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Nilpotent completions and Lie rings associated to link groups

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§1. Introduction

The nilpotent completion and the Lie ring associated to a group with finitely generated abelianization are nilpotent invariants derived from its lower central series. In classical link theory, several authors have studied those for a link group, the fundamental group of the complement of a link, since it is much more practical rather than studying a group itself.

On the other hand, Sullivan gave a cohomological and infinitesimal method to compute these invariants when the group is the fundamental group of a polyhedron. Thus as he suggested in [17], Problem 5, it seems interesting to apply his theory to the link theory. In this paper, we are concerned with this vague question. Of course it is hopeless to expect complete algebraic characterization of these invariants for link groups, however it is possible to obtain some general results from infinitesimally computable cases. Such computations are attained in § 4 and § 5.

In § 4, we construct a minimal model for a polyhedron which is cohomologically equivalent to a bouquet of circles. We establish, as Corollary 6.3, the equivalence between the freeness of the nilpotent completion of its fundamental group and the vanishing of every Massey product on H^1. Now, a link complement can never be cohomologically equivalent to a bouquet of circles since H^2 is non trivial. However, to apply this construction, we do not need trivial H^2, but we do need just non-existence of decomposable elements in H^2, and it still has some significance in the link theory. Actually, Milnor [10] proved that the nilpotent completion of a link group is isomorphic to that of a free group iff all the $\bar{\mu}$-invariants vanish, and Porter [15] succeeded in expressing the $\bar{\mu}$-invariant in terms of the Massey product. In particular, we get the equivalence which is eventually a special case of Corollary 6.3.

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In § 5, we explicitly construct a family of minimal models for polyhedra which are cohomologically equivalent to the product of a bouquet of circles with a circle. In the special case where the polyhedron is the complement of a link, this cohomological condition is a condition on the linking numbers. Our construction asserts that the structure of the Lie ring associated to the fundamental group of such a polyhedron is very simple while the nilpotent completion is not. Corollary 6.4, which has been conjectured by K. Murasugi, came up as an application of the construction.

Besides these, several corollaries of the constructions are established in § 6. We review nilpotent completions and Lie rings associated to groups in § 2, and Sullivan's theory in § 3.

The content of § 4 is from my thesis supervised by Professor John Morgan. I would like to express my great appreciation for his constant encouragement.

§ 2. Nilpotent completions and Lie rings

Let G be a group and let $G = G_0 \supset G_1 \supset G_2 \supset \cdots$ be the lower central series of G where $G_p = [G, G_{p-1}]$ for $p \geqslant 1$. Here are two invariants of G which come from the lower central series. The first one is the nilpotent completion of G. It is the tower of nilpotent groups:

$$\cdots \to G/G_2 \to G/G_1 \to \{e\}.$$

We will simply denote it by Nil(G). Nil(G) is said to be isomorphic to Nil(H) of a group H up to the pth stage if there is an isomorphism: $G/G_p \to H/H_p$. Then it induces an isomorphism: $G/G_q \to H/H_q$ for each $q \leqslant p$ and we get isomorphic towers up to the pth stage. We might say that Nil(G) is isomorphic to Nil(H) if these are isomorphic up to any stage.

Now, each G/G_p is a nilpotent group of index p and a central extension of G/G_{p-1} by the abelian group G_{p-1}/G_p. The second invariant is formed by these abelian groups. Let $\mathscr{L}_p(G) = G_{p-1}/G_p$ and $\mathscr{L}(G) = \oplus_{p \geqslant 1} \mathscr{L}_p(G)$. Then, the commutator operation determines a well defined bilinear mapping, $[\ ,\ ]: \mathscr{L}_p(G) \otimes \mathscr{L}_q(G) \to \mathscr{L}_{p+q}(G)$ such that

(1) $[\alpha, \beta] = -[\beta, \alpha]$ and

(2) $[[\alpha, \beta]\gamma] + [[\beta, \gamma]\alpha] + [[\gamma, \alpha]\beta] = 0$.

Hence $\mathscr{L}(G)$ admits a graded Lie ring structure generated by $\mathscr{L}_1(G)$. See [7].

Both concepts have rational versions. Say, the rational nilpotent completion of G, which will be denoted by $\mathbb{Q}$-nil(G), is the tower of $\mathbb{Q}$-nilpotent groups:

$$\cdots \to G/G_2 \otimes \mathbb{Q} \to G/G_1 \otimes \mathbb{Q} \to \{e\},$$

where each $G/G_p \otimes \mathbb{Q}$ stands for the Malcev completion [8] of the nilpotent group G/G_p. Also taking tensor product by $\mathbb{Q}$ in usual sense, we get a graded Lie algebra $\mathscr{L}(G) \otimes \mathbb{Q}$ associated to G.

The first important result concerning the structure of $\mathscr{L}(G)$ may be one for a free group by Witt [18]. See also [7].

PROPOSITION 2.1 (Witt). *Let F_n be a free group of rank n. Then $\mathscr{L}(F_n)$ is a free Lie ring generated by n elements. Here, free means that there are no relations except those generated by* (1) *and* (2). *Furthermore, $\mathscr{L}_p(F_n)$ is a free abelian group of rank*

$$W(n, p) = \frac{1}{p} \sum_{d|p} \mu(d) n^{p/d},$$

where $\mu(d)$ is the Mobius function.

$W(n, p)$ is called the Witt number.

In general, the nilpotent completion is a stronger invariant than the associated Lie ring, however,

LEMMA 2.2. *For any $p \geqslant 1$, if $\mathscr{L}(G)$ is isomorphic to $\mathscr{L}(F_n)$ up to the* pth *stage, then* Nil(G) *is isomorphic to* Nil(F_n) *up to the* pth *stage.*

Proof. Since G/G_p is a nilpotent group for any $p \geqslant 1$, it is generated by n elements (see [7], Lemma 5.9) and hence we have an epimorphism $\phi: F_n \to G/G_p$ which induces an isomorphism: $\mathscr{L}(F_n) \to \mathscr{L}(G/G_p)$ up to the pth stage. Looking at the commutative diagram for $q \leqslant p$,

$$\begin{array}{ccccccccc}
1 & \to & \mathscr{L}_q(F_n) & \to & F_n/(F_n)_q & \to & F_n/(F_n)_{q-1} & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mathscr{L}_q(G/G_p) & \to & G/G_q & \to & G/G_{q-1} & \to & 1,
\end{array}$$

we notice that ϕ induces an isomorphism: $F_n/(F_n)_q \to G/G_q$ until q becomes p by the five lemma and the induction on q.

LEMMA 2.3. *For any $p \geqslant 1$, if $\mathscr{L}(G) \otimes \mathbb{Q}$ is isomorphic to $\mathscr{L}(F_n) \otimes \mathbb{Q}$ up to the* pth *stage, then* $\mathbb{Q}$-nil(G) *is isomorphic to* $\mathbb{Q}$-nil(F_n) *up to the* pth *stage.*

Proof. Since for any $p \geqslant 1$, there is a homomorphism $\phi: F_n \to G/G_p$ which induces an isomorphism: $\mathscr{L}(F_n) \otimes \mathbb{Q} \to \mathscr{L}(G/G_p) \otimes \mathbb{Q}$ up to the pth stage, the same argument can be applied.

The next lemma will be used in § 5.

LEMMA 2.4. *Suppose that* $\mathscr{L}_1(G)$ *is generated by* $g_1, \ldots, g_{n+r}$ *such that*
(1) $[g_i, g_j] \neq 0$ *in* $\mathscr{L}_2(G)$ *for all* $i, j \leq n$ *and*
(2) $[g_i, g_{n+j}] = 0$ *in* $\mathscr{L}_2(G)$ *for all* $i, j \geq 1$.
Then $\mathscr{L}_p(G)$ *is generated by at most* $W(n, p)$ *elements. If* $\mathscr{L}_p(G)$ *is a free abelian group of rank* $W(n, p)$ *for all* $p \geq 2$, *then* $\mathscr{L}(G)$ *is isomorphic to* $\mathscr{L}(F_n \times \mathbb{Z}^r)$, *where* $\mathbb{Z}^r$ *is a free abelian group of rank* r

Proof. Let $h_1, \ldots, h_n$ be a basis of $\mathscr{L}_1(F_n)$ and define the homomorphism $\phi: \mathscr{L}_1(F_n) \to \mathscr{L}_1(G)$ by $\phi(h_i) = g_i$. Then ϕ naturally induces a homomorphism $\phi_p: \mathscr{L}_p(F_n) \to \mathscr{L}_p(G)$ for each p. What we want to show then is that ϕ_p is onto for $p \geq 2$.

Now, any element of $\mathscr{L}_p(G)$ can be written down as a linear combination of simple p-fold brackets of $g_1, \ldots, g_{n+r}$, like $[[\cdots[[g_{i_1} g_{i_2}] g_{i_3}] \cdots] g_{i_p}]$. Let us simply denote it by $(g_{i_1} \cdots g_{i_p})$. Suppose that $i_1, \ldots, i_p \leq n$, then this is the image of $(h_{i_1} \cdots h_{i_p})$ by ϕ_p. If $i_q > n$ for some $q < p$, then $(g_{i_1} \cdots g_{i_q}) = 0$ by induction hypothesis and therefore $(g_{i_1} \cdots g_{i_p}) = 0$. When $i_p > n$, We have the Jacobi identity,

$$(g_{i_1} \cdots g_{i_p}) = -((g_{i_{p-1}} g_{i_p})(g_{i_1} \cdots g_{i_{p-2}})) - ((g_{i_p}(g_{i_1} \cdots g_{i_{p-2}}) g_{i_{p-1}}).$$

The both terms of the right side are zero in $\mathscr{L}_p(G)$, and we get $(g_{i_1} \cdots g_{i_p}) = 0$. Hence ϕ_p is an epimorphism for $p \geq 2$. If $\mathscr{L}_p(G)$ is a free abelian group of rank $W(n, p)$ for all $p \geq 2$, ϕ_p must be an isomorphism and we are done.

The rational version of this is also established.

LEMMA 2.5. *Suppose that* $\mathscr{L}_1(G) \otimes \mathbb{Q}$ *is generated by* $g_1, \ldots, g_{n+r}$ *such that*
(1) $[g_i, g_j] \neq 0$ *in* $\mathscr{L}_2(G) \otimes \mathbb{Q}$ *for all* $i, j \leq n$ *and*
(2) $[g_i, g_{n+j}] = 0$ *in* $\mathscr{L}_2(G) \otimes \mathbb{Q}$ *for all* $i, j \geq 1$.
Then $\mathscr{L}_p(G) \otimes \mathbb{Q}$ *is generated by at most* $W(n, p)$ *elements. If* $\dim \mathscr{L}_p(G) \otimes \mathbb{Q} = W(n, p)$ *for all* $p \geq 2$, *then* $\mathscr{L}(G) \otimes \mathbb{Q}$ *is isomorphic to* $\mathscr{L}(F_n \times \mathbb{Z}^r) \otimes \mathbb{Q}$.

§ 3. Differential graded algebras

A differential graded algebra $\mathscr{A}$ is a graded vector space $A = \bigoplus_{p \geq 0} A^p$ over a field (always $\mathbb{Q}$ in this paper) with differential $d: A^p \to A^{p+1}$ and associative multiplication $\wedge: A^p \otimes A^q \to A^{p+q}$ so that
(1) $d^2 = 0$,

(2) $d(x \wedge y) = dx \wedge y + (-1)^{\deg x} x \wedge dy$ and
(3) $x \wedge y = (-1)^{\deg x \deg y} y \wedge x$.

A d.g.a. is minimal if d is decomposable. This means that the image of any element by d can be written down as a sum of decomposable elements. A Hirsch extension of a d.g.a. $\mathscr{A}$ is an inclusion $\mathscr{A} \to \mathscr{B}$ of a d.g.a.'s which, when we ignore the differentials, is isomorphic to $\mathscr{A} \to \mathscr{A} \otimes \Lambda(V)^p$ and where the differential of $\mathscr{B}$ sends $V \to A^{p+1}$. The integer p is the degree of the extension. From now on, we consider a series of Hirsch extensions:

$$\mathbb{Q} \subset \mathscr{A}_1 \subset \mathscr{A}_2 \subset \cdots \subset \bigcup_{p \geq 1} \mathscr{A}_p = \mathscr{A}$$

of degree 1. We should point out here that a d.g.a. generated by elements of degree 1 is always minimal, so is $\mathscr{A}$. The series is called canonical if $\mathscr{A}_1$ is generated by all closed 1-forms of $\mathscr{A}$ and $\mathscr{A}_{p+1}$ is generated by $\mathscr{A}_p$ and all 1-forms x such that $dx \in \mathscr{A}_p$ for each p. The following lemma, which is an immediate consequence of the definition, characterizes a canonical series.

LEMMA 3.1. *If* $\mathscr{A}: \mathbb{Q} \subset \mathscr{A}_1 \subset \mathscr{A}_2 \subset \cdots$ *is canonical, then*
(1) $H^1(\mathscr{A}_p) \to H^1(\mathscr{A}_{p+1})$ *is an isomorphism for all* p *and hence* $H^1(\mathscr{A}) = H^1(\mathscr{A}_1)$, *and*
(2) $H^2(\mathscr{A}_p) \to H^2(\mathscr{A}_{p+1})$ *is a monomorphism if we restrict it to the image of* $H^2(\mathscr{A}_{p-1})$.

Let us now consider the 1-minimal model of Sullivan. Let X be a polyhedron and let $\varepsilon(X)$ be $\mathbb{Q}$-polynomial forms on X. The 1-minimal model for X is a minimal d.g.a. $\mathscr{M}_X$ with a mapping $\rho: \mathscr{M}_X \to \varepsilon(X)$ of d.g.a.'s such that $\rho^*: H(\mathscr{M}_X) \to H(\varepsilon(X))$ is an isomorphism in degree 1 and injective in degree 2. We can find for instance in [11] § 5 how to construct $\mathscr{M}_X$ and its several properties. It turns out to be generated by elements of degree 1 and to have a canonical series:

$$\mathscr{M}_X: \mathbb{Q} \subset \mathscr{M}_1 \subset \mathscr{M}_2 \subset \cdots$$

Dualizing the part of degree 1, we get a tower of $\mathbb{Q}$-Lie algebras:

$$0 \leftarrow \mathfrak{G}_1 \leftarrow \mathfrak{G}_2 \leftarrow \cdots$$

Each of the Lie algebras $\mathfrak{G}_i$ is nilpotent and hence the Campbell-Hausdorff formula defines a group structure C.H.$(\mathfrak{G}_i)$ on each $\mathfrak{G}_i$.

THEOREM (Sullivan). *If* X *is arcwise connected and* $H^1(\varepsilon(X))$ *is finite*

dimensional, then the tower of nilpotent groups:

$$\{e\} \leftarrow \mathrm{C.H.}(\mathfrak{G}_1) \leftarrow \mathrm{C.H.}(\mathfrak{G}_2) \leftarrow \cdots$$

is isomorphic to $\mathbb{Q}$-nil$(\pi_1(X))$.

Thus knowing the rational nilpotent completion of $\pi_1(X)$ is equivalent to knowing the 1-minimal model for X. The proof of this theorem can be found in [2].

Let $\mathcal{M}_X : \mathbb{Q} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$ be the 1-minimal model for a polyhedron X and suppose that $\mathcal{M}_p$ is isomorphic to $\mathcal{M}_{p-1} \otimes \Lambda(V_p)$ as a vector space. Sullivan's theorem implies

$$\dim V_p = \operatorname{rank} \mathcal{L}_p(\pi_1(X)).$$

When $\dim H^1(\varepsilon(X)) = n$, the number above is bounded by the Witt number $W(n, p)$. Since $\mathcal{L}(\pi_1(X)) \otimes \mathbb{Q}$ is free up to the pth stage if and only if $\dim \mathcal{L}_q(\pi_1(X)) \otimes \mathbb{Q} = W(n, q)$ for all $q \leqslant p$, we have by Lemma 2.3 that

LEMMA 3.2. *If* $\dim V_q = W(n, q)$ *for all* $q \leqslant p$, *then* $\mathbb{Q}$-nil$(\pi_1(X))$ *is isomorphic to* $\mathbb{Q}$-nil(F_n) *up to the* pth *stage.*

This can be also proved by constructing isomorphisms for extensions of each stage.

LEMMA 3.3. *If* $\mathcal{L}_1(\pi_1(X))$ *is a free abelian group of rank* n, *then* $\dim V_q = W(n, q)$ *for all* $q \leqslant p$ *iff* $\mathrm{Nil}(\pi_1(X))$ *is isomorphic to* $\mathrm{Nil}(F_n)$ *up to the* pth *stage.*

Proof. Since $\mathcal{L}_1(\pi_1(X))$ is generated by n elements, $\mathcal{L}_q(\pi_1(X))$ is generated by at most $W(n, q)$ elements by Proposition 2.1. Thus if $\dim V_q = W(n, q)$ for each $q \leqslant p$ and hence $\dim \mathcal{L}_q(\pi_1(X)) \otimes \mathbb{Q} = W(n, q)$, then $\mathcal{L}_q(\pi_1(X))$ must be a free abelian group of rank $W(n, q)$, which means that $\mathcal{L}(\pi_1(X))$ is isomorphic to $\mathcal{L}(F_n)$ up to the pth stage. The result follows from Lemma 2.2.

The next lemma will be used in § 5.

LEMMA 3.4. *Suppose that* $\mathcal{L}_1(\pi_1(X))$ *admits a system of generators as in Lemma* 2.5. *Then if* $\dim V_p = W(n, p)$ *for all* $p \geqslant 2$, *then* $\mathcal{L}(G) \otimes \mathbb{Q}$ *is isomorphic to* $\mathcal{L}(F_n \times \mathbb{Z}^r) \otimes \mathbb{Q}$.

Proof. Since $\dim V_p = \dim \mathcal{L}_p(\pi_1(X)) \otimes \mathbb{Q}$, this is a corollary of Lemma 2.5.

LEMMA 3.5. *Suppose that $\mathscr{L}_1(\pi_1(X))$ is a free abelian group of rank $n+r$ and admits a system of generators as in Lemma 2.4. Then if* $\dim V_p = W(n, p)$ *for all $p \geqslant 2$, then $\mathscr{L}(\pi_1(X))$ is isomorphic to $\mathscr{L}(F_n \times \mathbb{Z}^r)$.*

Proof. Since $\dim V_p = W(n, p)$ means that $\mathscr{L}_p(\pi_1(X))$ is a free abelian group of rank $W(n, p)$ in this case, this is a corollary of Lemma 2.4.

§ 4. The 1-minimal model for $S^1 \vee \cdots \vee S^1$

Our goal of this section is to construct the 1-minimal model for a cohomology bouquet of n circles.

Let A_p be the vector space over $\mathbb{Q}$ generated by the n^p elements, $x_{i_1 \cdots i_p}$'s where $i_1 \cdots i_p$ ranges over all sequences of integers of length p such that $1 \leqslant i_j \leqslant n$ for all $1 \leqslant j \leqslant p$. Consider the exterior algebra of the direct sum $A = \bigoplus_{p \geqslant 1} A_p$. We define the differential d by

$$dx_{i_1 \cdots i_p} = \sum_{k=1}^{p-1} x_{i_1 \cdots i_k} \wedge x_{i_{k+1} \cdots i_p},$$

on a basis of A and extend it linearly to all of A and then extend it to all of $\Lambda(A)$ by the Leibnitz rule. Then

LEMMA 4.1. $d^2 = 0$

Proof. It suffices to check this for a generator.

$$\begin{aligned} d(dx_{i_1 \cdots i_p}) &= d\left(\sum_{k=1}^{p-1} x_{i_1 \cdots i_k} \wedge x_{i_{k+1} \cdots i_p}\right) \\ &= \sum_{k=1}^{p-1} \sum_{m=1}^{k-1} x_{i_1 \cdots i_m} \wedge x_{i_{m+1} \cdots i_k} \wedge x_{i_{k+1} \cdots i_p} \\ &\quad - \sum_{k=1}^{p-1} \sum_{m=k+1}^{p-1} x_{i_1 \cdots i_k} \wedge x_{i_{k+1} \cdots i_m} \wedge x_{i_{m+1} \cdots i_p} \\ &= \left(\sum_{m=1}^{p-1} \sum_{k=1}^{m-1} - \sum_{k=1}^{p-1} \sum_{m=k+1}^{p-1}\right) x_{i_1 \cdots i_k} \wedge x_{i_{k+1} \cdots i_m} \wedge x_{i_{m+1} \cdots i_p} \\ &= 0. \end{aligned}$$

Let I_p be the subspace of A_p $(p \geqslant 2)$ inductively defined by $\{u \in A_p;\ du = 0$ in $\Lambda^2(A_1 \oplus A_2/I_2 \oplus \cdots \oplus A_{p-1}/I_{p-1})\}$, denote A_p/I_p by $\bar{A}_p$ and also denote $A_1 \oplus$

$\bar{A}_2 \oplus \cdots \oplus \bar{A}_p$ by $\bar{\bar{A}}_p$. Then, $\mathcal{M}_p = \Lambda(\bar{\bar{A}}_p)$ with the induced differential (we use the same symbol d) produces a series of Hirsch extensions of minimal d.g.a.'s:

$$\mathbb{Q} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \bigcup_{p \geq 1} \mathcal{M}_p = \mathcal{M}$$

of degree 1. Our first claim is

LEMMA 4.2. *The inclusion induces an isomorphism*: $H^1(\mathcal{M}_{p-1}) \to H^1(\mathcal{M}_p)$ *for all* $p > 1$.

Proof. We use the induction on p. Suppose that this is true for $p-1$, which means that any closed 1-form of $\mathcal{M}_{p-1}$ is contained in $\mathcal{M}_1$. Now, $\mathcal{M}_p = \mathcal{M}_{p-1} \otimes \Lambda(\bar{A}_p)$ as a vector space and since I_p is nothing but the kernel of $d|_{A_p}: A_p \to \Lambda^2(\bar{\bar{A}}_{p-1})$, the induced differential $d|_{\bar{A}_p}: \bar{A}_p \to \Lambda^2(\bar{\bar{A}}_{p-1})$ is injective. The image of this is contained in $\oplus_{i+j=p} \bar{A}_i \wedge \bar{A}_j$, however the image of $\Lambda^1(\bar{\bar{A}}_{p-1})$ by d is contained in $\oplus_{i+j<p} \bar{A}_i \wedge \bar{A}_j$, and they have no common points except zero. In other words, the Hirsch extension $\mathcal{M}_{p-1} \subset \mathcal{M}_p$ does not produce new closed 1-forms.

Let W_p be the image of A_p by d in $\Lambda^2(\bar{\bar{A}}_{p-1})$. That is to say, W_p is a subspace of $\Lambda^2(\bar{\bar{A}}_{p-1})$ generated by the closed 2-forms, $\sum_{k=1}^{p-1} x_{i_1 \cdots i_k} \wedge x_{i_{k+1} \cdots i_p}$'s. Since the subspace of exact forms in $\Lambda^2(\bar{\bar{A}}_{p-1})$ is contained in $\oplus_{i+j<p} \bar{A}_i \wedge \bar{A}_j$, W_p can be identified with a subspace of $H^2(\mathcal{M}_{p-1})$, and also since W_p is the image of A_p by d, it is mapped to 0 in $H^2(\mathcal{M}_p)$ by the inclusion.

LEMMA 4.3. $\dim \bar{A}_p \geq W(n, p)$.

Proof. Define the multiplication $\cdot$ on A by

$$x_{i_1 \cdots i_p} \cdot x_{j_1 \cdots j_q} = x_{i_1 \cdots i_p j_1 \cdots j_q}.$$

Then, A becomes an associative but not commutative graded algebra. The usual bracket operation on A:

$$[x, y] = x \cdot y - y \cdot x,$$

defines a graded Lie algebra structure on A. Let L be the graded Lie subalgebra of A generated by $x_1, \ldots, x_n$. Then, L_p, the intersection of L and A_p, is the set of Lie elements of degree p.

The symmetric group $\mathfrak{S}_p$, which consists of the permutations of integers $1, \ldots, p$, naturally acts on A_p by $\sigma x_{i_1 \cdots i_p} = x_{i_{\sigma(1)} \cdots i_{\sigma(p)}}$ for $\sigma \in \mathfrak{S}_p$. This extends to the

action of the group ring $\mathbb{Q}[\mathfrak{S}_p]$ on A_p. We now define a specific element $\Omega_p \in \mathbb{Q}[\mathfrak{S}_p]$ in terms of the cyclic permutations $\sigma_j = (12 \cdots j)$ for $j = 2, \ldots, p$, by

$$\Omega_p = (1-\sigma_2)(1-\sigma_3^2) \cdots (1-\sigma_p^{p-1}).$$

Ω_p then determines a linear mapping: $A_p \to A_p$.

It is known that $\mathscr{L}_p(F_n)$ is generated by simple brackets $(g_{i_1} \ldots g_{i_p})$, where $g_1, \ldots, g_n$ are generators of F_n, and the mapping:

$$\mathscr{L}_p(F_n) \otimes \mathbb{Q} \to L_p \subset A_p$$

$$(g_{i_1} \cdots g_{i_p}) \qquad \Omega_p x_{i_1 \cdots i_p}$$

is an isomorphism. See for instance [7], Theorem 5.12. In particular the linear mapping Ω_p maps A_p onto L_p and we have

(1) $\operatorname{rank} \Omega_p = \dim L_p = W(n, p)$.

Take an element $u = \sum_{i_1 \cdots i_p} a^{i_1 \cdots i_p} x_{i_1 \cdots i_p}$ of A_p where the summation ranges over all sequences of length p, and let us compute a necessary condition for $du = 0$ in $\Lambda^2(\bar{\bar{A}}_{p-1})$, i.e. $u \in I_p$. Suppose that $du = 0$ there. Then since

$$\begin{aligned} du &= \sum_{i_1 \cdots i_p} a^{i_1 \cdots i_p} dx_{i_1 \cdots i_p} \\ &= \sum_{i_1 \cdots i_p} a^{i_1 \cdots i_p} (x_{i_1} \wedge x_{i_2 \cdots i_p} + \cdots + x_{i_1 \cdots i_{p-1}} \wedge x_{i_p}) \\ &= \sum_{i_p} \Big(\sum_{i_1 \cdots i_{p-1}} ((1-\sigma_p^{p-1}) a^{i_1 \cdots i_p}) x_{i_1 \cdots i_{p-1}} \Big) \wedge x_{i_p} \\ &\quad + \Big(\text{the terms contained in } \bigoplus_{i,j \geq 2} \bar{A}_i \wedge \bar{A}_j \Big), \end{aligned}$$

if we let

$$u_{i_p} = \sum_{i_1 \cdots i_{p-1}} ((1-\sigma_p^{p-1}) a^{i_1 \cdots i_p}) x_{i_1 \cdots i_{p-1}},$$

u_{i_p} must be an element of I_{p-1} for all $1 \leq i_p \leq n$. Repeating the same procedure $p-1$ times, we eventually obtain the condition that $\Omega_p a^{i_1 \cdots i_p} = (1-\sigma_2)(1-\sigma_3^2) \cdots (1-\sigma_p^{p-1}) a^{i_1 \cdots i_p} = 0$ for all sequences $i_1 \cdots i_p$.

We now think of the conjugate element $\bar{\Omega}_p$ of $\Omega_p \in \mathbb{Q}[\mathfrak{S}_p]$ by the conjugation $\sigma \leftrightarrow \sigma^{-1}$. Again $\bar{\Omega}_p$ determines a linear mapping: $A_p \to A_p$ which can be iden-

tified with the induced mapping of Ω_p on the dual space $A_p^* = \mathrm{Hom}\,(A_p, \mathbb{Q})$, and we have

$$\bar{\Omega}_p u = \sum_{i_1 \cdots i_p} a^{i_1 \cdots i_p} \bar{\Omega}_p x_{i_1 \cdots i_p}$$

$$= \sum_{i_1 \cdots i_p} (\Omega_p a^{i_1 \cdots i_p}) x_{i_1 \cdots i_p}.$$

Suppose that $d\bar{\Omega}_p u = 0$ in $\Lambda^2(\bar{\bar{A}}_{p-1})$, then recalling the formula $\Omega_p^2 = p\Omega_p$ in $\mathbb{Q}[\mathfrak{S}_p]$ (see [7], p. 365) and the necessary condition above, we get

$$\Omega_p(\Omega_p a^{i_1 \cdots i_p}) = p\Omega_p a^{i_1 \cdots i_p} = 0$$

for all sequences $i_1 \cdots i_p$. This means nothing but $\bar{\Omega}_p u$ being zero itself and hence the restriction of d to the image of $\bar{\Omega}_p$, $d|_{\bar{\Omega}_p(A_p)} : \bar{\Omega}_p(A_p) \to W_p \subset \Lambda^2(\bar{\bar{A}}_{p-1})$, is injective. In particular we have

(2) $\dim \bar{A}_p = \dim W_p \geqslant \dim \bar{\Omega}_p(A_p)$.

On the other hand, we have

(3) $\dim \bar{\Omega}_p(A_p) = \mathrm{rank}\, \bar{\Omega}_p = \mathrm{rank}\, \Omega_p$.

Combining (1), (2) and (3), we complete the proof.

The main result of this section is

THEOREM 4.4. *Let X be a polyhedron whose cohomology ring with rational coefficients is isomorphic to $H^*(S^1 \vee \cdots \vee S^1; \mathbb{Q})$. Then $\mathcal{M}: \mathbb{Q} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$ is isomorphic to the canonical series of the* 1*-minimal model $\mathcal{M}_X$ for X.*

Proof. We prove this by induction on the length of a series. Suppose that $\mathbb{Q} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M}_{p-1}$ is isomorphic to the $p-1$st stage of the canonical series of $\mathcal{M}_X$. Then we have a d.g.a. mapping $\rho_{p-1} : \mathcal{M}_{p-1} \to \varepsilon(X)$ such that $\rho_{p-1}(x_{i_1 \cdots i_q}) = \omega_{i_1 \cdots i_q}$ for $q \leqslant p-1$, and $H^2(\mathcal{M}_{p-2}) \to H^2(\mathcal{M}_{p-1})$ is a zero map by the property of a canonical series, Lemma 3.1, (2). Also since $\dim H^1(\mathcal{M}_1) = n$ and $H^2(\mathcal{M}_{p-1})$ is generated by decomposable elements, Sullivan's theorem implicitly says that $\dim H^2(\mathcal{M}_{p-1})$ cannot exceed $\dim \mathcal{L}_p(F_n) \otimes \mathbb{Q} = W(n, p)$, that is

(1) $W(n, p) \geqslant \dim H^2(\mathcal{M}_{p-1})$.

To see that $\mathcal{M}_{p-1} \subset \mathcal{M}_p$ is isomorphic to the pth stage of the canonical series of $\mathcal{M}_X$, we need to construct a d.g.a. mapping $\rho_p : \mathcal{M}_p \to \varepsilon(X)$ and to show that the restriction of $H^2(\mathcal{M}_p) \to H^2(\varepsilon(X))$ to the image of $H^2(\mathcal{M}_{p-1})$ is injective. First of

all, since $\rho_{p-1}(\sum_{k=1}^{p-1} x_{i_1\cdots i_k} \wedge x_{i_{k+1}\cdots i_p}) = \sum_{k=1}^{p-1} \omega_{i_1\cdots i_k} \wedge \omega_{i_{k+1}\cdots i_p}$ is a closed 2-form of $\varepsilon(X)$ and $H^2(\varepsilon(X)) = 0$, there exists a 1-form $\omega_{i_1\cdots i_p}$ of $\varepsilon(X)$ so that

$$d\omega_{i_1\cdots i_p} = \sum_{k=1}^{p-1} \omega_{i_1\cdots i_k} \wedge \omega_{i_{k+1}\cdots i_p}.$$

We define $\rho_p : \mathcal{M}_p \to \varepsilon(X)$ as an extension of ρ_{p-1} by mapping $x_{i_1\cdots i_p}$ to $\omega_{i_1\cdots i_p}$. Since $d|_{\bar{A}_p} : \bar{A}_p \to W_p$ is an isomorphism and W_p can be identified with a subspace of $H^2(\mathcal{M}_{p-1})$, by Lemma 4.3 we have

$$\dim H^2(\mathcal{M}_{p-1}) \geqslant \dim W_p = \dim \bar{A}_p \geqslant W(n, p).$$

These inequalities become equalities by (1) and W_p can be identified with $H^2(\mathcal{M}_{p-1})$ itself. Since W_p was the image of A_p by d, $H^2(\mathcal{M}_{p-1}) \to H^2(\mathcal{M}_p)$ turns out a zero mapping, and we are done by induction.

Remark. Since $\dim \bar{A}_p = W(n, p)$, $\mathbb{Q}$-nil$(\pi_1(X))$ turns out to be isomorphic to $\mathbb{Q}$-nil(F_n) by Lemma 3.2. This consequence also follows from the result of [16].

§ 5. The 1-minimal model for $(S^1 \vee \cdots \vee S^1) \times S^1$

In this section, we consider a family of minimal models for polyhedra which are cohomologically equivalent to the product of a bouquet of n circles with a circle.

We define the vector space B_1 over $\mathbb{Q}$ by adding one more generator, x_{n+1}, to A_1 and let B_p be equal to A_p for $p \geqslant 2$. The specific basis, $x_{i_1\cdots i_p}$'s, of A_p determines a homomorphism: $A_p \to A_p^* = \mathrm{Hom}(A_p, \mathbb{Q})$ and let I_p^* be the image of I_p by this mapping. Consider the subset $\Delta_p = \{x \in A_p;\ f(x) = 0 \text{ for all } f \in I_p^*\}$. Choose n elements from Δ_p for each $p \geqslant 3$ to form a set θ which will be called a system of twisting coefficients. Let us denote by $\theta_j^{i_1\cdots i_p}$ the coefficient of the $x_{i_1\cdots i_p}$-component of the jth element of degree p in θ. The system of twisting coefficients with $\theta_j^{i_1\cdots i_p} = 0$ for all $1 \leqslant j \leqslant n$, $1 \leqslant i_1 \cdots i_p \leqslant n$ and $p \geqslant 3$, will be denoted by 0. We now define the differential d_θ by

$$d_\theta x_{i_1\cdots i_p} = \sum_{k=1}^{p-1} x_{i_1\cdots i_k} \wedge x_{i_{k+1}\cdots i_p} - \sum_{j=1}^{n} \sum_{m=2}^{p-1} \sum_{k=1}^{p-m} \theta_j^{i_k\cdots i_{k+m}} x_{i_1\cdots i_{k-1} j i_{k+m+1}\cdots i_p} \wedge x_{n+1}$$

on a basis of $B = \oplus_{p \geqslant 1} B_p$ first of all and extend it to all of the exterior algebra $\Lambda(B)$ by linearity and the Leibnitz rule. Notice that d_0 is the same as d in § 4 for $p \geqslant 2$.

LEMMA 5.1. $d_\theta^2 = 0$

Proof. It suffices to show this for a generator.

$$d_\theta^2 x_{i_1\cdots i_p} = d_\theta(d_0 x_{i_1\cdots i_p}) - \sum_{j=1}^{n}\sum_{m=2}^{p-1}\sum_{k=1}^{p-m} \theta_j^{i_k\cdots i_{k+m}}\, d_\theta(x_{i_1\cdots i_{k-1}ji_{k+m+1}\cdots i_p}\wedge x_{n+1}).$$

Since $d_0^2 = 0$ by Lemma 4.1, the first term of the right side becomes

$$\begin{aligned} &d_\theta(d_0 x_{i_1\cdots i_p}) \\ &= \sum_{s=1}^{p-1}((d_\theta - d_0)x_{i_1\cdots i_s}\wedge x_{i_{s+1}\cdots i_p} - x_{i_1\cdots i_s}\wedge(d_\theta - d_0)x_{i_{s+1}\cdots i_p}) \\ &= \sum_{s=1}^{p-1}\sum_{j=1}^{n}\Bigg(\sum_{m=2}^{s-1}\sum_{k=1}^{s-m}\theta_j^{i_k\cdots i_{k+m}} x_{i_1\cdots i_{k-1}ji_{k+m+1}\cdots i_s}\wedge x_{i_{s+1}\cdots i_p} \\ &\quad + \sum_{m=2}^{p-s-1}\sum_{k=s+1}^{p-m}\theta_j^{i_k\cdots i_{k+m}} x_{i_1\cdots i_s}\wedge x_{i_{s+1}\cdots i_{k-1}ji_{k+m+1}\cdots i_p}\Bigg)\wedge x_{n+1}. \end{aligned}$$

On the other hand, since

$$\begin{aligned} &d_\theta(x_{i_1\cdots i_{k-1}ji_{k+m+1}\cdots i_p}\wedge x_{n+1}) \\ &= d_0 x_{i_1\cdots i_{k-1}ji_{k+m+1}\cdots i_p}\wedge x_{n+1} \\ &= \Bigg(\sum_{s=1}^{k-1} x_{i_1\cdots i_s}\wedge S_{i_{s+1}\cdots i_{k-1}ji_{k+m+1}\cdots i_p} \\ &\quad + \sum_{s=k+m}^{p-1} x_{i_1\cdots i_{k-1}ji_{k+m+1}\cdots i_s}\wedge x_{i_{s+1}\cdots i_p}\Bigg)\wedge x_{n+1}, \end{aligned}$$

the second term of the first identity becomes

$$\begin{aligned} &\sum_{j=1}^{n}\sum_{m=2}^{p-1}\sum_{k=1}^{p-m}\theta_j^{i_k\cdots i_{k+m}} d_\theta(x_{i_1\cdots i_{k-1}ji_{k+m+1}\cdots i_p}\wedge x_{n+1}) \\ &= \sum_{j=1}^{n}\sum_{m=2}^{p-1}\sum_{k=1}^{p-m}\theta_j^{i_k\cdots i_{k+m}}\Bigg(\sum_{s=1}^{k-1} x_{i_1\cdots i_s}\wedge x_{i_{s+1}\cdots i_{k-1}ji_{k+m+1}\cdots i_p} \\ &\quad + \sum_{s=k+m}^{p-1} x_{i_1\cdots i_{k-1}ji_{k+m+1}\cdots i_s}\wedge x_{i_{s+1}\cdots i_p}\Bigg)\wedge x_{n+1} \end{aligned}$$

Thus since

$$\sum_{s=1}^{p-1}\sum_{m=2}^{s-1}\sum_{k=1}^{s-m} = \sum_{m=2}^{p-1}\sum_{k=1}^{p-m}\sum_{s=k+m}^{p-1}$$

and

$$\sum_{s=1}^{p-1}\sum_{m=2}^{p-s-1}\sum_{k=s+1}^{p-m} = \sum_{m=2}^{p-1}\sum_{k=1}^{p-m}\sum_{s=1}^{k-1},$$

both are cancelled each other and we are done.

Let J_p be the subspace of B_p $(p \geqslant 2)$ inductively defined by $\{u \in B_p;\, d_\theta u = 0$ in $\Lambda^2(B_1 \oplus B_2/J_2 \oplus \cdots \oplus B_{p-1}/J_{p-1})\}$. We simply denote it without θ because J_p actually does not depend on θ as we will see in Lemma 5.3. Again denote B_p/J_p by $\bar{B}_p$ and $B_1 \oplus \bar{B}_2 \oplus \cdots \oplus \bar{B}_p$ by $\bar{\bar{B}}_p$. Then $\mathcal{M}_p^\theta = \Lambda(\bar{\bar{B}}_p)$ with the induced differential (we use the same symbol d_θ) produces a series of Hirsch extensions of d.g.a.'s:

$$\mathbb{Q} \subset \mathcal{M}_1^\theta \subset \mathcal{M}_2^\theta \subset \cdots$$

of degree 1. Let us denote $\bigcup_{p \geqslant 1} \mathcal{M}_p^\theta$ by $\mathcal{M}^\theta$. Then

LEMMA 5.2. *The inclusion induces an isomorphism:* $H^1(\mathcal{M}_{p-1}^\theta) \to H^1(\mathcal{M}_p^\theta)$ *for all* $p \geqslant 2$.

LEMMA 5.3. J_p *is equal to* I_p. *In other words,* $d_0 u = 0$ *in* $\Lambda^2(\bar{\bar{B}}_{p-1})$ *for some* $u \in B_p$ *iff* $d_\theta u = 0$ *in* $\Lambda^2(\bar{\bar{B}}_{p-1})$. *In particular,* $\dim \bar{B}_p = \dim \bar{A}_p = W(n, p)$ *for* $p \geqslant 2$.

Both lemmas are obvious when $p = 2$. We prove these by mixed induction on p. Let us assume that both are true for $p-1$.

Proof of Lemma 5.3. Suppose that u is an element of B_p so that $d_0 u = 0$ in $\Lambda^2(\bar{\bar{B}}_{p-1})$. By the definition of d_θ, we can decompose $d_\theta u$ as

$$d_\theta u = d_0 u + \nabla \wedge x_{n+1}$$

where ∇ is an element of $\Lambda^1(\bar{\bar{B}}_{p-1})$. Since $d_\theta^2 = 0$ and $d_0 u = 0$, we have

$$0 = d_\theta^2 u = d_\theta(\nabla \wedge x_{n+1}) = d_\theta \nabla \wedge x_{n+1} = d_0 \nabla \wedge x_{n+1},$$

which implies that $d_0 \nabla = 0$. Because ∇ was in $\Lambda^1(\bar{\bar{B}}_{p-1})$, we get $d_\theta \nabla = 0$ by induction hypothesis. Since we also assumed that Lemma 5.2 is true for $p-1$, any closed 1-form of $\mathcal{M}_{p-1}^\theta$ is contained in $\mathcal{M}_1^\theta$, in particular so is ∇. Therefore if we let $\theta_j^p = \sum_{i_1 \cdots i_p} \theta_j^{i_1 \cdots i_p} x_{i_1 \cdots i_p}$, then $\nabla = \sum_{j=1}^n u^*(\theta_j^p) x_j$. However since $u^* \in I_p^*$ and $\theta_j^p \in \Delta_p$, $u^*(\theta_j^p)$ must be zero for all $1 \leqslant j \leqslant n$, which means that $\nabla = 0$ itself. The converse is obvious and we are done

Proof of Lemma 5.2. Let $U_p^\theta \subset \Lambda^2(\bar{\bar{B}}_{p-1})$ be the image of B_p by d_θ. W_p of the last section can be naturally identified with a subspace of $\bigoplus_{i+j=p} \bar{B}_i \wedge \bar{B}_j$, and we have the commutative diagram:

$$\begin{tikzcd} B_p \arrow[r, "d_\theta"] \arrow[dr, "d_0"'] & U_p^\theta \subset \Lambda^2(\bar{\bar{B}}_{p-1}) \arrow[d] \\ & W_p \subset \bigoplus_{i+j=p} \bar{B}_i \wedge \bar{B}_j \end{tikzcd}$$

where the vertical line is the projection to the direct summand. Then since J_p is the kernel of $d_\theta|_{B_p}: B_p \to \Lambda^2(\bar{\bar{B}}_{p-1})$ and J_p is equal to I_p by Lemma 5.3,

$$\begin{tikzcd} \bar{B}_p \arrow[r, "d_\theta"] \arrow[dr, "d_0"'] & U_p^\theta \subset \Lambda^2(\bar{\bar{B}}_{p-1}) \arrow[d] \\ & W_p \subset \bigoplus_{i+j=p} \bar{B}_i \wedge \bar{B}_j \end{tikzcd}$$

becomes the commutative diagram of isomorphisms. In particular, U_p^θ and $\bigoplus_{i+j<p} \bar{B}_i \wedge \bar{B}_j$ have no common points except zero. And since $\mathcal{M}_p^\theta = \mathcal{M}_{p-1}^\theta \otimes \Lambda(\bar{B}_p)$ and the image of $\Lambda^1(\bar{\bar{B}}_{p-1})$ by d_θ is contained in $\bigoplus_{i+j<p} \bar{B}_i \wedge \bar{B}_j$, the Hirsch extension $\mathcal{M}_{p-1}^\theta \subset \mathcal{M}_p^\theta$ produces no new closed 1-forms.

LEMMA 5.4. *The image of* $H^2(\mathcal{M}_1^\theta) \to H^2(\mathcal{M}_2^\theta)$ *is injectively mapped to* $H^2(\mathcal{M}^\theta)$.

Proof. U_p^θ and $\bigoplus_{i+j<p} \bar{B}_i \wedge \bar{B}_j$ have no common points except zero, and hence the new exact 2-forms of $\mathcal{M}_p^\theta$ have no common points with $\bigoplus_{i+j=2} \bar{B}_i \wedge \bar{B}_j$ except zero for $p \geqslant 3$. Since the image of $H^2(\mathcal{M}_1^\theta) \to H^2(\mathcal{M}_2^\theta)$ is actually generated by $x_i \wedge x_{n+1}$'s by the definition of d_θ, these do not become exact in $\mathcal{M}_p^\theta$ for any $p \geqslant 3$ and hence in $\mathcal{M}^\theta = \bigcup_{p \geqslant 1} \mathcal{M}_p^\theta$

The main theorem of this section is

THEOREM 5.5. *Let X be a polyhedron whose cohomology ring with rational coefficients is isomorphic to* $H^*((\overbrace{S^1 \vee \cdots \vee S^1}^{n}) \times S^1; \mathbb{Q})$. *Then there exists a system of twisting coefficients θ so that* $\mathcal{M}^\theta: \mathbb{Q} \subset \mathcal{M}_1^\theta \subset \mathcal{M}_2^\theta \subset \cdots$ *is isomorphic to the canonical series of the* 1-*minimal model* $\mathcal{M}_X$ *for* X.

Proof. By the assumption, there are 1-forms $\omega_1, \ldots, \omega_{n+1}$ of $\varepsilon(X)$ which

generate $H^1(\varepsilon(X))$ such that

(i) $[\omega_i \wedge \omega_j] = 0$ for all $i, j \leqslant n$ and

(ii) $[\omega_i \wedge \omega_{n+1}]$'s form a basis of $H^2(\varepsilon(X))$.

The dual basis $g_1, \ldots, g_{n+1}$ of $\mathscr{L}_1(\pi_1(X))$ with respect to $\omega_1, \ldots, \omega_{n+1}$ satisfies the conditions of Lemma 2.5. We then prove this theorem by induction on the length of a series.

Suppose that $\mathbb{Q} \subset \mathcal{M}_1^\theta \subset \cdots \subset \mathcal{M}_{p-1}^\theta$ is isomorphic to the $p-1$st stage of the canonical series of $\mathcal{M}_X$ for some θ. Notice that since $\mathcal{M}_{p-1}^\theta = \Lambda(\bar{\bar{B}}_{p-1})$, we only need a system of twisting coefficients up to degree $p-1$. Then we have a d.g.a. mapping $\rho_{p-1}: \mathcal{M}_{p-1}^\theta \to \varepsilon(X)$ so that $\rho_{p-1}(x_{i_1 \cdots i_q}) = \omega_{i_1 \cdots i_q}$ for $q \leqslant p-1$, and the image of $H^2(\mathcal{M}_{p-2}^\theta) \to H^2(\mathcal{M}_{p-1}^\theta)$ is equal to the image of $H^2(\mathcal{M}_1^\theta) \to H^2(\mathcal{M}_{p-1}^\theta)$ by the inclusions because of Lemma 3.1, (2), Lemma 5.4 and the structure of $H^2(\varepsilon(X))$. Also by Lemma 2.5 and Sullivan's theorem, we have

(1) $W(n, p) \geqslant \dim H^2(\mathcal{M}_{p-1}^\theta) - n$

where n means the dimension of $H^2(\varepsilon(X))$.

To see that $\mathcal{M}_{p-1}^\theta \subset \mathcal{M}_p^\theta$ for some θ is isomorphic to the pth stage of the canonical series of $\mathcal{M}_X$, we need to find appropriate n elements of Δ_p for θ, to construct a d.g.a. mapping $\rho_p: \mathcal{M}_p^\theta \to \varepsilon(X)$ and to show that the restriction of $H^2(\mathcal{M}_p^\theta) \to H^2(\varepsilon(X))$ to the image of $H^2(\mathcal{M}_{p-1}^\theta)$ is injective. First of all, since

$$\begin{aligned}
&\rho_{p-1}\Bigg(\sum_{k=1}^{p-1} x_{i_1 \cdots i_k} \wedge x_{i_{k+1} \cdots i_p} \\
&\quad - \sum_{j=1}^{n} \left(\sum_{m=2}^{p-1} \sum_{k=1}^{p-m} - \begin{pmatrix} k=1 \text{ and} \\ m=p-1 \end{pmatrix} \right) \theta_j^{i_k \cdots i_{k+m}} x_{i_1 \cdots i_{k-1} j i_{k+m+1} \cdots i_p} \wedge x_{n+1} \Bigg) \\
&= \sum_{k=1}^{p-1} \omega_{i_1 \cdots i_k} \wedge \omega_{i_{k+1} \cdots i_p} \\
&\quad - \sum_{j=1}^{n} \left(\sum_{m=2}^{p-1} \sum_{k=1}^{p-m} - \begin{pmatrix} k=1 \text{ and} \\ m=p-1 \end{pmatrix} \right) \theta_j^{i_k \cdots i_{k+m}} \omega_{i_1 \cdots i_{k-1} j i_{k+m+1} \cdots i_p} \wedge \omega_{n+1}
\end{aligned}$$

is a closed 2-form of $\varepsilon(X)$ for each $i_1 \cdots i_p$, and $H^2(\varepsilon(X))$ is generated by $\omega_i \wedge \omega_{n+1}$'s, it is cohomologous to

$$\sum_{j=1}^{n} \theta_j^{i_1 \cdots i_p} \omega_j \wedge \omega_{n+1}$$

for some $\{\theta_j^{i_1 \cdots i_p}\}_{j=1}^n$. For each j, $\sum_{i_1 \cdots i_p} \theta_j^{i_1 \cdots i_p} x_{i_1 \cdots i_p} \in A_p$ must be contained in Δ_p because ρ_{p-1} is a d.g.a. mapping. Adding $\theta_j^{i_1 \cdots i_p}$'s to θ, we get a system of twisting

coefficients up to degree p. Then

$$\sum_{k=1}^{p-1} \omega_{i_1\cdots i_k} \wedge \omega_{i_{k+1}\cdots i_p} - \sum_{j=1}^{n} \sum_{m=2}^{p-1} \sum_{k=1}^{p-m} \theta_j^{i_1\cdots i_p} \omega_{i_1\cdots i_{k-1} j i_{k+m+1}\cdots i_p} \wedge \omega_{n+1}$$

becomes an exact form and there exists a 1-form $\omega_{i_1\cdots i_p}$ of $\varepsilon(X)$ such that $d\omega_{i_1\cdots i_p}$ is equal to it, where d is the differential of $\varepsilon(X)$. Mapping $x_{i_1\cdots i_p}$ to $\omega_{i_1\cdots i_p}$, we define $\rho_p : \mathcal{M}_p^\theta \to \varepsilon(X)$ as an extension of ρ_{p-1}.

We finally show that the image of $H^2(\mathcal{M}_{p-1}^\theta) \to H^2(\mathcal{M}_p^\theta)$ is equal to the image of $H^2(\mathcal{M}_1^\theta) \to H^2(\mathcal{M}_p^\theta)$ because if so, the proof is completed by Lemma 5.4. Since U_p^θ can be identified with a subspace of $H^2(\mathcal{M}_{p-1}^\theta)$ and has no common points with the image of $H^2(\mathcal{M}_1^\theta)$ except zero, we have by Lemma 5.3 that

$$\dim H^2(\mathcal{M}_{p-1}^\theta) - n \geqslant \dim U_p^\theta = \dim \bar{B}_p = W(n, p).$$

Thus by (1), the inequality becomes an equality and $H^2(\mathcal{M}_{p-1}^\theta)$ can be identified with the direct sum of the image of $H^2(\mathcal{M}_1^\theta)$ and U_p^θ. Since U_p^θ is the image of B_p by d_θ, the image of $H^2(\mathcal{M}_{p-1}^\theta) \to H^2(\mathcal{M}_p^\theta)$ turns out the image of $H^2(\mathcal{M}_1^\theta) \to H^2(\mathcal{M}_p^\theta)$, and we are done.

Here are corollaries of Theorem 5.5, Lemma 3.4 and Lemma 3.5.

COROLLARY 5.6. *Let X be a polyhedron such that $H^*(X; \mathbb{Q})$ is isomorphic to $H^*((S^1 \vee \cdots \vee S^1) \times S^1; \mathbb{Q})$ as a ring. Then $\mathcal{L}(\pi_1(X)) \otimes \mathbb{Q}$ is isomorphic to $\mathcal{L}(F_n \times \mathbb{Z}) \otimes \mathbb{Q}$.*

COROLLARY 5.7. *Let X be a polyhedron as in Corollary 5.6. If $\mathcal{L}_p(\pi_1(X))$ is free abelian for $p = 1$ and 2, then $\mathcal{L}(\pi_1(X))$ is isomorphic to $\mathcal{L}(F_n \times \mathbb{Z})$.*

Proof. Since $\mathcal{L}_1(\pi_1(X))$ is free abelian, we can choose a set of generators of $\mathcal{L}_1(\pi_1(X))$ as in Lemma 2.5. Furthermore since $\mathcal{L}_2(\pi_1(X))$ is also free abelian, it satisfies the conditions in Lemma 2.4. Thus this is an corollary of Theorem 5.5 and Lemma 3.5.

Remark. The condition of this corollary seems equivalent to saying that X is an integral cohomology $(S^1 \vee \cdots \vee S^1) \times S^1$ while I have no proof for this.

§ 6. Applications

To state corollaries of Theorem 4.4, following Kraines [6], let us define the Massey product on the first cohomology group. Given elements $\gamma_1, \ldots, \gamma_p \in H^1(\varepsilon(X))$, suppose that a collection of 1-forms $S = \{\omega_{ij} \in \varepsilon(X);\ 1 \leq i \leq j \leq p,\ j-i < p-1\}$ satisfies the conditions

(1) ω_{ii} is a closed form representing γ_i for $1 \leq i \leq p$, and

(2) $d\omega_{ij} = \sum_{k=i}^{j-1} \omega_{ik} \wedge \omega_{k+1,j}$ if $i < j$.

Then the $\mathbb{Q}$-polynomial 2-form $\sum_{k=1}^{p-1} \omega_{1k} \wedge \omega_{k+1,p}$ turns out to be closed. We call S a defining system. The Massey product $\langle \gamma_1, \ldots, \gamma_p \rangle$ is defined as a subset of $H^2(\varepsilon(X))$ consisting of all elements produced by such systems. When $p = 2$, it is nothing but the wedge (cup) product. The Massey product $\langle \gamma_1, \ldots, \gamma_p \rangle$ will be understood as a cohomology class if it contains a single element. It is known that if any $(p-1)$-tuple Massey product on $H^1(\varepsilon(X))$ vanishes, that is, contains only the zero element, then every p-tuple Massey product contains a single element. See [9], Proposition 2.4 for the proof. We now have equivalent conditions for the vanishing of every p-tuple Massey products.

LEMMA 6.1. *Every p-tuple Massey product vanishes iff every q-tuple Massey product for any $1 < q \leq p$ vanishes.*

Proof. If every p-tuple Massey product vanishes, then for each $1 < q \leq p$, every q-tuple Massey product must contain the zero element. Thus any binary Massey product vanishes because it has no indeterminacy. Assume by induction that every $(q-1)$-tuple Massey product on $H^1(\varepsilon(X))$ vanishes, then every q-tuple Massey product contains a single element which is zero and we are done.

Here are corollaries of Theorem 4.4.

COROLLARY 6.2. *Let X be a polyhedron of* $\dim H^1(\varepsilon(X)) = n$. *Then, every p-tuple Massey product on $H^1(\varepsilon(X))$ vanishes iff $\mathbb{Q}$-nil$(\pi_1(X))$ is isomorphic to* $\mathbb{Q}$-nil(F_n) *up to the* pth *stage.*

Proof. To construct the 1-minimal model for X, we can use the vanishing of Massey products instead of the vanishing of $H^2(\varepsilon(X))$. Actually, the closed 2-forms in $\mathcal{M}_q$ were generated by $\sum_{k=1}^{q-1} x_{i_1 \cdots i_k} \wedge x_{i_{k+1} \cdots i_q}$'s for $q \leq p$ which are mapped to $\sum_{k=1}^{q-1} \omega_{i_1 \cdots i_k} \wedge \omega_{i_{k+1} \cdots i_q}$ of $\varepsilon(X)$ by ρ_q. This is nothing but the Massey product $\langle \omega_{i_1}, \ldots, \omega_{i_q} \rangle$.

Conversely, suppose that, for some $q \leq p$, some q-tuple Massey product does not vanish while every r-tuple Massey product does vanish for all $1 < r < q$. Then

$\rho_q^*: H^2(\mathcal{M}_q) \to H^2(\varepsilon(X))$ is not a zero map and rank $\mathcal{L}_q(\pi_1(X)) = \dim \operatorname{Ker} \rho_q^*$ is strictly less than $\dim H^2(\mathcal{M}_q) = W(n, q)$. Thus $\mathbb{Q}$-nil$(\pi_1(X))$ cannot be isomorphic to $\mathbb{Q}$-nil(F_n) at the qth stage.

By virtue of Lemma 3.3, we have

COROLLARY 6.3. *Let X be a polyhedron such that $\mathcal{L}_1(\pi_1(X))$ is a free abelian group of rank n, then every p-tuple Massey product on $H^1(\varepsilon(X))$ vanishes iff* Nil$(\pi_1(X))$ *is isomorphic to* Nil(F_n) *up to the* pth *stage.*

Remark. If we start with the Massey product on $H^1(\pi_1(X))$, this has been known by Dwyer [4], Corollary 4.5. I would like to thank the referee for pointing out this reference. For the link complement, there are much more detailed studies by Milnor [10] and Porter [15].

Remark. In [14] and [5], some higher intersectional properties of compact 4-manifolds have been detected by the nilpotent completion and the Massey product respectively. This corollary shows that these results are equivalent.

Let $L = K_1 \cup \cdots \cup K_n$ be a link of n components in S^3. Then $H^1(S^3 - L; \mathbb{Q})$ is generated by the Alexander duals ξ_i to the component K_i for $i = 1, \ldots, n$, and $H^2(S^3 - L; \mathbb{Q})$ is generated by the Lefshetz duals γ_{ij} to the path which connects K_i with K_j. These are subject to the relations in H^*:

$$\xi_i \wedge \xi_j = lk(K_i, K_j)\gamma_{ij}$$

and

$$\gamma_{ij} + \gamma_{jk} = \gamma_{ik}.$$

The next corollary has been conjectured by Murasugi.

COROLLARY 6.4. *Let G be a link group, $\pi_1(S^3 - L)$. If $lk(K_i, K_j) = 1$ for all $i \neq j$, then $\mathcal{L}(G)$ is isomorphic to $\mathcal{L}(F_{n-1} \times \mathbb{Z})$. In particular, rank $\mathcal{L}_p(G) = W(n-1, p)$ for all $p \geq 2$.*

Proof. Let $\omega_i = \xi_i - \xi_n$ for $i = 1, \ldots, n-1$ in this case. Then $\omega_i \wedge \omega_j = 0$ for all $i, j \leq n-1$ and $\omega_i \wedge \xi_n$'s form a basis of $H^2(S^3 - L: \mathbb{Q})$, and hence $S^3 - L$ is clearly a rational cohomology $(S^1 \vee \cdots \vee S^1) \times S^1$. Also $\mathcal{L}_1(\pi_1(X))$ and $\mathcal{L}_2(\pi_1(X))$ are free abelian because of Alexander duality and Chen's computations [3], Corollary 2, respectively. Thus we can apply Corollary 5.7 to this case.

COROLLARY 6.5. *Let L be a link of 3 component such that linking numbers of any two components are zero. Then $\mathscr{L}(G) \otimes \mathbb{Q}$ is isomorphic to $\mathscr{L}(F_2 \times \mathbb{Z}) \otimes \mathbb{Q}$.*

Proof. Let $\omega_1 = lk(K_2, K_3)\xi_1 - lk(K_1, K_2)\xi_3$ and $\omega_2 = lk(K_1, K_3)\xi_2 - lk(K_1, K_2)\xi_3$. Then $\omega_1 \wedge \omega_2 = 0$ and $\omega_1 \wedge \xi_3$, $\omega_2 \wedge \xi_3$ form a basis of $H^2(S^3 - L; \mathbb{Q})$ and hence $S^3 - L$ is a rational cohomology $(S^1 \vee S^1) \times S^1$. Applying Corollary 5.6, we are done.

$\mathscr{L}(G)$ is nilpotent if $\mathscr{L}_p(G) = 0$ for some p. This is equivalent to G/G_ω being nilpotent, where $G_\omega = \bigcap_{p \geqslant 1} G_p$.

COROLLARY 6.6. *Let G be the link group of a link L. Then*

(1) *$\mathscr{L}(G)$ is nilpotent iff either L is a knot or L is of 2 components whose mutual linking number is equal to ± 1.*

(2) *$\mathscr{L}(G) \otimes \mathbb{Q}$ is nilpotent iff either L is a knot or L is of 2 components whose mutual linking number is not zero.*

Proof. When L is a knot, $\mathscr{L}(G)$ is nilpotent of index 1 since $S^3 - L$ is a homology circle. If L has two components, then "if" part is obvious for both cases, (1), (2), because $\mathscr{L}_2(G)$ is isomorphic to a cyclic group of order $= |lk(K_1, K_2)|$. To see "only if" part, recall Murasugi's explicit computation [12] of the Chen groups. That is, roughly speaking, the Chen group $\mathrm{Ch}_p(G) = G_{p-1}[G_1, G_1]/G_p[G_1, G_1]$ of a 2 component link group G is infinite for all $p \geqslant 1$ if $lk(K_1, K_2) = 0$, and is nontrivially finite for all $p \geqslant 2$ if $lk(k_1, K_2) \neq 0, \pm 1$. Since there is an epimorphism of $\mathscr{L}_p(G)$ to $\mathrm{Ch}_p(G)$ for each p, $\mathscr{L}(G)$ cannot be nilpotent except when $lk(K_1, K_2) = \pm 1$. Also $\mathscr{L}(G) \otimes \mathbb{Q}$ cannot be nilpotent except when $lk(K_1, K_2) \neq 0$.

Let us think of the case where L has more than 3 components. When L contains two components whose mutual linking number is zero, then by forgetting the other components, we get a homomorphism of G onto the group of a link of 2 components whose mutual linking number is zero. When the linking numbers of any 2 components of L are not zero, then by forgetting some components, we get a homomorphism of G onto the group of a link of 3 components as in Corollary 6.5. Thus $\mathscr{L}(G) \otimes \mathbb{Q}$ cannot be nilpotent in either case. Of course neither does $\mathscr{L}(G)$ and we are done.

Remark. This remark was pointed out by Murasugi. It can be known by [1] and [12] that a link group itself is nilpotent iff it is abelian. Such a link must be either a trivial knot or a Hopf link by Dehn's lemma and Neuwirth's theorem [13].

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