The Conway potential function for links.

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The Conway potential function for links

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Conway introduced the potential function of a link in 1970, [1]. This potential function, closely allied to the Alexander link polynomial, nevertheless has important properties which the Alexander polynomial does not have. However, despite this fact, no proof has appeared either for the properties, or even for the existence of Conway's potential function. That, then, is the purpose of this paper. Kauffman [3] showed how to define what may be called the reduced potential function of a link in terms of a Seifert matrix. This reduced potential function is an L-polynomial in one variable. However, the potential function is essentially a function of several variables, and I can see no way of generalising Kauffman's method to obtain the full potential function. Quite a different approach is therefore indicated.

The potential function is determined except for sign by the Alexander polynomial, since for a link with n components,

$$\begin{aligned}&(t_1-t_1^{-1})\cdot \nabla(t_1)=\Delta(t_1^2)\cdot t_1^{\mu_1} \quad \text{if} \quad n=1\\ &\nabla(t_1,\ldots,t_n)=\Delta(t_1^2,\ldots,t_n^2)\cdot t_1^{\mu_1}\cdots t_n^{\mu_n} \quad \text{if} \quad n>1\end{aligned} \tag{1.1}$$

where ∇ is the potential function, Δ is the Alexander polynomial properly chosen with correct sign and μ_i are integers which are uniquely determined by the requirement that ∇ should satisfy the symmetry condition (5.5). But the Alexander polynomial is not usually defined with a well determined sign. It is shown here, however, how by defining a simple correspondence between the rows and columns of an Alexander matrix obtained from a Wirtinger presentation, the Alexander polynomial can be defined with a well determined sign. Then, one may define a symmetric potential function using (1.1).

However, in order to derive properties of the potential function, and in particular the replacement relations which are of central importance, it is necessary to be able to determine in advance the values of the μ_i in (1.1) directly from the link projection. This is perhaps the most delicate step in the definition of the potential function. The values of the μ_i turn out to depend on the *curvature* of the projection of the i-th component of the link.

The method of proof of invariance of the potential function is somewhat old fashioned, by means of the three PL moves of Reidemeister [5]. This is perhaps justified by the fact that the potential function is not an algebraic invariant, and a proof of its invariance must contain some geometric element. It is often the case that a theorem is easily proven once one makes the correct definition. This is the case here, and for that reason, tedious detail is often omitted.

The contents of this paper overlap in part with some of the results of a recent monograph of Kauffman, [4], in which the Conway polynomial is treated from a different point of view. Kauffman also notes the connection with what is in fact the Whitney degree of the planar knot projection, called here the curvature, and by Kauffman, curliness.

Finally, the notion of defining a correspondence between rows and columns in an Alexander matrix was suggested to me by J. H. Conway in a brief conversation in Galway in 1973, and this paper has developed as an expansion of that idea. It was written down while I was a visitor at the J. W. Goethe University in Frankfurt am Main in the summer of 1982, and I should like to express my appreciation for the hospitality that was extended to me there.

§2. Definition of the potential function

We consider an oriented link, the components of which are numbered 1 to n ($n \geqslant 1$) in some way. It will be described how a potential function is assigned to the link. If the link has more than one component, the potential function will be an integral L-polynomial in the variables $t_1, \ldots, t_n$, that is an element of the polynomial ring $Z[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$. If the link has one component, then the potential function is of the form $f(t_1)/(t_1 - t_1^{-1})$, where $f(t_1) \in Z[t_1, t_1^{-1}]$.

We start with a regular projection of the link. If some connected component of the link projection has no crossing points, define $\nabla(t_1) = (t_1 - t_1^{-1})^{-1}$ if L has one component (L is a trivial knot) and $\nabla(t_1, \ldots, t_n) = 0$ if L has more than one component (L is a split link). From now on we exclude this possibility. At a crossing point of the projection, two arcs meet, one passing under and one over. By cutting the undercrossing arc at the point where it crosses under, the link is cut into m arcs (where m is equal to the number of crossing points) called *generating arcs*. Thus at each crossing point, P, of the projection three generating arcs meet, one arc passing over at P, one arc terminating at P and one exiting from P (with regard to the link orientation). These last two arcs together make up the undercrossing arc at P. Now, number the crossing points $P_1, \ldots, P_m$ and the generating arcs $u_1, \ldots, u_m$ in such a way that u_i is the generating arc which exits from P_i. If generating arc u_i belongs to the j-th link component, then give u_i the *label* t_j.

To a path, a, in the plane of projection, which does not start or end at a point on the link projection, and which avoids the crossing points, we can associate an element, $\mathfrak{a}$, of the free group, $F(u_1, \ldots, u_m)$ generated by the u_i as follows. One moves along the path writing down the sequence of generating arcs crossed, more precisely writing u_i if u_i is crossed from right to left and u_i^{-1} if it is crossed from left to right. Using this, we read off a Wirtinger relator, R_i, at each crossing point, P_i, of the projection, as follows. R_i is the word in the u_i corresponding to a small loop which starts at a point to the right of both over- and undercrossing arcs at P_i and proceeds anticlockwise around P_i. Thus, for a positive crossing, P_i (the undercrossing arc crosses under the overcrossing arc from right to left), relator R_i is $u_k u_i u_k^{-1} u_j^{-1}$ and for a negative crossing (the undercrossing crosses under from left to right), R_i is $u_i u_k u_j^{-1} u_k^{-1}$, where in each case u_k is the overcrossing arc. Now let θ be the map from $ZF(u_1, \ldots, u_m)$ to $Z[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ which takes each u_i to its label, and define the $m \times m$ *Jacobian matrix*, M, by $M_{ij} = (\partial R_i/\partial u_j)^\theta$.

From a basic formula of the free differential calculus, we have

$$\sum_{j=1}^{m} (u_j^\theta - 1) \cdot (j\text{-th column of } M) = 0 \tag{2.1}$$

The link projection divides the plane of projection into regions. Let w_i be a path from a base point b in the unbounded region to a point close to P_i and to the right of both under- and overcrossing arcs, $\mathfrak{w}_i$ the corresponding word in $F(u_1, \ldots, u_m)$. If the w_i are chosen so that they do not intersect except at b, then for some permutation, σ, of degree m representing the anticlockwise order of the w_i about b we have

$$\mathfrak{w}_{\sigma(1)} R_{\sigma(1)} \mathfrak{w}_{\sigma(1)}^{-1} \cdot \mathfrak{w}_{\sigma(2)} R_{\sigma(2)} \mathfrak{w}_{\sigma(2)}^{-1} \cdots \mathfrak{w}_{\sigma(m)} R_{\sigma(m)} \mathfrak{w}_{\sigma(m)}^{-1} = \mathrm{id} \quad \text{in} \quad F(u_1, \ldots, u_m)$$

from which it follows that

$$\sum_{i=1}^{m} \mathfrak{w}_i^\theta \cdot (i\text{-th row of } M) = 0 \tag{2.2}$$

Now, if $M^{(ij)}$ denotes the matrix obtained from M by deleting the i-th row and j-th column, then from (2.1) and (2.2) we have that

$$(-1)^{i+j} \det (M^{(ij)})/\mathfrak{w}_i^\theta (u_j^\theta - 1) = (-1)^{k+l} \det (M^{(kl)})/\mathfrak{w}_k^\theta (u_l^\theta - 1)$$

So, defining

$$D(t_1, \ldots, t_n) = (-1)^{i+j} \det (M^{(ij)})/\mathfrak{w}_i^\theta (u_j^\theta - 1) \tag{2.3}$$

for any i and j, we see that D is independent of the choice of i and j. (If M is a 1×1 matrix, define $\det(M^{(11)}) = 1$.) It is also clear that D does not depend on the original numbering of the crossing points and generating arcs, since a renumbering corresponds to a simultaneous identical permutation of the rows and columns of M. Thus, D depends only on the link projection and numbering of the components of the link. By its very definition, if $n > 1$, $D(t_1, \ldots, t_n)$ is the Alexander polynomial of the link, and if $n = 1$, then $D(t_1) = (t_1 - 1)^{-1} \cdot \Delta(t_1)$. It will turn out that for different projections of the same link, the value of D differs only by a factor $t_1^{\beta_1} \cdots t_n^{\beta_n}$. Hence the value of D is determined as to sign, and so represents a signed form of the Alexander polynomial.

We now need to determine the factor $t_1^{\mu_1} \cdots t_n^{\mu_n}$ in (1.1) required to make the potential function symmetric. For each component of the link, trace out the Seifert circuits in the projection of that component, and let its *curvature* equal (number of anticlockwise circuits) − (number of clockwise circuits). Let κ_i be the curvature of the i-th component. Further, for each i, let ν_i equal the number of crossing points in the link projection for which the overcrossing arc has label t_i (belongs to the i-th component of the link). Now define

$$\nabla(t_1, \ldots, t_n) = D(t_1^2, \ldots, t_n^2) \cdot t_1^{\mu_1} \cdots t_n^{\mu_n} \quad \text{where} \quad \mu_i = \kappa_i - \nu_i. \tag{2.4}$$

This, then, is Conway's potential function.

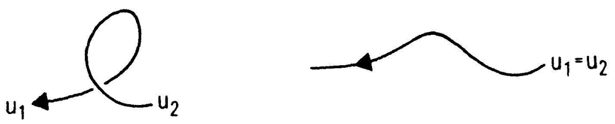
§3. The potential function is a link invariant

We have shown in the previous section that the potential function defined there is uniquely determined by the link projection. We now show that it remains invariant under transition from one projection to another via the three basic Reidemeister moves, and hence it is a link invariant.

In the definition of the potential function the numbering of the generating arcs is immaterial and may be suited to our convenience. Similarly, since we may choose to delete any row and column from the Jacobian matrix we will assume that the row and column deleted are not among those specifically considered. This is always possible as long as the projection has at least one more generating arc besides those explicitly shown. Once we have shown that the introduction or removal of trivial loops (first basic move) does not change the potential function, this desirable situation may be achieved by the introduction of redundant trivial loops. For the same reason we may always assume that the generating arcs shown in diagrams are all different. The only exceptions to these rules, therefore, are in the verification of invariance for the removal of trivial loops from a component which has at most one other crossing point (which must also belong to a trivial loop). This must be treated as a rather trivial special case. Details are omitted.

In that part of the link projection which is altered by the Reidemeister move there are at most three link components involved. For convenience we give them labels r, s, t instead of t_{i_1}, t_{i_2}, t_{i_3}, write κ_r, κ_s, κ_t instead of κ_{i_j} and ν_r, ν_s, ν_t instead of ν_{i_j}. For each of the three Reidemeister moves one must consider various cases depending on the orientation of the link components, and in the case of removal of trivial loops, whether the loop is clockwise or anticlockwise. We consider explicitly only one representative case for each type of move. Quantities with primes (′) refer to the diagram on the left, unprimed quantities the diagram on the right in each case.

First basic move:

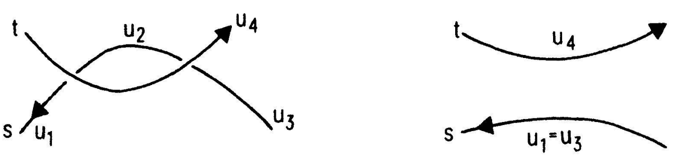


Here $R_1' = u_2 u_1 u_2^{-1} u_2^{-1}$ and so (with c_i standing for column i),

$$\det(M'^{(ij)}) = \left\| \begin{array}{cc|c} t & -t & 0 \\ \hline c_1 & c_2 & * \end{array} \right\| = t \cdot \| c_1 + c_2 \mid * \| = t \cdot \det(M^{(ij)}).$$

Since the factor $(-1)^{i+j}/\mathfrak{w}_i^\theta(u_j^\theta - 1)$ is unchanged we have $D' = t \cdot D$. However, $\nu_t' = \nu_t + 1$, $\kappa_t' = \kappa_t - 1$, so $\nabla' = \nabla$ from (2.4).

Second basic move:



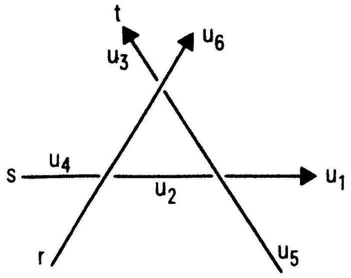
Now, $R_1' = u_1 u_4 u_2^{-1} u_4^{-1}$, $R_2' = u_4 u_2 u_4^{-1} u_3^{-1}$. Thus,

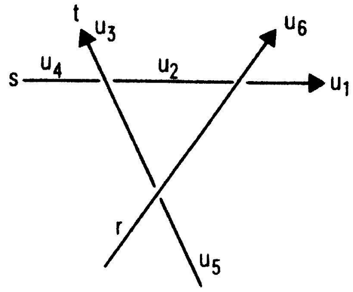
$$\det(M'^{(ij)}) = \left\| \begin{array}{cccc|c} 1 & -t & 0 & s-1 & 0 \\ 0 & t & -1 & 1-s & 0 \\ \hline c_1 & 0 & c_3 & c_4 & * \end{array} \right\| = \left\| \begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & t & -1 & 1-s & 0 \\ \hline c_1 & 0 & c_3 & c_4 & * \end{array} \right\|$$

$$= \left\| \begin{array}{ccc|c} t & -1 & 1-s & 0 \\ \hline 0 & c_1 + c_3 & c_4 & * \end{array} \right\| = t \cdot \| c_1 + c_3 \quad c_4 \mid * \| = t \cdot \det(M^{(ij)})$$

Hence as before, $D' = t \cdot D$. However, $\nu'_t = \nu_t + 2$ and other values are unchanged. Thus, $\nabla' = \nabla$.

Third basic move:





Here

$$R'_1 = u_1u_5u_2^{-1}u_5^{-1}, \qquad R'_2 = u_2u_6u_4^{-1}u_6^{-1}, \qquad R'_3 = u_6u_3u_6^{-1}u_5^{-1},$$
$$R_1 = u_1u_6u_2^{-1}u_6^{-1}, \qquad R_2 = u_2u_3u_4^{-1}u_3^{-1}, \qquad R_3 = u_6u_3u_6^{-1}u_5^{-1}.$$

Thus,

$$\det(M'^{(ij)}) = \left\| \begin{array}{cccccc|c} 1 & -t & 0 & 0 & s-1 & 0 & 0 \\ 0 & 1 & 0 & -r & 0 & s-1 & 0 \\ 0 & 0 & r & 0 & -1 & 1-t & 0 \\ \hline c_1 & 0 & c_3 & c_4 & c_5 & c_6 & * \end{array} \right\|$$
$$= \left\| \begin{array}{ccccc|c} 1 & 0 & -rt & s-1 & st-t & 0 \\ 0 & r & 0 & -1 & 1-t & 0 \\ \hline c_1 & c_3 & c_4 & c_5 & c_6 & * \end{array} \right\|$$
$$= \left\| \begin{array}{ccccc|c} 1 & r(s-1) & -rt & 0 & s-1 & 0 \\ 0 & r & 0 & -1 & 1-t & 0 \\ \hline c_1 & c_3 & c_4 & c_5 & c_6 & * \end{array} \right\| = A.$$

The first step is by adding t times the second row to the first then eliminating the second row and column. The second step is by adding $s-1$ times the (now) second row to the first. Similarly,

$$\det(M^{(ij)}) = \left\| \begin{array}{cccccc|c} 1 & -r & 0 & 0 & 0 & s-1 & 0 \\ 0 & 1 & s-1 & -t & 0 & 0 & 0 \\ 0 & 0 & r & 0 & -1 & 1-t & 0 \\ \hline c_1 & 0 & c_3 & c_4 & c_5 & c_6 & * \end{array} \right\|$$

which is transformed to A by adding r times the second row to the first and eliminating the second row and column. Thus, $D' = D$, and since the values of the ν's and the κ's are unchanged, $\nabla' = \nabla$.

From this we conclude that the potential function is a link invariant.

§4. The reduced potential function

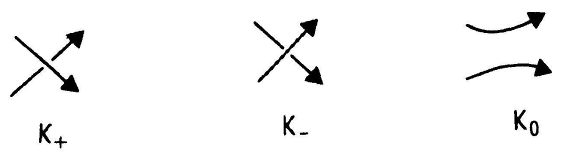
We may define a reduced potential function, $\bar{\nabla}$, for a link by

$$\bar{\nabla}(t) = (t - t^{-1}) \cdot \nabla(t, \ldots, t).$$

This is an integral L-polynomial. It has two basic properties.

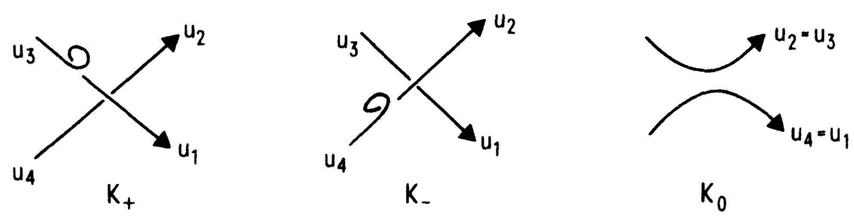
(4.1) *For the trivial knot,* $\bar{\nabla}(t) = 1$.

(4.2) (*Replacement relation.*) *For three links,* K_+, K_- *and* K_0 *which differ only in one place as shown,*



the potential functions satisfy $\bar{\nabla}_+(t) = \bar{\nabla}_-(t) + (t - t^{-1})\, \bar{\nabla}_0(t)$.

Proof of (4.2). For convenience we introduce an extra trivial loop in K_+ and K_-, which does not alter the potential function.



Let $\theta : ZF(u_1, \ldots, u_n) \to Z[t, t^{-1}]$ take all u_i to t, and denote $(\partial R_i/\partial u_j)^\theta$ by $\bar{M}$. If $\bar{D}(t) = (-1)^{i+j} \det(\bar{M}^{(ij)})/\mathfrak{w}_i^\theta$, then $\bar{\nabla}(t) = \bar{D}(t^2) \cdot t^{\kappa - \nu - 1}$ where now ν is the number of crossing points and κ is the sum of curvatures of all components. Then,

$$\det(\bar{M}_+^{(ij)}) = \left\| \begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 1-t & t & 0 & -1 & 0 \\ \hline c_1 & c_2 & c_3 & c_4 & * \end{array} \right\| = \left\| \begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 1 & t & -t & -1 & 0 \\ \hline c_1 & c_2 & c_3 & c_4 & * \end{array} \right\|$$

and

$$\det(\bar{M}^{(ij)}) = \left\| \begin{array}{cccc|c} 1 & t-1 & -t & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ \hline c_1 & c_2 & c_3 & c_4 & * \end{array} \right\| = \left\| \begin{array}{cccc|c} 1 & t & -t & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ \hline c_1 & c_2 & c_3 & c_4 & * \end{array} \right\|$$

Hence,

$$\det(\bar{M}_+^{(ij)}) - \det(\bar{M}_-^{(ij)}) = \left\| \begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 1 & t & -t & -1 & 0 \\ \hline c_1 & c_2 & c_3 & c_4 & * \end{array} \right\|$$

$$= \left\| \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & t & 0 & 0 & 0 \\ \hline c_1 & c_2 & c_3+c_2 & c_1+c_4 & * \end{array} \right\| = (t-1)\, \| c_2+c_3 \quad c_1+c_4 \mid * \|$$

$$= (t-1) \cdot \det(\bar{M}_0^{(ij)}).$$

This shows that $\bar{D}_+(t) - \bar{D}_-(t) = (t-1) \cdot \bar{D}_0(t)$. However, $\nu_+ = \nu_- = 2 + \nu_0$, and $\kappa_+ = \kappa_- = \kappa_0 + 1$, and so (4.2) follows.

We now show that the properties (4.1) and (4.2) characterise the reduced potential function. (This was also proven by Kauffman [3].) The following part of the proof deserves to be singled out.

(4.3) (*Induction principle*). *Let* $\mathfrak{C}$ *be a class of links satisfying* (i) *the trivial knot is in* $\mathfrak{C}$, (ii) *all split links are in* $\mathfrak{C}$, (iii) *if* K_0 *is in* $\mathfrak{C}$ *and one of* K_+ *and* K_- *is in* $\mathfrak{C}$, *then both* K_+ *and* K_- *are in* $\mathfrak{C}$. *Then* $\mathfrak{C}$ *contains all links.*

Proof. Consider a link, L, with m crossings. By interchanging overcrossing and undercrossing for some number, $h(L)$ of crossing points, L may be transformed either to a split link or a trivial knot. Consider one of these crossings. Suppose it is positive and denote L by L_+. Then L_0 has $m-1$ crossings, whereas L_- has m crossings but $h(L_-) < h(L_+)$. By induction on m and h one deduces that L_+ is in $\mathfrak{C}$.

Now we prove

(4.4) (*Uniqueness of the reduced potential function.*) $\bar{\nabla}(t)$ *is the unique link invariant, an* L*-polynomial defined for all links, which satisfies* (4.1) *and* (4.2).

We assume that $\bar{\nabla}'(t)$ also satisfies (4.1) and (4.2) and let $\mathfrak{C}$ be the class of links for which $\bar{\nabla}'_L(t) = \bar{\nabla}_L(t)$. It follows easily from (4.1) and (4.2) that $\bar{\nabla}'_L(t) = \bar{\nabla}_L(t) = 0$ for split links. (See for instance Kauffman [3].) By (4.3), then, $\mathfrak{C}$ contains all links.

From (4.4) we deduce the following important corollary.

(4.5) (*Symmetry of the reduced potential function.*) *For a link of n components,* $\bar{\nabla}(t)=(-1)^{n-1}\,\bar{\nabla}(t^{-1})$.

Proof. Let $\bar{\nabla}'(t)=(-1)^{n-1}\,\bar{\nabla}(t^{-1})$. It is easily verified that $\bar{\nabla}'(t)$ satisfies (4.1) and (4.2) since $\bar{\nabla}(t)$ does. The vital point is that K_+ and K_- have the same number of components, whereas the number of components of K_0 differs by one.

Similar in style is the following proposition.

(4.6) *If* $\sim L$ *is the mirror image of* L, *a link with* n *components, then* $\bar{\nabla}_L(t)=(-1)^{n-1}\bar{\nabla}_{\sim L}(t)$.

To prove this, observe that $(-1)^{n-1}\,\bar{\nabla}_{\sim L}(t)$ satisfies (4.1) and (4.2).

Let L be a link with n components and let G_n be the complete graph on n vertices. We give the edge joining the vertices i and j of G_n a *weight* equal to λ_{ij}, the linking number of the i-th and j-th components of L. We say that G_n is *weighted by* L. Define the weight of a subgraph of G_n to be the product of the weights of all its edges. We can now determine more exactly the form of $\bar{\nabla}(t)$.

(4.7) *For a link* L *of* n *components,* $\bar{\nabla}(t)=(t-t^{-1})^{n-1}H(t)$ *where* $H(t)$ *is an integral* L*-polynomial in even powers of* t *and* t^{-1}. *For* $n=1$, $H(1)=1$. *For* $n>1$, $H(1)$ *is equal to the sum of the weights of all spanning trees in* G_n *weighted by* L.

Let $\mathfrak{C}$ be the class of links for which the proposition is true. It is trivially true for the trivial knot and for split links. We assume (4.7) holds for K_0 and K_+ (or K_-). If the two arcs crossing in K_+ are from the same link component, then that component splits into two components in K, and it follows that $H_+(t)=H_-(t)+(t-t^{-1})^2H_0(t)$. If however the two arcs are from different components, then these two components are amalgamated to one component in K, and one has $H_+(t)=H_-(t)+H_0(t)$. Thus, all statements but the last are easily proven. The value of $H(1)$ may be deduced by induction continuing this line of argument, however the details are omitted as the result will not be used further.

Of course, $H(t)$ is nothing but a disguised and signed form of the Hosokawa polynomial (see [2]), just as $\nabla(t)$ is a disguised form of the Alexander polynomial. In fact, with this viewpoint, (4.7) contains the main results of [2]. Corresponding to Theorem 2 of Hosokawa, we may give a different description of $H(1)$ as follows: Let L be the matrix given by

$$L_{ij}=-\lambda_{ij}\quad\text{if}\quad i\neq j$$

$$L_{jj}=\sum_{\substack{i=1\\ i\neq j}}^{n}\lambda_{ij}$$

Let $L^{(kl)}$ be the minor obtained by deleting the k-th row and l-th column of L. Then $H(1)=(-1)^{k+l}\cdot\det(L^{(kl)})$. It is a simple matter to prove this by induction using the recursion relations for H derived above.

It follows from (4.7) that for knots of one component, $\bar{\nabla}(1)=1$, so $\nabla(t)$ is determined uniquely by the Alexander polynomial. For $n\geq 2$, if $H(1)\neq 0$, in particular if all the linking numbers are positive, then the sign of $H(1)$ determines the correct sign for the potential function.

From the uniqueness of the reduced potential function it follows that our $\bar{\nabla}(t)$ is equal to Kauffman's $\Omega(t)$. In particular, $\bar{\nabla}(t)=\det(tV-t^{-1}V^*)$ where V is a Seifert matrix and V^* its transpose. An important property of $\bar{\nabla}(t)$ which is most easily proven using the Seifert matrix is

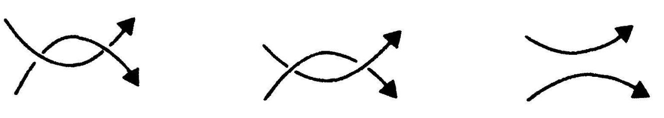
(4.8) (*Signature and nullity of links.*) $\bar{\nabla}_L(i)=0$ *if and only if* nullity$(L)>1$. *Otherwise,* $\bar{\nabla}_L(i)=R\cdot i^\sigma$. *Here* $i^2=-1$, *R is a positive real number and σ is the signature of the link.*

Proof. Suppose V is a $k\times k$ matrix. Then $\bar{\nabla}(i)=\det(iV+iV^*)=i^k\cdot\det(V+V^*)$. Now $V+V^*$ is congruent to a diagonal matrix, J, with p ones, q minus-ones and r zeros on the diagonal, $r=$ nullity$(L)-1$. Further, $\det(V+V^*)=R\cdot\det(J)$ for a positive real R. Now $\det(J)=0$ if and only if $r\neq 0$. If $r=0$, then $\bar{\nabla}(i)=i^k\cdot(-1)^q=i^k\cdot(-1)^{-q}=i^{k-2q}=i^{p-q}$ (since $k=p+q$) $=i^\sigma$.

§5. Properties of the potential function

Similar to the replacement relation (4.2) we have for the (unreduced) potential function

(5.1) (*Replacement relation.*) $\nabla_{++}+\nabla_{--}=(t_{i_1}t_{i_2}+t_{i_1}^{-1}t_{i_2}^{-1})\,\nabla_{00}$ *for links containing the tangles*



K_{++} K_{--} K_{00}

the components having labels t_{i_1} *and* t_{i_2}.

Similarly,

(5.2) (*Replacement relation.*) $\nabla_{++}+\nabla_{--}=(t_{i_1}t_{i_2}^{-1}+t_{i_1}^{-1}t_{i_2})\,\nabla_{00}$ *in the case where one of the two arcs in* (5.1) *is oppositely oriented.*

The proof of these relations is similar to the proof of (4.2) and is omitted. A further property which is easily proven is

(5.3) *If L is a link with n components, then*

$$\nabla(1, t_2, \ldots, t_n) = (t_2^{\lambda_{12}} \cdots t_n^{\lambda_{1n}} - t_2^{-\lambda_{12}} \cdots t_n^{-\lambda_{1n}}) \cdot \nabla'(t_2, \ldots, t_n)$$

where $\nabla'(t_2, \ldots, t_n)$ *is the potential function of the link obtained by eliminating the first component of L and* λ_{ij} *is the linking number between the i-th and j-th components of the link.*

Indeed, if we number the generating arcs of L such that $u_1, \ldots, u_k$ are the consecutive generating arcs of the first component, we obtain, setting $t_1 = 1$ in the matrix M, a matrix of the form $\begin{pmatrix} A & 0 \\ * & B \end{pmatrix}$. A is a $k \times k$ matrix which gives rise to the first half of the expression on the right of (5.3) and B gives rise to $\nabla'(t_2, \ldots, t_n)$. See Torres [6] for a proof of this result for the Alexander polynomial.

Applying (5.3) $n-1$ times we have the formula

(5.4) $\nabla_L(1, \ldots, 1, t_i, 1, \ldots, 1) = \nabla_i(t_i) \cdot \prod_{k=1, k\neq i}^{n} (t_i^{\lambda_{ik}} - t_i^{-\lambda_{ik}})$ *where* $\nabla_i(t_i)$ *is the potential function of the i-th component of L.*

We are now able to prove

(5.5) (*Symmetry of the potential function.*) $\nabla(t_1, \ldots, t_n) = (-1)^n \nabla(t_1^{-1}, \ldots, t_n^{-1})$.

Proof. We assume the well known symmetry of the Alexander polynomial [6] which implies that $\nabla(t_1, \ldots, t_n) = \varepsilon t_1^{\gamma_1} \cdots t_n^{\gamma_n} \cdot \nabla(t_1^{-1}, \ldots, t_n^{-1})$. From (5.4) $\nabla(1, \ldots, t_i, \ldots, 1) = \nabla_i(t_i) G_i(t_i)$ where $G_i(t_i) = (-1)^{n-1} G_i(t_i^{-1})$ and $\nabla_i(t_i) = -\nabla_i(t_i^{-1})$ from (4.6). Then $\nabla_i(t_i)G_i(t_i) = \nabla(1, \ldots, t_i, \ldots, 1) = \varepsilon t_i^{\gamma_i} \nabla(1, \ldots, t_i^{-1}, \ldots, 1) = \varepsilon t_i^{\gamma_i} \nabla_i(t_i^{-1}) G_i(t_i^{-1}) = \varepsilon t_i^{\gamma_i} \cdot (-1)^n \nabla_i(t_i) G_i(t_i)$. Now $\nabla_i(t_i) \neq 0$, and $G_i(t_i) \neq 0$ as long as all linking numbers are non-zero. In this case, therefore, $\gamma_i = 0$ and $\varepsilon = (-1)^n$, and (5.5) is proven for the case where all λ_{ij} are non-zero.

Now assume $\lambda_{i_0 j_0} = 0$. From (5.1) we have a formula $\nabla_{++++} + \nabla_{00} = (t_{i_0} t_{j_0} + t_{i_0}^{-1} t_{j_0}^{-1}) \nabla_{++}$. Identifying the link L as K_{00} we see that $\nabla_L = \nabla_{00}$ may be expressed in terms of the potential functions of K_{++++} (for which $\lambda_{i_0 j_0} = 2$) and K_{++} (for which $\lambda_{i_0 j_0} = 1$). If ∇_{++++} and ∇_{++} satisfy (5.5) then so does $\nabla_{00} = \nabla_L$. So, (5.5) follows by induction on the number of λ_{ij} equal to zero.

Next we consider the mirror image of L.

(5.6) *If* $\sim L$ *is the mirror image of* L *then* $\nabla_{\sim L}(t_1, \ldots, t_n) = (-1)^{n-1}\, \nabla_L(t_1, \ldots, t_n)$.

As is well known, the Alexander polynomials of L and $\sim L$ are equal, so $\nabla_{\sim L}(t_1, \ldots, t_n) = \varepsilon\, \nabla_L(t_1, \ldots, t_n)$. Using (5.4) we deduce that $\varepsilon = (-1)^{n-1}$ as long as all linking numbers are non-zero, since for a knot of one component, $\nabla_K(t) = \nabla_{\sim K}(t)$ by (4.6). This may be extended to all links using (5.1) just as in the previous proof.

Finally, we consider the effect of changing the orientation of one component of a link.

(5.7) *If* L^* *is obtained from* L *by reversing the orientation of the first component, then* $\nabla_{L^*}(t_1, t_2, \ldots, t_n) = -\nabla_L(t_1^{-1}, t_2, \ldots, t_n)$.

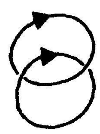
Once again from the properties of the Alexander polynomial we have $\nabla_{L^*}(t_1, \ldots, t_n) = \varepsilon\, \nabla_L(t_1^{-1}, t_2, \ldots, t_n)$. Using (5.4) we deduce that $\varepsilon = -1$ for links with all linking numbers non-zero and extend to all links using (5.1) and (5.2).

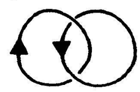
§6. Axiomatic determination of the potential function?

The proofs in the last section of properties of the potential function unfortunately rely on properties of the Alexander polynomial. Hence, they are more cumbersome than the proofs of properties of the reduced potential function which rely only on the two properties (4.1) and (4.2). For links with more than one component, however, a simple set of defining "axioms" for the potential function are not known, at least to me. As an exercise the reader may like to attempt to calculate the potential function of the Borromean rings using the derived properties of potential functions but without resorting to matrix calculations. (I cannot do it.)

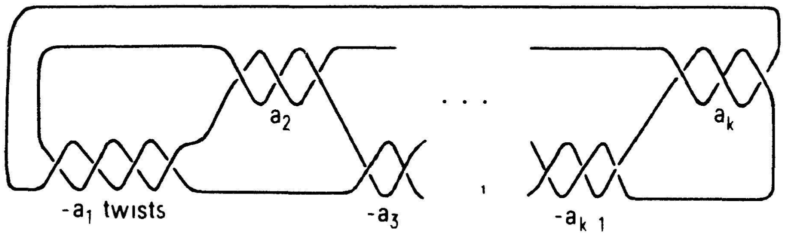
However, for many links, a simple set of properties suffice for the determination of the potential function. As an example, we show that the two properties

(6.1) *For a split link,* $\nabla = 0$.

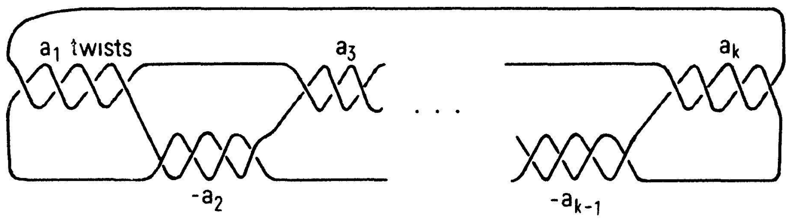
(6.2) *For a simple positive clasp,*  , $\nabla = 1$.

along with the replacement relations (5.1) and (5.2) are enough to determine the potential function of a 2-bridged link of two components. Note first that these conditions imply that $\nabla=-1$ for a negative clasp  .

Following Conway [1], one denotes a 2-bridged link by a sequence of integers $[a_1 \cdots a_k]$ which represents the link



if k is even, and



if k is odd. (We do not worry too much about link orientation in this explanation.)

The links $[a_1 \cdots a_k]$ and $[b_1 \cdots b_l]$ are the same if the continued fractions

$$a_k+\frac{1}{a_{k-1}}+\cdots+\frac{1}{a_1} \quad \text{and} \quad b_l+\frac{1}{b_{l-1}}+\cdots+\frac{1}{b_1}$$

are equal. Every 2-bridged link has a notation $[a_1 \cdots a_k]$ with all a_i positive, and since $[1\quad a_1 \cdots a_k]=[a_1+1\quad a_2 \cdots a_k]$ we may assume $a_1>1$. Now using (5.1) or (5.2) we see that

$$\nabla_{[a_1\quad a_2\cdots a_k]}=-\nabla_{[a_1-4\quad a_2\cdots a_k]}+A(t_1, t_2)\cdot\nabla_{[a_1-2\quad a_2\cdots a_k]} \tag{**}$$

where $A(t_1, t_2)$ is one of $(t_1t_2+t_1^{-1}t_2^{-1})$ or $(t_1t_2^{-1}+t_1^{-1}t_2)$ depending on the orientation of the strings crossing in the part of the diagram represented by a_1. (The two strings must belong to different components if L is to have two components.) However, $[0\quad a_2 \cdots a_k]=[a_3 \cdots a_k]$, $[-1\quad a_2 \cdots a_k]=[a_2-1\quad a_3 \cdots a_k]$ and $[-2\quad a_2 \cdots a_k]=[2\quad a_2-1\quad a_3 \cdots a_k]$. (If $a_2=1$, this last one is equal to $[2+a_3\quad a_4 \cdots a_k]$.) Therefore, in all cases, the two links on the right hand side of (**) have smaller crossing number than the left hand side. Eventually, the calculation reduces to the potential functions of [0] (split link) and [2] (simple clasp) given by (6.1) and (6.2).

As a result of this calculation we see that

(6.4) *The potential function of a 2-component 2-bridged link is an integral polynomial in* $t_1t_2+t_1^{-1}t_2^{-1}$ *and* $t_1t_2^{-1}+t_1^{-1}t_2$.

In view of the success for 2-bridged links, one is disposed to hope that the potential function of any two-component link is uniquely determined by simple "axioms." It indeed seems possible that the replacement relations, (5.1), (5.2) and (4.1) along with values for the trivial knot, split links and the simple clasp may uniquely determine the potential function of a two component link.

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