Quasipsherical knots with infinitely many ends.

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Quasispherical knots with infinitely many ends

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A smooth n-knot K in S^{n+2} is called *quasiaspherical* [3] if $H_{n+1}(U)=0$ where U is the universal cover of the exterior of K. Let G be a finitely generated group such that $G/G' \approx Z$ and let H be a subgroup of G which is not contained in G'. We say that (G, H) is *unsplitable* if G does not have a free product with amalgamation decomposition $A \underset{F}{*} B$ with F finite and H contained in A.[1]

THEOREM 1. *K is quasiaspherical if and only if $(\pi_1(S^{n+2}-K), H)$ is unsplitable, where H is the subgroup generated by a meridian.*

The "only if" part of this theorem was proved by Swarup [7]. A sketch of the "if" part was given in [2]; for the sake of completeness we give the details in § 1.

A knot K has *infinitely many ends* if for each integer m there is a compact set in U whose complement has more than m components with non compact closure. The property of having infinitely many ends depends only on $\pi_1(S^{n+2}-K)$.

THEOREM 2. [5]. *K has infinitely many ends if and only if either*

(i) $\pi_1(S^{n+2}-K)=A \underset{F}{*} B$ *where F is finite; or*

(ii) $\pi_1(S^{n+2}-K)=A \underset{F}{\leftarrow\!\supset} \phi$ *where F is finite and properly contained in A and $\phi: F \to A$ is a monomorphism.*[2]

Therefore, a knot which is not quasiaspherical has infinitely many ends. There are examples of n-knots which are not quasiaspherical, for $n \geq 2$ [2] [4].

Ratcliffe conjectures ([4, p. 323], [3, Problem 3]) that n-knots with infinitely many ends are not quasiaspherical. We give counter-examples to this conjecture for $n \geq 2$. Thus, by the results of Lomonaco [3; Theorem 10.1], even in the class

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[1] Whenever we write $A \underset{C}{*} B$ it is understood that C is a proper subgroup of A and B.

[2] The HNN group $A \underset{F}{\leftarrow\!\supset} \phi$ is $(A * \|t: -\|)/N$, where N is the normal closure of $\{tft^{-1}\phi(f)^{-1} : f \in F\}$. Here $\|t: -\|$ is an infinite cyclic group generated by t.

of infinitely many ended knots there are knots for which the homotopy type of the complement is determined by its algebraic 2-type.

First we obtain sufficient conditions for a pair $(A \underset{F}{\hookleftarrow} \phi, H)$ to be unsplitable; then we realize geometrically examples of such pairs. An affirmative answer to the question we ask in § 1 would characterize unsplitable pairs $(A \underset{F}{\hookleftarrow} \phi, H)$. We settle it when A has at most one end and H is generated by the stable letter. In § 2 we construct a 2-knot whose group is $(Z_m \ltimes Z_{2^m-1}) \underset{Z_m}{\hookleftarrow} \psi$ where $Z_m \cup \psi(Z_m)$ generates the semidirect product $Z_m \ltimes Z_{2^m-1}$, a meridian being represented by the stable letter. Using § 1 one shows that this is a quasiaspherical knot with infinitely many ends.

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§ 1. Algebraic part

Let G be a finitely generated group and let H be a subgroup of G. Viewing ZG as a left G-module by left multiplication, we consider the restriction homomorphism $r: H^1(G; ZG) \to H^1(H; ZG)$. Swarup [7, Th. 4] proved:

PROPOSITION 1. *If* $r: H^1(G; ZG) \to H^1(H; ZG)$ *is not injective then* $G = A \underset{F}{*} B$ *or* $G = A \underset{F}{\hookleftarrow} \phi$ *where* F *is finite and* $H \subset A$.

The converse of this theorem is valid [10, Theorem 5.2]:

PROPOSITION 2. *If* $G = A \underset{F}{*} B$ *or* $G = A \underset{F}{\hookleftarrow} \phi$ *with* F *finite and if* $H \subset A$ *then* $r: H^1(G; ZG) \to H^1(H; ZG)$ *is not injective.*

COROLLARY 1. *Let* G *be a finitely generated group such that* $G/G' \approx Z$ *and let* H *be a subgroup of* G *such that* $H \not\subset G'$. *Then* (G, H) *is unsplitable if and only if the restriction* $r: H^1(G; ZG) \to H^1(H; ZG)$ *is injective.*

Proof. G cannot be of the form $A \underset{F}{\hookleftarrow} \phi$ with $H \subset A$ because $A \subset G'$. The result then follows from Propositions 1 and 2.

Now if U is the universal cover of the exterior of a knot K then using the exact sequence of $(U, \partial U)$, Poincaré duality and the isomorphisms $H^1_c(U) \approx H^1(G; ZG)$ $H^1_c(\partial U) \approx H^1(H; ZG)$ it follows that $H_{n+1}(U)$ is isomorphic to the kernel of r.

From these observations and Corollary 1, Theorem 1 follows.

If $G = A \underset{F}{*} B$, where F is finite, we say that A is a *factor* of G.

In the remainder of this section we let $G = A \underset{F}{\hookleftarrow} \phi$ where F is finite and

$G/G' \approx Z$, let $m = a_0 t^{\varepsilon_1} a_1 \cdots t^{\varepsilon_n} a_n$ $a_i \in A$ $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \varepsilon_i = 1$ and let H be the (infinite cyclic) subgroup of G generated by m.

PROPOSITION 3. *Let C be the subgroup of A generated by $F \cup \phi(F) \cup \{a_0, \ldots, a_n\}$. If C is a finite proper subgroup of A or if C is contained in a factor of A then (G, H) is not unsplitable.*

Proof. Suppose C is a finite proper subgroup of A. Then the homomorphism from $G = A \underset{F}{\hookleftarrow} \phi$ to $(C \underset{F}{\hookleftarrow} \phi) \underset{C}{*} A$ whose restriction to A is the natural inclusion and which sends the stable letter of $A \underset{F}{\hookleftarrow} \phi$ to the stable letter of $C \underset{F}{\hookleftarrow} \phi$ is easily seen to be an isomorphism. Since $C \underset{F}{\hookleftarrow} \phi$ contains the image of H it follows that (G, H) is not unsplitable.

Similarly one shows that if C is contained in a factor P of $A = P \underset{E}{*} Q$ then there is an isomorphism from G onto $(P \underset{F}{\hookleftarrow} \phi) \underset{E}{*} Q$ where E is finite and H is mapped into $P \underset{F}{\hookleftarrow} \phi$.

Question. Is the converse of Proposition 3 valid?

A partial answer is the following:

THEOREM 3. *Let $G = A \underset{F}{\hookleftarrow} \phi$ where F is finite and $G/G' \approx Z$; let H be the subgroup generated by the stable letter t and let C be the subgroup of A generated by $F \cup \phi(F)$. Assume*

(i) A has at most one end, and

(ii) C is not a finite proper subgroup of A. Then (G, H) is unsplitable.

Proof. Associated to a *HNN*-group there is a natural exact sequence of cohomology groups [1, Th. 3.1]. The homomorphism of the *HNN* group $H = 1 \underset{1}{\hookleftarrow}$ to the *HNN* group $G = A \underset{F}{\hookleftarrow} \phi$ sending the stable letter t of H to the stable letter t of G induces a commutative diagram with exact rows

$$\begin{tikzcd}
 & ZG \arrow[r, "(1-t)\cdot"] \arrow[d, equal] & ZG \arrow[d, equal] & & \\
0 \arrow[r] & H^0(1; ZG) \arrow[r, "(1-t)\cdot"] & H^0(1; ZG) \arrow[r] & H^1(H; ZG) \arrow[r] & 0 \\
0 \arrow[r] & H^0(A; ZG) \arrow[r, "(1-t)\cdot"] \arrow[u, "i"'] \arrow[d, equal] & H^0(F; ZG) \arrow[r] \arrow[u, "j"'] \arrow[d, equal] & H^1(G; ZG) \arrow[r] \arrow[u, "r"'] & H^1(A; ZG) = 0 \\
 & (ZG)^A \arrow[r, "(1-t)\cdot"] & (ZG)^F & &
\end{tikzcd}$$

Here i can be identified with the inclusion of $(ZG)^A$ in ZG and j, with the inclusion of $(ZG)^F$ in ZG. Notice that $H^1(A; ZG) \approx H^1(A; ZA) \otimes_{ZA} ZG = 0$ because A has at most one end [8, page 145].

LEMMA. *Let $w \in ZG$. If $(1-t) \cdot w \in (ZG)^F$ then $w \in (ZG)^C$.*

Proof. Write $w = \sum_{g \in G} n_g \cdot g$. Then $(1-t)w = \sum_{g \in G} m_g \cdot g$ where $m_g = n_g - n_{t^{-1}g}$. Since $(1-t) \cdot w \in (ZG)^F$ we have $m_g = m_{fg}$ that is

$$n_g - n_{t^{-1}g} = n_{fg} - n_{t^{-1}fg}, \qquad g \in G, \qquad f \in G. \tag{$*$}$$

We only need to show (i) $n_g = n_{fg}$ and (ii) $n_g = n_{\phi(f)g}$ for $f \in F$, $g \in G$.

For a sufficiently large k we have $n_{t^{-k}g} = n_{t^{-k}fg} = 0$. From $(*)$ it follows that $n_{t^{-i}g} = n_{t^{-i}fg}$ for $k \geq i \geq 0$. This proves (i).

To prove (ii) notice that $n_{tg} - n_g = n_{tg} - n_{t^{-1}tg} = n_{f(tg)} - n_{t^{-1}f(tg)} = n_{ftg} - n_{\phi(f)g}$. By (i) $n_{tg} = n_{ftg}$. Hence $n_g = n_{\phi(f)g}$. This proves the lemma.

An element $x \in \ker r$ is the image of an element $y \in (ZG)^F$. Then $j(y) = y$ is of the form $(1-t) \cdot w$ where $w \in ZG$. By the lemma $w \in (ZG)^C$. If C is infinite then $w = 0$ so that $x = 0$; if $C = A$ then y is in the image of $(ZG)^A$ and therefore $x = 0$. Hence, r is injective and, by Corollary 1, (G, H) is unsplitable. This completes the proof of the theorem.

§2. Geometric realization

Let L be a smooth n-link in S^{n+2}, $n > 1$, with components $L_1, \dots, L_r$. L has a unique framing. Denote by N^{n+2} the manifold obtained by surgery on L. Then L is replaced by $M = m_1 \cup \cdots \cup m_r$ where each m_i is a 1-sphere. M has a natural framing so that if we perform surgery on M using this framing we recover S^{n+2}.

If G is a group, a *cyclic word* of G is a subset of G which is the union $[g]$ of the conjugacy classes of g and g^{-1}, for some $g \in G$. The cyclic word of $\pi_1 N^{n+2}$ determined by m_i will also be denoted by m_i and will be called a *meridian*. It corresponds to a meridian of $\pi_1(S^{n+2} - L)$ under the isomorphism $\pi_1(S^{n+2} - L) = \pi_1(N^{n+2} - M) \approx \pi_1(N^{n+2})$. We remark that a finite system of cyclic words $c_1, \dots, c_r$ of $\pi_1 N$ determines disjoint 1-spheres (which we also denote by $c_1, \dots, c_r$), well defined up to isotopy, which represent them.

Let (G, m, c) be a triple where G is a group, m is a system of r cyclic words $m_1, \dots, m_r$ of G, and c is also a system of r cyclic words $c_1, \dots, c_r$ of G.

If, for some i, we replace c_i by $c_i' = [g_i g_j]$ where $g_i \in c_i$ $g_j \in c_j$ $i \neq j$ we obtain a new system c' of cyclic words of G. We say that (G, m, c') is obtained from (G, m, c) by a *band move*.

If in the triple (G, m, c) some cyclic word m_i of m coincides with a cyclic word c_j of c consider the projection $G \to \hat{G}$ where $\hat{G} = G/\langle m_i \rangle$.[4] Let $\hat{m}$ be the system $\hat{m}_1, \ldots, \hat{m}_{i-1}, \hat{m}_{i+1}, \ldots, \hat{m}_r$ and let $\hat{c}$ be the system $\hat{c}_1, \ldots, \hat{c}_{j-1}, \hat{c}_{j+1}, \ldots, \hat{c}_r$. Then we say that $(\hat{G}, \hat{m}, \hat{c})$ is obtained from (G, m, c) by a *collapse*.

PROPOSITION 4. *Let $c = \{c_1, \ldots, c_r\}$ be a system of cyclic words of $\pi_1 N^{n+2}$; let $m = \{m_1, \ldots, m_r\}$ be the system of meridians of $\pi_1 N^{n+2}$. Assume the triple $(1, \emptyset, \emptyset)$ can be obtained from the triple (G, m, c) by a finite sequence of band moves and collapses. Then, if we perform surgery on $c_1 \cdots c_r$ using suitable framings, we obtain S^{n+2}.*

Proof. Consider the $(n+2)$-manifold $\chi(L_1, L_2, \ldots, L_r;\ c_1, \ldots, c_r)$ obtained from S^{n+2} by surgery on $L_1, L_2, \ldots, L_r$ and then by surgery on $c_1, \ldots, c_r$; the framing of $L_1, \ldots, L_r$ is unique; the framings of $c_1, \ldots, c_r$ are specified later.

A band move on $c_1, \ldots, c_r$ can be realized by a "band move" among the 1-dimensional surgeries. By this we understand the effect on the boundary of a cobordism when we perform handle slidings; these handle slidings do not change the cobordism. Thus if $c' = \{c'_1, \ldots, c'_r\}$ is obtained from $c = \{c_1, \ldots, c_r\}$ by band moves then $\chi(L_1, \ldots, L_r;\ c_1, \ldots, c_r) = \chi(L_1, \ldots, L_r;\ c'_1, \ldots, c'_r)$.

If now some cyclic word of c', say c'_r, equals some cyclic word of m, say m_r, then if we endow m_r with the natural framing $\chi(L_1, \ldots, L_r;\ c'_1, \ldots, c'_{r-1}, m_r) = \chi(L_1, \ldots, L_{r-1};\ c'_1, \ldots, c'_{r-1})$ because the surgeries on L_r and m_r cancel. We want the framings of $c_1, \ldots, c_r$ be such that the framing of c'_r coincides with the framing of m_r. Then we have

$$\chi(L_1, \ldots, L_r; c_1, \ldots, c_r) \approx \chi(L_1, \ldots, L_{r-1}; c'_1, \ldots, c'_{r-1})$$

Proceeding this way we eventually obtain

$$\chi(L_1, \ldots, L_r, c_1, \ldots, c_r) = \chi(\emptyset; \emptyset) = S^{n+2}.$$

This proves the proposition because we can find the framings of $c_1, \ldots, c_r$ working all the process backwards.

Suppose $c_1, \ldots, c_r$ are cyclic words of $\pi_1 N^{n+2}$ such that by a finite sequence of band moves and collapses, it is possible to obtain the triple $(1, \emptyset, \emptyset)$ from $(\pi_1 N; m_1, \ldots, m_r;\ c_1, \ldots, c_r)$. Perform surgery on $c_1 \cup \cdots \cup c_r$ using suitable framings to obtain S^{n+2}. Then $c_1 \cup \cdots \cup c_r$ is replaced by a disjoint union of n-spheres $S_1, \ldots, S_r$ in S^{n+2}.

The following proposition is clear.

[4] $\langle\ \rangle$ denotes normal closure.

PROPOSITION 5. *Let* $1 \leq k \leq r$. *Then* $\bigcup_{i=1}^{k} S_i$ *is a link in* S^{n+2} *with group* $\pi_1 N / \bigcup_{i>k} \langle c_i \rangle$. *The meridian corresponding to* S_i, $i \leq k$, *is represented by* c_i.

Remark. This construction of links generalizes the construction introduced in [2, § 1].

Now, we will construct quasiaspherical knots with infinitely many ends. Let $L = L_1 \cup L_2$ be a smooth 2-link in S^4 such that $\pi_1 N^4 \approx \|a, t, x: a^m = 1, t^{-1}at = a^{-1}\|$ where m is odd and t, x are the meridians. For example L can be taken to be a split link one of whose components is a 2-twist spun torus knot and the other one is trivial. Now let c_1, c_2 be the cyclic words of $\pi_1 N^4$ represented by xt^{-1} and $a^{-1}xax^{-2}$ respectively. It is easy to find a sequence of band moves changing $\{c_1, c_2\}$ into $\{x, t\}$. According to Proposition 5 there is a knot K_m in S^4 whose group is $\|a, t, x: a^m = 1,\ t^{-1}\ at = a^{-1},\ a^{-1}xax^{-2} = 1\| \approx (Z_m \ltimes Z_{2^m-1}) \underset{Z_m}{\leftrightarrows} \phi$ where $Z_m \ltimes Z_{2^m-1}$ is the semidirect product $\|a, t: a^m = x^{2^m-1} = 1,\ a^{-1}xa = x^2\|$; the domain of ϕ is the subgroup generated by a; and $\phi(a) = a^{-1}$. Moreover xt^{-1} represents a meridian of K_m.

THEOREM 4. *The* 2-*knot* K_m *is quasiaspherical and has infinitely many ends.*

Proof. By Theorem 2 ii) K_m has infinitely many ends. To see that it is quasiaspherical notice that $\pi_1(S^4 - K_m) \approx \|a, x, t: a^m = a^{-1}xax^{-2} = 1,\ t^{-1}at = a^{-1}\| \xrightarrow[\approx]{f} \|a, x, s: a^m = a^{-1}xax^{-2} = 1\ \ s^{-1}as = x^{-1}a^{-1}\| \approx (Z_m \ltimes Z_{2^m-1}) \underset{Z_m}{\leftrightarrows} \psi$ where $f(a) = a$, $f(x) = x$, $f(t) = sx$; the domain of ψ is the subgroup generated by a and $\psi(a) = x^{-1}a^{-1}$. Since $Z_m \cup \psi(Z_m)$ generates $Z_m \ltimes Z_{2^m-1}$ and the stable letter s is a meridian, it follows from Theorems 3 and 1 that K_m is quasiaspherical. This proves the theorem.

Since the spinning construction preserves meridian, we have:

COROLLARY 2. *For* $n \geq 2$ *there are quasiaspherical* n-*knots with infinitely many ends.*

Remark. The knot K_m has the same group as the corresponding knot in [2, pag. 95]. However, the latter is not quasiaspherical (see [4] or Proposition 3).

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