}}} \frac{1}{|S_i|}. \tag{4.5}$$

Proof. If K is a finite group then the trivial $\mathbb{Q}K$-module $\mathbb{Q}$ is projective and has Euler characteristic $\mu(\mathbb{Q}) = 1/|U|$. If U is a subgroup of G then $\mathbb{Q}G$ is free as a $\mathbb{Q}U$-module, hence $\mathbb{Q} \otimes_{\mathbb{Q}U} \mathbb{Z}G \cong \mathbb{Q}(U\backslash G)$ is $\mathbb{Q}G$-projective; and by the covariance property of χ (and μ) we get $\mu(\mathbb{Q}(U\backslash G)) = \mu(\mathbb{Q})$. Using the behaviour of χ (and μ) with respect to exact sequence yields formula (4.5).

4.5. *Remark.* For the proof of the main result we shall actually only need the case $\mathcal{S} = \varnothing$ of Theorem 4.1. In this case (4.2) is irrelevant and hence the projective dimension of $H^1(G; \mathbb{Q}G)$ is even ≤ 1.

5. The cokernel $C(G, S)$ of res is free-Abelian

5.1. Next we observe that H. Müller's result [9] on the cokernel of the restriction map extends to the case of a family of subgroups:

THEOREM 5.1 (H. Müller). *Let G be a finitely generated accessible group and $\mathcal{S}=\{S_1, S_2, \ldots, S_m\}$ a finite family of finitely generated subgroups. Then the cokernel $C(G, \mathcal{S})$ of the restriction map $H^1(G;\mathbb{Z}G)\to H^1(\mathcal{S};\mathbb{Z}G)$ is free-Abelian.*

Proof. Following the proof of [9], Corollary 1.9 one can embedded S_1 into a certain accessible group $\bar{S}_1$ with $C(\bar{S}_1, S_1)$ free-Abelian and such that there is a short exact sequence

$$C(\bar{G}, \bar{\mathcal{S}}) \rightarrowtail C(G, \mathcal{S}) \otimes_G \mathbb{Z}\bar{G} \twoheadrightarrow C(\bar{S}_1, S_1) \otimes_{S_1} \mathbb{Z}\bar{G},$$

where $\bar{G}$ stands for the amalgamated free product $\bar{G}=G *_{S_1} \bar{S}_1$ and $\bar{\mathcal{S}}$ for the family $\bar{\mathcal{S}}=\{\bar{S}_1, S_2, \ldots, S_m\}$ of subgroups of $\bar{G}$. Hence it suffices to prove that $C(\bar{G}, \bar{\mathcal{S}})$ is free-Abelian. Repeating the argument shows that we may assume that all subgroups $S_1, \ldots, S_m$ are accessible. The proof of [9], Corollary 1.4 now carries over.

6. The case when $0<\operatorname{rk} C(G, \mathcal{S})<\infty$

6.1. Throughout this section we assume G to be a finitely generated accessible group and $\mathcal{S}=\{S_1, \ldots, S_m\}$ a finite non-empty family of finitely generated infinite subgroups such that the cokernel $C(G, \mathcal{S})$ of (1.1) is of finite $\mathbb{Z}$-rank>0.

LEMMA 6.1. *Under these assumptions the restriction map* (1.1) *is injective, so that one has the short exact sequence of G-modules.*

$$H^1(G;\mathbb{Z}G) \rightarrowtail H^1(\mathcal{S};\mathbb{Z}G) \twoheadrightarrow C(G, \mathcal{S}). \tag{6.1}$$

Proof. If not, then by Swarup's relative version of Stalling's structure theorem [14], after replacing the groups S_i by suitable conjugates, the pair $(G, \mathcal{S})$ decomposes non-trivially as an amalgamated product of two pairs $(G_i, \mathcal{S}_i)$, $i=1,2$ or as an HNN-extension over a pair $(G_1, \mathcal{S}_1)$, where in either case the amalgamated (associated) subgroup is finite. Writing C_i for the cokernel $C(G_i, \mathcal{S}_i)$ we obtain the following commutative diagram with exact rows.

$$\begin{tikzcd}
H^1(G;\mathbb{Z}G) \arrow[r] \arrow[d, "\alpha"'] & H^1(\mathcal{S};\mathbb{Z}G) \arrow[r] \arrow[d, "\beta"'] & C(G, \mathcal{S}) \arrow[r] \arrow[d, "\gamma"'] & 0 \\
\prod H^1(G_i;\mathbb{Z}G) \arrow[r] & \prod H^1(\mathcal{S}_i;\mathbb{Z}G) \arrow[r] & \prod C_i \otimes_{G_i} \mathbb{Z}G \arrow[r] & 0
\end{tikzcd}$$

α is the restriction which occurs in the Mayer–Vietoris sequence for G; hence, as the amalgamated subgroup is finite, α is epimorphic. $\mathcal{S}$ is the disjoint union of $\mathcal{S}_1$

and $\mathcal{S}_2$; hence β is the identity. It follows by the 5-Lemma that γ is an isomorphism. Therefore one of the G-modules $C_i \otimes_{G_i} \mathbb{Z}G$ is of finite $\mathbb{Z}$-rank >0. But this implies that G_i is of finite index in G which is impossible.

6.2. Dunwoody's accessibility criterion [4] asserts that a group G is accessible if and only if the cohomology group $H^1(G;\mathbb{Z}G)$ is finitely generated as a right G-module. From our assumption that G is acessible and $C(G,\mathcal{S})$ free-Abelian of finite rank it thus follows that $H^1(\mathcal{S};\mathbb{Z}G)$ and hence each $H^1(S_i;\mathbb{Z}G)\cong H^1(S_i;\mathbb{Z}S_i)\otimes_{S_i}\mathbb{Z}G$ is finitely generated over $\mathbb{Z}G$. As $\mathbb{Z}G$ is a free $\mathbb{Z}S_i$-module we can infer that $H^1(S_i;\mathbb{Z}S_i)$ is finitely generated over $\mathbb{Z}S_i$. Hence all groups S_i, $1\leq i\leq m$, are accessible by the criterion again.

Thus the absolute version of Theorem 4.1 applies for both G and S_i, $1\leq i\leq m$. Hence the G-modules $H^1(G;\mathbb{Z}G)$ and $H^1(\mathcal{S};\mathbb{Z}G)$ are of type $(FP)_\infty$, and in view of the short exact sequence (6.1) so is $C(G,\mathcal{S})$. Moreover the $\mathbb{Q}G$-modules $H^1(G;\mathbb{Q}G)$ and $H^1(\mathcal{S};\mathbb{Q}G)$ are of type (FP) and of projective dimension ≤ 1. Hence the short exact sequence (6.1), when tensored with $\mathbb{Q}$, shows that $C_{\mathbb{Q}}(G,\mathcal{S})=C(G,\mathcal{S})\otimes_{\mathbb{Z}}\mathbb{Q}$ is a $\mathbb{Q}G$-module of type (FP) and of projective dimension ≤ 2.

By Lemma 3.1 we can now infer that the group G is of type $(FP)_\infty$ over $\mathbb{Z}$ and of type (FP) with $\mathrm{cd}_{\mathbb{Q}}\, G\leq 2$ over $\mathbb{Q}$.

6.3. Our next aim is to show that the kernel $\Delta=\Delta_{G/\mathcal{S}}$ of the augmentation map $\varepsilon:\mathbb{Z}G/\mathcal{S}\twoheadrightarrow\mathbb{Z}$ (4.1) is a G-module of type $(FP)_1$. To that end take an arbitrary direct power $\prod\mathbb{Z}G$ of copies of $\mathbb{Z}G$, and apply $\mathrm{Tor}_n^{\mathbb{Z}G}(\prod\mathbb{Z}G,-)$ to the short exact sequence (4.1). This yields the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\mathrm{Tor}_1^{\mathbb{Z}G}\left(\prod\mathbb{Z}G,\mathbb{Z}\right) & \to & \left(\prod\mathbb{Z}G\right)\otimes_G\Delta & \to & \left(\prod\mathbb{Z}G\right)\otimes_G\mathbb{Z}(G/\mathcal{S}) & \to & \left(\prod\mathbb{Z}G\right)\otimes_G\mathbb{Z} & \to & 0\\
 & & {\scriptstyle\mu_1}\downarrow & & {\scriptstyle\mu_2}\downarrow & & {\scriptstyle\mu_3}\downarrow & & \\
0 & \to & \prod\Delta & \to & \prod\mathbb{Z}G/\mathcal{S} & \to & \prod\mathbb{Z} & \to & 0
\end{array}$$

where the vertical arrows stand for the limiting homomorphism (e.g., $\mu_1(\prod\lambda_i\otimes d)=\prod\lambda_i d$, $\lambda_i\in\mathbb{Z}G$, $d\in\Delta$). Since $\mathbb{Z}$ is of type $(FP)_\infty$ as a G-module $\mathrm{Tor}_1^{\mathbb{Z}G}(\prod\mathbb{Z}G,\mathbb{Z})=0$ and μ_3 is an isomorphism. $\mathcal{S}$ is a finite family of finitely generated subgroups of G, hence $\mathbb{Z}G/\mathcal{S}$ is of type $(FP)_1$ and μ_2 is an isomorphism. It follows that μ_1 is an isomorphism, whence Δ is of type $(FP)_1$ (see e.g. [1], chapter I).

6.4. From Section 6.3. we infer that the $\mathbb{Q}G$-module $\Delta_{\mathbb{Q}}=\Delta\otimes\mathbb{Q}$ is of type $(FP)_1$. So let us choose a $\mathbb{Q}G$-projective resolution

$$P_2\xrightarrow{\partial_2}P_1\xrightarrow{\partial_1}P_0\twoheadrightarrow\Delta_{\mathbb{Q}} \tag{6.2}$$

which is finitely generated in dimensions 0 and 1, and which we can use to compute the relative cohomology groups $H^n(G, \mathcal{S}; \mathbb{Q}G)$ for $n = 1$ and 2. Also, we have the long exact sequence for the pair $(G, \mathcal{S})$

$$\cdots \to H^0(\mathcal{S}; \mathbb{Q}G) \to H^1(G, \mathcal{S}; \mathbb{Q}G) \to H^1(G; \mathbb{Q}G) \xrightarrow{\text{res}}$$

$$H^1(\mathcal{S}; \mathbb{Q}G) \to H^2(G, \mathcal{S}; \mathbb{Q}G) \to$$

where res is injective by Lemma 6.1. Since all groups in $\mathcal{S}$ are infinite $H^0(\mathcal{S}; \mathbb{Q}G) = 0$ and hence $H^1(G, \mathcal{S}; \mathbb{Q}G) = 0$. This shows that

$$0 \to P_0^* \to P_1^* \to \operatorname{coker}(\partial_1^*) \to 0,$$

with $P_i^* = \operatorname{Hom}_{\mathbb{Q}G}(P_i, \mathbb{Q}G)$ is a short exact sequence. But P_0^* and P_1^* are finitely generated projective right $\mathbb{Q}G$-modules, hence $\operatorname{coker}(\partial_1^*)$ is a $\mathbb{Q}G$-module of projective dimension ≤ 1. Clearly $\operatorname{coker}(\partial_1^*)$ contains $\ker \partial_2^* / \operatorname{im} \partial_1^* = H^2(G, \mathcal{S}; \mathbb{Q}G)$ which, in turn, contains the submodule $C_{\mathbb{Q}}(G, \mathcal{S})$ of finite $\mathbb{Q}$-dimension. By Corollary 3.2 this implies that the cohomology dimension of G over $\mathbb{Q}$ is in fact ≤ 1.[(2)] Hence by Dunwoody's generalization of Stallings' theorem [4] G contains a free subgroup of finite index.

We summarize

THEOREM 6.2. *Let G be a finitely generated accessible group and $\mathcal{S}$ a finite family of finitely generated subgroups of G. If the cokernel $C(G, \mathcal{S})$ of the restriction map*

$$H^1(G; \mathbb{Z}G) \to H^1(\mathcal{S}; \mathbb{Z}G)$$

is of finite $\mathbb{Z}$-rank >0 then G contains a free subgroup of finite index.

Remark. It follows, in particular, that in the situation of Theorem 6.2 one has $H^2(G; \mathbb{Z}G) = 0$. Hence the long exact sequence for $(G, \mathcal{S})$ shows that $H^2(G, \mathcal{S}; \mathbb{Z}G) \cong C(G, \mathcal{S})$.

6.5. It remains to examine the situation when G is a finitely generated infinite free-by-finite group and $\mathcal{S}$ a finite family of m infinitely generated, infinite subgroups. Then G can be thought of as the fundamental group of a finite graph

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[2] This type of argument was used by Farrell [6]

$(\mathfrak{G}, X)$ of finite groups. Let V denote the set of vertices and E the set of positive edges of X. Then Theorem 4.1 yields the formula

$$\mu(H^1(G;\mathbb{Q}G)) = \sum_E \frac{1}{|G_e|} - \sum_V \frac{1}{|G_v|}.$$

But this is precisely the negative of the formula for the Euler characteristic $\mu(G) = \mu(\mathfrak{G})$ (e.g. [3], Theorem 2). Hence we have

$$\mu(H^1(G;\mathbb{Q}G)) = -\mu(G),$$

and similar for S_i,

$$\begin{aligned}\mu(H^1(S_i;\mathbb{Q}G)) &= \mu(H^1(S_i;\mathbb{Q}S_i)\otimes_{S_i}\mathbb{Z}G)\\ &= \mu(H^1(S_i;\mathbb{Q}S_i)) = -\mu(S_i).\end{aligned}$$

From the short exact sequence (6.1) we now obtain the formula

$$\mu(C_{\mathbb{Q}}(G,\mathcal{S})) = \mu(G) - \sum_{i=1}^{m} \mu(S_i) \tag{6.3}$$

On the other hand $C_{\mathbb{Q}}(G,\mathcal{S})$ is a $\mathbb{Q}G$-module of finite $\mathbb{Q}$-dimension, whence $\mu(C_{\mathbb{Q}}(G,\mathcal{S})) = \dim C_{\mathbb{Q}}(G,\mathcal{S})\cdot\mu(G)$ (see e.g. [3], Lemma 8). Together with (6.3) this yields the equation

$$\mu(G)(\operatorname{rk} C(G,\mathcal{S}) - 1) + \sum_{i=1}^{m} \mu(S_i) = 0. \tag{6.4}$$

Let F be a free subgroup of finite index in G and n the rank of F. Then $\mu(F) = 1 - n = |G:F|\cdot\mu(G)$. This shows that $\mu(G)$ is ≤ 0 and $\mu(G) = 0$ if and only if G is infinite cyclic-by-finite. Of course the same holds for $\mu(S_i)$; hence we can deduce from (6.4) that $\mu(S_i) = 0$ for $1 \leq i \leq m$ and either $\mu(G) = 0$ or $\operatorname{rk} C(G,\mathcal{S}) = 1$. In other words: all groups S_i, $1 \leq i \leq m$, contain an infinite cyclic subgroup of finite index and either the same holds for G itself or one has $C(G,\mathcal{S}) \cong \mathbb{Z}$.

Remark. Instead of using Euler characteristics A. Freudenberger [Diplomarbeit 1982, University of Freiburg im Breisgau, Germany] obtains formula (6.4) by computing the $\mathbb{Q}$-dimensions in the long exact homology sequence of G with coefficients in (6.1) tensored with $\mathbb{Q}$.

6.6. The proof of Theorems A and B is now easily completed: If G is infinite cyclic-by-finite then the index $|G:S_i|$ is finite for all $1 \leq i \leq m$, $H^1(G;\mathbb{Z}G)\cong\mathbb{Z}$, and $H^1(\mathcal{S};\mathbb{Z}G)=\prod H^1(S_i;\mathbb{Z}S_i)\otimes_{S_i}\mathbb{Z}G=\mathbb{Z}(\mathcal{S}\backslash G)$ is free-Abelian of rank $\sum|G:S_i|$. By the short exact sequence (6.1) we thus have

$$\operatorname{rk} C(G,\mathcal{S})=\sum_{i=1}^{m}|G:S_i|-1.$$

On the other hand, if $\mu(G)\neq 0$ and hence $C(G,\mathcal{S})\cong\mathbb{Z}$ we consider a free subgroup F of finite index in G and the full subpair $(F,\mathcal{S}')$ of $(G,\mathcal{S})$ given by F (c.f. Section 2.2). By [2], Proposition 7.5, we have

$$H^2(F,\mathcal{S}';\mathbb{Z}F)\cong H^2(G,\mathcal{S};\mathbb{Z}G)\cong C(G,\mathcal{S})\cong\mathbb{Z}.$$

Hence $(F,\mathcal{S}')$ is a 2-dimensional Poincaré duality pair by the PD^2-criterion [2] Theorem 9.3.

7. The case when $C(G,\mathcal{S})=0$.

7.1. Here we have to consider compatible decompositions of the pair $(G,\mathcal{S})$ as defined in Section 2.3. That is, both G and $\mathcal{S}$ are "fundamental groups of graphs of groups" $G=\pi_1(\mathfrak{G},X)$ $\mathcal{S}=\pi_1(\mathfrak{S},Y)$ – where the graph Y is not necessarily connected – and there is given an orientation preserving graph map $f:Y\to X$ and for each vertex v of Y a group element $c_v\in G$ such that the equations (2.1) and (2.2) are satisfied.

One feature of compatible decompositions is a natural homomorphism between the Mayer–Vietories sequences of G and $\mathcal{S}$. Indeed one has the commutative diagram of G-modules.

$$\begin{array}{ccccc}
\bigoplus_{E(Y)}\mathbb{Z}G/S_e & \rightarrowtail & \bigoplus_{V(Y)}\mathbb{Z}G/S_v & \twoheadrightarrow & \mathbb{Z}G/S \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \varepsilon} \\
\bigoplus_{E(X)}\mathbb{Z}G/G_e & \rightarrowtail & \bigoplus_{V(X)}\mathbb{Z}G/G_v & \twoheadrightarrow & \mathbb{Z}
\end{array} \qquad (7.1)$$

where $V(X)$, $V(Y)$ stand for the set of vertices and $E(X)$, $E(Y)$ for the set of positive edges of X resp. Y. The rows are the short exact sequences [9], p. 168

and the vertical maps are given by

$$\left.\begin{aligned} f_1(gS_e) &= gc_{o(e)}G_{f(e)} \\ f_0(gS_v) &= gc_vG_{f(v)} \end{aligned}\right\} g \in G, \quad v \in V(Y), \quad e \in E(Y)$$

Commutativity of the diagram is guaranteed by (2.2). Applying the functor $\mathrm{Ext}^*_{\mathbb{Z}G}(-, A)$, A an arbitrary G-module, and using the Shapiro Lemma thus yields the commutative ladder

$$\begin{tikzcd} \cdots H^k(G;A) \arrow[r] \arrow[d] & \prod\limits_{V(X)} H^k(G_v;A) \arrow[r] \arrow[d] & \prod\limits_{E(X)} H^k(G_e;A) \arrow[r] \arrow[d] & H^{k+1}(G;A) \arrow[r] \arrow[d] & \cdots \\ \cdots H^k(\mathcal{S};A) \arrow[r] & \prod\limits_{V(Y)} H^k(S_v;A) \arrow[r] & \prod\limits_{E(Y)} H^k(S_e;A) \arrow[r] & H^{k+1}(\mathcal{S};A) \arrow[r] & \cdots \end{tikzcd} \tag{7.2}$$

7.2. We are now in a position to prove

PROPOSITION 7.1. *Let G be a group and $\mathcal{S}$ a family of subgroups. Assume that G and $\mathcal{S}$ have compatible decompositions $G = \pi_1(\mathfrak{G}, X)$, $\mathcal{S} = \pi_1(\mathfrak{S}, Y)$, via a graph map $f: Y \to X$ which is injective on the edges, and such that the following two conditions hold*

(i) $S_e^{c_{o(e)}} = G_{f(e)}$ *for every edge* $e \in E(y)$

(ii) *all vertex groups S_v of $\mathcal{S}$ have ≤ 1 end.*

Then the restriction map $H^1(G;\mathbb{Z}G) \to H^1(\mathcal{S};\mathbb{Z}G)$ is surjective.

Proof. The condition (i) implies that the vertical map f_1 in the diagram (7.1) is the injection of a direct summand. Hence $\mathrm{Hom}_G(f_1, \mathbb{Z}G) = f_1^*$ is surjective and (7.2) with $A = \mathbb{Z}G$ yields the commutative diagram with exact rows

$$\begin{tikzcd} \prod H^0(G_e;\mathbb{Z}G) \arrow[r] \arrow[d, "f_1^*"'] & H^1(G;\mathbb{Z}G) \arrow[r] \arrow[d, "\mathrm{res}"'] & \prod H^1(G_v;\mathbb{Z}G) \arrow[d] \\ \prod H^0(S_e;\mathbb{Z}G) \arrow[r, "\delta"] & H^1(\mathcal{S};\mathbb{Z}G) \arrow[r] & \prod H^1(S_v;\mathbb{Z}G) \end{tikzcd}$$

Now, condition (ii) asserts that $H^1(S_v;\mathbb{Z}G) = 0$ for all $v \in V(Y)$; hence δ is an epimorphism and so is res.

7.3. It remains to prove the following converse of Proposition 7.1.

PROPOSITION 7.2. *Let G be a finitely generated group and $\mathcal{S}=\{S_i \mid i\in I\}$ a finite family of finitely generated accessible subgroups. If the restriction map* $\mathrm{res}: H^1(G;\mathbb{Z}G)\to H^1(\mathcal{S};\mathbb{Z}G)$ *is surjective then G and $\mathcal{S}$ have compatible decompositions $G=\pi_1(\mathfrak{G}, X)$, $\mathcal{S}=\pi_1(\mathfrak{S}, Y)$ via an orientation preserving graph map $f: Y\to X$ which is bijective on the edges and such that*

(i) *for every edge $e\in E(Y)$ the edge group G_e is finite and coincides with $S_e^{c_{\alpha(e)}}$,*

(ii) *all vertex groups S_v of $\mathcal{S}$ have ≤ 1 end.*

Proof. Since $\mathcal{S}$ is a finite family of finitely generated accessible groups $\mathcal{S}$ can be written as the "fundamental group" of some finite graph of groups $(\mathfrak{S}, Y)$ with all edge groups S_e finite and all vertex groups S_v having ≤ 1 end. If we arrange $(\mathfrak{S}, Y)$ such that all embedlings $S_e < G_{o(e)}$ are proper, then the number of edge pairs of Y is an invariant of $\mathcal{S}$ which we call the *complexity*.

We shall prove Proposition 7.2 by induction on the complexity of $\mathcal{S}$. If the complexity is $=0$ then every S_i has ≤ 1 end and the proposition holds with X consisting of one vertex and no edges and Y consisting of an isolated vertex for every $i\in I$. If $\mathcal{S}$ has complexity >0 then $H^1(\mathcal{S};\mathbb{Z}G)\neq 0$. So assume $H^1(S_1;\mathbb{Z}G)\neq 0$, and put $\mathcal{T}=\mathcal{S}-\{S_1\}$. Since res is surjective H. Müller's first decomposition Theorem applies ([7], Corollary 3.1.). Thus after replacing the groups in $\mathcal{T}$ by suitable conjugates the pair $(G,\mathcal{T})$ and the subgroup S_1 have a proper simultaneous decomposition in the following sense. Either $G=G_1 *_K G_2$ and $S_1=S_{11} *_K S_{12}$ where K is finite. $S_{1i}\leq G_i$ $(i=1,2)$, and $\mathcal{T}$ is the disjoint of families $\mathcal{T}_1, \mathcal{T}_2$ of subgroups of G_1, resp. G_2; or $G=G_1 *_{K,p}$ is an HNN-group with stable letter p and finite associated subgroups K, pKp^{-1}, $\mathcal{T}$ consists of subgroups of G_1, and S_1 is either $=S_{11} *_{K,p}$ or $=S_{11} *_{K} pS_{12}p^{-1}$, with $S_{11}, S_{12}\leq G_1$. Note that all these decompositions of G and S_1 are compatible in the sense of Section 2.3.

We restrict the discussion to the first case, the other cases being similar. By (7.2) we have a map between the Mayer–Vietoris sequences for G and S_1, and adding $H^1(\mathcal{T}, \mathbb{Z}G)$ to the latter yields the commutative diagram with exact rows

$$\begin{tikzcd}
H^0(K;\mathbb{Z}G) \arrow[r] \arrow[d, "="'] & H^1(G;\mathbb{Z}G) \arrow[r] \arrow[d, "\mathrm{res}"'] & \bigoplus_{i=1}^{2} H^1(G_i;\mathbb{Z}G) \arrow[r] \arrow[d, "\oplus \mathrm{res}_i"'] & 0 \\
H^0(K;\mathbb{Z}G) \arrow[r] & H^1(\mathcal{S};\mathbb{Z}G) \arrow[r] & \bigoplus_{i=1}^{2} H^1(S_{1i};\mathbb{Z}G)\oplus H^1(\mathcal{T};\mathbb{Z}G) \arrow[r] & 0
\end{tikzcd}$$

from which we deduce that the restriction map

$$\mathrm{res}_i : H^1(G_i;\mathbb{Z}G_i)\to H^1\left(\mathcal{T}\cup\{S_{1i}\};\mathbb{Z}G_i\right)$$

is surjective for $i=1, 2$. Now the complexity of $\mathcal{T}_i \cup \{S_{1i}\}$ is less than that of $\mathcal{S}$. Hence, by induction, G_i and $\mathcal{T}_i \cup \{S_{1i}\}$ have a compatible decomposition satisfying the assertion of Proposition 7.2. Putting these together yields a compatible decomposition of G and $\mathcal{S}$ with the required properties.

Remark. Instead of assuming that the subgroups in $\mathcal{S}$ are accessible in Proposition 7.2 one could also assume that the group G is accessible. Indeed, by Dunwoody's criterion [4] this would mean that $H^1(G; \mathbb{Z}G)$ is finitely generated as a right G-module. Since $\mathrm{res}: H^1(G; \mathbb{Z}G) \to H^1(\mathcal{S}; \mathbb{Z}G)$ is assumed to be surjective the same holds for $\mathcal{S}$, implying that every $S_i \in \mathcal{S}$ is accessible.

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