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Some existence theorems for closed geodesics

W. BALLMANN, G. THORBERGSSON AND W. ZILLER[1]

In the present paper we use elementary methods to prove some results on the existence of closed geodesics. One of the main results is as follows:

THEOREM. *Let g be a metric on* $P^n\mathbb{R}$ *such that* $\frac{1}{4} \leq \delta \leq K \leq 1$, *where* K *denotes the sectional curvature of* M. *Then* g *has at least* $g(n) = 2n - s - 1$, $0 \leq s = n - 2^k < 2^k$, *closed geodesics without self-intersections, with lengths in* $[\pi, \pi/\sqrt{\delta}] \subset [\pi, 2\pi]$, *and which are not null-homotopic. If all closed geodesics of length* $\leq 2\pi$ *are non-degenerate (an open and dense condition on the set of metrics with respect to the* C^2 *topology), then* g *has at least* $n(n+1)/2$ *such closed geodesics.*

Notice that this theorem does not follow from the corresponding existence theorem for S^n in [BTZ2]. Moreover the proof is more elementary, since no use is made of loop space methods.

The proof of the Lusternik–Schnirelmann theorem [LS] can be used to show that any metric on $P^2\mathbb{R}$ has three closed geodesics without self-intersections which are not null-homotopic, see [Ba]. The ellipsoid with pairwise different principal axes sufficiently close to one induces a metric on $P^n\mathbb{R}$ with $n(n+1)/2$ closed geodesics without self-intersections which are non-degenerate and not null-homotopic and have length approximately π. One can achieve that the lengths of all other closed geodesics are greater than any given number by choosing the axes sufficiently close to 1.

Some of the other results in this paper are

(i) If M is homeomorphic to S^n and $\frac{1}{4} \leq K \leq 1$, then there exists a closed geodesic without self-intersections, with lengths in $[2\pi, 4\pi]$, and with index $n-1$.

(ii) If M is homeomorphic to S^n and $\frac{4}{9} \leq K \leq 1$, then there exist two closed geodesics c and d without self-intersections, with lengths in $[2\pi, 3\pi]$, such that $\operatorname{ind}(c) = n-1$ and $\operatorname{ind}(d) + \operatorname{null}(d) = 3(n-1)$.

(iii) If M is homeomorphic to S^n and $\frac{1}{4} \leq \delta \leq K \leq 1$, and if K is not constant,

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then there does not exist any closed geodesic with length in $[2\pi/\sqrt{\delta}, 4\pi]$. If $n=2$ and if $\frac{1}{9}<\delta\leq K\leq 1$, then there does not exist any prime closed geodesic with length in $(2\pi/\sqrt{\delta}, 6\pi)$.

(iv) If g_0 is the metric on $P^n\mathbb{R}$ of constant curvature 1 and if g satisfies the Morse condition $g_0<g<9g_0$ then there exist at least $g(n)$ closed geodesics with lengths in $(\pi, 3\pi)$.

(v) A convex hypersurface in E^{n+1} which contains a ball of radius r and is contained in a ball of radius R has at least $g(n)$ closed geodesics with lengths in the interval $[2\pi r, 2\pi R]$ if $2r>R$.

(ii) and (iii) are partially proved in [Ka] and [Ts] respectively [Su].

Some of these elementary results are existence results needed in [BTZ1], where we examined stability properties of closed geodesics. Thus the present paper can be used as an introduction to [BTZ1].

In Chapter 1 we examine closed geodesics on S^n, in Chapter 2 closed geodesics on $P^n\mathbb{R}$, in Chapter 3 the Morse condition, in Chapter 4 closed geodesics on convex surfaces, and in Chapter 5 closed geodesics on convex hypersurfaces.

1. Closed geodesics on spheres

We first review the definitions of a few concepts and some of their properties. M will always denote a compact Riemannian manifold.

Let Λ be the space of closed piecewise C^∞ curves $c: I=[0,1]\to M$, and for $p\in M$ let Ω_p be the subspace of Λ consisting of curves with $c(0)=p$. To a C^∞ map $f:(I^k,\partial I^k)\to(M,p)$ we associate the maps $f_\Omega:(I^{k-1},\partial I^{k-1})\to(\Omega_p,p)$ and $f_\Lambda:(I^{k-1},\partial I^{k-1})\to(\Lambda,\Lambda^0)$ defined by $f_\Omega(x_1,\ldots,x_{k-1})(t)=f(x_1,\ldots,x_{k-1},t)$ and $f_\Lambda=j\circ f_\Omega$, where $j:(\Omega_p,p)\to(\Lambda,\Lambda^0)$ is the inclusion, Λ^0 the space of constant curves, and p the constant curve with image p. (We have also used the convention $I^0=\{0\}$ and $\partial I^0=\emptyset$.)

Let $h\in\pi_k(M)$, $k\geq 1$, be a non-trivial homotopy class, and let $f:(I^k,\partial I^k)\to(M,p)$ be a C^∞ representative. We define

$$\alpha_{\Omega_p}(h)=\inf\left\{\max_{x\in I^{k-1}} E(g(x))\mid g \text{ homotopic to } f_\Omega\right\}$$

$$\alpha_\Lambda(h)=\inf\left\{\max_{x\in I^{k-1}} E(g(x))\mid g \text{ homotopic to } f_\Lambda\right\}$$

where E is the energy functional $E(c)=\frac{1}{2}\int_0^1\langle\dot c,\dot c\rangle\,dt$. For a compact manifold M there always exists a $k\geq 1$ such that $\pi_k(M)\neq 0$.

Let $V = V(c)$ denote the space of piecewise C^∞ vector fields along a closed geodesic $c \in \Lambda$ satisfying $\langle X(t), \dot c(t)\rangle = 0$ for all $t \in I$ and $X(0) = X(1)$. On V we define the index form of c by

$$H(X, Y) = \int_0^1 (\langle \nabla X, \nabla Y\rangle - \langle R(X, \dot c)\dot c, Y\rangle)\, dt.$$

The index (resp. extended index) of c as a closed geodesic, denoted by $\operatorname{ind}(c)$ (resp. $\operatorname{ind}_0(c)$), is the maximal dimension of a subspace U of V such that $H \mid U$ is negative definite (resp. negative semi-definite). The index (resp. extended index) of c as a geodesic segment, denoted by $\operatorname{ind}_\Omega(c)$ (resp. $\operatorname{ind}_\Omega(c) + \operatorname{null}_\Omega(c)$) is the maximal dimension of a subspace U of V consisting of vector fields X satisfying $X(0) = X(1) = 0$ such that $H \mid U$ is negative definite (resp. negative semi-definite). Obviously

$$\operatorname{ind}(c) \geq \operatorname{ind}_\Omega(c) \quad \text{and} \quad \operatorname{ind}_0(c) \geq \operatorname{ind}_\Omega(c) + \operatorname{null}_\Omega(c).$$

In [BTZ1], (1.5) and (1.6), it is shown that

$$\operatorname{ind}(c) \leq \operatorname{ind}_\Omega(c) + n - 1 \quad \text{and} \quad \operatorname{ind}_0(c) \leq \operatorname{ind}_\Omega(c) + \operatorname{null}_\Omega(c) + n - 1.$$

These inequalities immediately give the following estimates of the index and the extended index of a closed geodesic on a Riemannian manifold whose sectional curvature satisfies $0 < \delta \leq K \leq 1$:

$$\text{(1.1)} \qquad \begin{aligned} L(c) > k\frac{\pi}{\sqrt{\delta}} &\Rightarrow \operatorname{ind}(c) \geq k(n-1) \\ L(c) < k\pi &\Rightarrow \operatorname{ind}_0(c) \leq k(n-1) \end{aligned}$$

see [BTZ1], (1.8) and (1.9).

We will frequently use the following injectivity radius estimate, see [CE], [CG], [KS]:

(1.2) (Klingenberg). If M^n is simply connected and the sectional curvature K of M satisfies $\frac{1}{4} \leq \delta \leq K \leq 1$, or if n is even and $0 < \delta \leq K \leq 1$, then the injectivity radius of M satisfies $i(M) \geq \pi$. In particular, biangles, geodesic loops, and closed geodesics have length $\geq 2\pi$.

The following theorem follows easily from critical point theory (e.g., apply Lemma 22.5 in [Mi] to a finite dimensional approximation of Λ, see [Mi] §16) and the fact that $f \to f_\Lambda$ induces an isomorphism $\pi_k(M) \to \pi_{k-1}(\Lambda, \Lambda^0)$ for $k \geq 2$.

1.3. THEOREM. *Suppose* $\pi_k(M) \neq \{0\}$, $k \geq 1$. *Then there exists a closed geodesic of index* $\leq k-1$ *and of length* $\alpha_\Lambda(h)$ *when* $h \in \pi_k$ (M) *is non-trivial. If* $k \geq 2$, *then there exists such a closed geodesic which is null-homotopic.* □

Remark. The existence of a closed geodesic if $\pi_1(M)=0$ is due to Birkhoff [Bi] for $M=S^n$ and Fet–Lusternik [LF] in general.

1.4. THEOREM. *Suppose that the sectional curvature* K *of* M^n *satisfies* $0 < \delta \leq K \leq 1$.

(i) *Then there exists a closed geodesic of index* $\leq n-1$ *and length* $\leq 2\pi/\sqrt{\delta}$. *If* M *is not homotopy equivalent to* S^n, *then there exists a closed geodesic of index* $< n/2$ *and length* $\leq \pi/\sqrt{\delta}$.

(ii) *If* M *is homeomorphic to* S^n *and* $\delta > \frac{1}{4}$, *then any closed geodesic of index* $\leq n-1$ *has index* $n-1$, *length in* $[2\pi, 2\pi/\sqrt{\delta}] \subset [2\pi, 4\pi)$, *and no self-intersections.*

Proof. (i) Since $\pi_k(M) \neq 0$ for some $1 \leq k \leq n$ there exists a closed geodesic c of index $\leq n-1$ by (1.3). $L(c) > 2\pi/\sqrt{\delta}$ would imply $\operatorname{ind}(c) \geq 2(n-1)$ by (1.1), which is a contradiction. If M is not homotopy equivalent to S^n there exists a k, $1 \leq k \leq n/2$, such that $\pi_k(M) \neq 0$ and therefore a closed geodesic c of index $< n/2 \leq n-1$, hence $L(c) \leq \pi/\sqrt{\delta}$ by (1.1).

(ii) Let c be a closed geodesic of index $\leq n-1$. Then $L(c) \leq 2\pi/\sqrt{\delta} < 4\pi$ by (1.1), and (1.2) implies $L(c) \geq 2\pi$. Since $2\pi > \pi/\sqrt{\delta}$, (1.1) implies $\operatorname{ind}(c) \geq n-1$, hence $\operatorname{ind}(c) = n-1$. If a closed geodesic has a self-intersection it is the union of two loops, each of which has length at least 2π. This contradicts $L(c) < 4\pi$. □

We now prove the existence of a second closed geodesic. Assume that $\pi_1(M^n)=0$ and $\frac{4}{9} \leq \delta \leq K \leq 1$. Then M is homeomorphic to S^n, and hence $\pi_n(M) = \mathbb{Z}$. Choose a fixed generator $h \in \pi_n(M)$. According to theorem (1.3) there exists a closed geodesic of length $l = \sqrt{2}\alpha_\Lambda(h)$.

1.5. THEOREM. *Suppose that* M *is homeomorphic to* S^n *and* $\frac{4}{9} \leq \delta \leq K \leq 1$.

(i) (Karcher [Ka]) *A geodesic loop* c *of maximal length* $L \leq 2\pi/\sqrt{\delta}$ *is a closed geodesic without self-intersections and* $\operatorname{ind}_0(c) = 3(n-1)$. *Furthermore,* c *is a geodesic triangle of maximal perimeter.*

(ii) $l \leq L$, *and* $l = L$ *implies that there exists a closed geodesic* c *of length* L *through every point* p *of* M *with* $\operatorname{ind}_0(c) = 3(n-1)$. *In particular, there exist two different closed geodesics without self-intersections and lengths in* $[2\pi, 3\pi]$.

Remarks. (a) The claim about $\operatorname{ind}_0(c)$ in (i) is not contained in [Ka] but was communicated to us by Karcher. The proof in [Ka] contains a mistake. In using the triangle inequality one has to make a separate discussion when equality occurs. In particular, it is not correct, as claimed there, that every geodesic triangle of maximal perimeter is a closed geodesic. But by replacing some sides in a triangle of maximal perimeter one gets a closed geodesic among the triangles of maximal perimeter.

(b) $L \geq 2d(M)$. If $L = 2d(M)$, then for every two points $p, q \in M$ at maximal distance $d(M)$ every geodesic through p is closed of length L and meets q. This follows since a geodesic triangle $(\gamma_1, \gamma_2, \gamma_3)$, where γ_1 connects p and q, obviously has perimeter $\geq 2d(M)$ and $= 2d(M)$ if $\gamma_1 * \gamma_2$ is a geodesic.

Proof. (i) We only prove the claim about $\operatorname{ind}_0(c)$. A theorem of Toponogov states that the perimeter of every geodesic triangle is $\leq 2\pi/\sqrt{\delta}$, and if there exists a geodesic triangle of perimeter $2\pi/\sqrt{\delta}$, then K is constant. We first assume that K is not constant. Let c be a geodesic loop of maximal length $\leq 2\pi/\sqrt{\delta}$ and set $\gamma_i = c \mid [(i-1)/3, i/3]$, $1 \leq i \leq 3$. The triple $(\gamma_1, \gamma_2, \gamma_3)$ is a geodesic triangle since $L(\gamma_i) \leq \pi$. Since K is not constant the perimeter of $(\gamma_1, \gamma_2, \gamma_3)$ is $< 2\pi/\sqrt{\delta} \leq 3\pi$, hence $L(\gamma_i) < \pi$. Therefore there are no pairs of conjugate points on γ_i. Hence each triple of vectors (X_0, X_1, X_2), $X_i \in T_{c(t_i)}M$ and $X_i \perp \dot{c}(t_i)$, $t_i = i/3$, determines a unique broken Jacobi field J along c such that $J(t_i) = X_i$ and $J(1) = X_0$. $H(J, J) \leq 0$ since J corresponds to a variation of c through geodesic triangles and c is a geodesic triangle of maximal perimeter. Hence there exists a $3(n-1)$-dimensional subspace of $V(c)$ on which H is negative semi-definite which implies $\operatorname{ind}_0(c) \geq 3(n-1)$. Since $L(c) < 2\pi/\sqrt{\delta} \leq 3\pi$, (1.1) implies $\operatorname{ind}_0(c) \leq 3(n-1)$ and hence $\operatorname{ind}_0(c) = 3(n-1)$.

If K is constant, say $K \equiv 1$, it is clear that the great circles are triangles of maximal perimeter and the above arguments show that their extended index ind_0 is equal to $3(n-1)$.

(ii) Let h be a generator of $\pi_n(M)$. Then there exists a closed geodesic c_1 of length $l = \sqrt{2}\alpha_\Lambda(h)$. Theorem (1.3) also applies to geodesic loops, i.e., for every $p \in M$ there exists a geodesic loop c of length $\sqrt{2}\alpha_{\Omega_p}(h)$ and $\operatorname{ind}_\Omega(c) \leq n-1$, and hence $L(c) \leq 2\pi/\sqrt{\delta} \leq 3\pi$. Furthermore by definition $\alpha_\Lambda(h) \leq \alpha_{\Omega_p}(h)$ for every $p \in M$. Hence $2\pi \leq \sqrt{2}\alpha_\Lambda(h) \leq \sqrt{2}\alpha_{\Omega_p}(h) \leq 3\pi$. A geodesic loop of maximal length $\leq 2\pi/\sqrt{\delta}$ is a closed geodesic c_2, and c_1 and c_2 can be geometrically equal only if $L(c_1) = L(c_2) = L$. But then $\sqrt{2}\alpha_{\Omega_p}(h) = L$ for every p which implies by (i) that there exists a closed geodesic c of length L through every p and $\operatorname{ind}_0(c) = 3(n-1)$. $\square$

Remark. We remarked above that there is a closed geodesic among the geodesic triangles of maximal perimeter on a $\frac{4}{9}$-pinched manifold. More generally one can apply Lusternik–Schnirelmann theory to a $\mathbb{Z}_2$-quotient of a space of triangles on a δ-pinched manifold to obtain $n+1$ closed geodesics with lengths in the interval $[2\pi, 2\pi/\sqrt{\delta}]$ if $\delta \geq \frac{4}{9}$. We briefly sketch how this can be done. A similar proof will be carried out in more detail in chapter 2. In [BTZ2], theorem (4.1), using much more complicated methods, the existence of $g(n) \geq n+1$ such geodesics is proved on δ-pinched manifolds if $\delta \geq \frac{1}{4}$.

If there is a geodesic triangle of perimeter $2\pi/\sqrt{\delta}$ on M, then $K \equiv \delta$, and all geodesics are closed of length $2\pi/\sqrt{\delta}$. Hence we may assume that there exists an $\varepsilon > 0$ such that any geodesic triangle has perimeter $\leq (2\pi/\sqrt{\delta}) - 3\varepsilon$. Let v, w be unit tangent vectors with the same foot point, and let γ_v, γ_w be the geodesics determined by $\dot{\gamma}_v(0) = v$, $\dot{\gamma}_w(0) = w$. Then

$$d(\gamma_v(\pi-\varepsilon), \gamma_w(\pi-\varepsilon)) + 2(\pi-\varepsilon) \leq (2\pi/\sqrt{\delta}) - 3\varepsilon \leq 3\pi - 3\varepsilon.$$

Hence $d(\gamma_v(\pi-\varepsilon), \gamma_w(\pi-\varepsilon)) \leq \pi - \varepsilon < i(M)$. Denote by $U^2(M)$ the set $\{(v, w) \mid v, w$ are unit tangent vectors with the same foot point$\}$. The above inequality shows that the function $f: U^2M \to \mathbb{R}$, $(v, w) \to d^2(\gamma_v(\pi-\varepsilon), \gamma_w(\pi-\varepsilon))$ is C^∞. It is also invariant under the $\mathbb{Z}_2$-action $(v, w) \to (w, v)$ of U^2M. (v, w) is a minimum of f if and only if $(v, w) \in U^1(M) = \{(v, w) \in U^2(M) \mid v = w\}$. $U^1(M)$ is the fixed point set of the $\mathbb{Z}_2$-action on $U^2(M)$. Using the first variation formula and the fact that the sides of the geodesic triangles have no conjugate points it is easy to prove that (v, w) is a critical point of f if and only if $v = w$ or $v = -w$ and the triangle corresponding to (v, w) is a closed geodesic. Hence Lusternik–Schnirelmann theory implies that there are at least as many closed geodesics on M with lengths in $[2\pi, 2\pi/\sqrt{\delta})$ as the length of a maximal chain of homology classes $h_1, \ldots, h_s$ in $H_*(U^2(M)/\mathbb{Z}_2, U^1(M)/\mathbb{Z}_2; \mathbb{Z}_2)$ with the following properties: $\xi_i \cap h_i = h_{i-1}$ for some $\xi_i \in H^*(U^2(M)/\mathbb{Z}_2; \mathbb{Z}_2)$, $* > 0$, whose restriction to a sufficiently small neighborhood of $S^1c/\mathbb{Z}_2 = \{(v, w) \mid v = -w = \dot{c}(t)\}/\mathbb{Z}_2$ vanishes for every closed geodesic c which is a geodesic triangle, see [BTZ2], (1.2) and (1.3). The length of such a chain is $n+1$ since $(U^2(M), U^1(M))$ is an $n-1$ bundle over $T_1M/\mathbb{Z}_2$ (where $\mathbb{Z}_2$ acts by $v \to -v$) and since the cohomology ring of $T_1M/\mathbb{Z}_2$ is easily seen to be generated by $\theta \in H^1$ and $\omega \in H^{n-1}$ with the relations $\theta^n = 0$, $\omega^2 = 0$, and $\theta^{n-1} \cup \omega = [T_1M/\mathbb{Z}_2]$.

We now discuss certain gaps in the length spectra of closed geodesics and geodesic loops. If $\pi_1(M) = 0$ and $\frac{1}{4} \leq K \leq 1$, then we already know from (1.2) that there does not exist any closed geodesic or geodesic loop with length in $[0, 2\pi)$. As was observed by Tsukamoto [Ts] the proof of Berger's rigidity theorem implies the following result:

(1.6) Suppose $\pi_1(M) = 0$ and $\frac{1}{4} \leq K \leq 1$. If there exists a closed geodesic of length 2π, then M is isometric to a sphere with $K \equiv 1$, or to a projective space $P^k\mathbb{C}$, $P^k\mathbb{H}$, P^2Ca equipped with their standard metrics.

Remark. If $\pi_1(M)=0$, $i(M)\geq\pi$, and $0<K\leq 1$, then a closed geodesic of length 2π and of positive index is contained in a totally geodesic, embedded surface of constant curvature 1. This follows easily using arguments as in the proof of [Be], Theorem 4. For dim $M=2$ this was already proved by Klingenberg.

We already know that there exist closed geodesics in the interval $[2\pi, 2\pi/\sqrt{\delta}]$ if M is homeomorphic to S^n and $\frac{1}{4}\leq\delta\leq K\leq 1$. The next gap in the length spectrum is given by

1.7. THEOREM. *If M is homeomorphic to S^n and $\frac{1}{4}\leq\delta\leq K\leq 1$, and if K is not constant, then there does not exist any closed geodesic with length in $[2\pi/\sqrt{\delta}, 4\pi]$.*

Remarks. (a) In [Th] it was proved that there does not exist any closed geodesic with length in $[2\pi/\sqrt{\delta}, 4\pi)$ if $\frac{4}{9}<\delta\leq K\leq 1$ unless $K\equiv\delta$.

(b) Tsukamoto [Ts] claims that a closed geodesic without self-intersections on a simply connected manifold with $\frac{1}{4}\leq\delta\leq K\leq 1$ does not have length $2\pi/\sqrt{\delta}$ unless $K\equiv\delta$. But his proof contains a gap. For even dimensions a complete proof of his result was given by Sugimoto (now Goto) [Su], Theorem B and C. Using the injectivity radius estimate (1.2) the proof in [Su] can be shortened considerably and carries over directly to odd dimensions.

Proof. If there is a closed geodesic of length $\leq 4\pi$ and with self-intersections, then it is the union of loops one of which has length $\leq 2\pi$. Since $i(M)\geq\pi$ by (1.2) a geodesic loop of length $\leq 2\pi$ is a closed geodesic of length 2π. By (1.6) this implies $K\equiv 1$. If there exists a closed geodesic c of length $2\pi/\sqrt{\delta'}\in[2\pi/\sqrt{\delta}, 4\pi]$, $\frac{1}{4}\leq\delta'\leq\delta$, and $K\not\equiv 1$, then c has no self-intersections contradicting the theorem of Tsukamoto–Sugimoto quoted in Remark (b) above. This proves the theorem. □

1.8. COROLLARY. *If $\pi_1(M^n)=0$ and $\frac{1}{4}\leq K\leq 1$, then there exists a closed geodesic c without self-intersections,* ind$(c)\leq n-1$, *and length in $[2\pi, 4\pi]$. Unless $K\equiv 1$, or $K\equiv\frac{1}{4}$, or M isometric to $P^k\mathbb{C}$, $P^k\mathbb{H}$, P^2Ca equipped with their standard metrics, we have $2\pi<L(c)<4\pi$ and* ind$(c)=n-1$.

Proof. This follows by combining (1.7) with the proof of (1.4). □

Remark. Notice that on $P^k\mathbb{C}$, $P^k\mathbb{H}$, and P^2Ca, equipped with their standard metrics, the closed geodesics have index 1, 3, and 7 respectively.

2. Closed geodesics on real projective spaces

In this chapter we examine the existence of closed geodesics on $P^n\mathbb{R}$ and manifolds with $\pi_1(M) \cong \mathbb{Z}_2$.

2.1. THEOREM. *Suppose M is diffeomorphic to $P^n\mathbb{R}$ and g is a metric such that $\frac{1}{4} \le \delta \le K \le 1$. Then g has at least $g(n)$ closed geodesics without self-intersections, with lengths in $[\pi, \pi/\sqrt{\delta}] \subset [\pi, 2\pi]$, and which are not null-homotopic. If all closed geodesics of length $\le 2\pi$ are non-degenerate, then g has at least $n(n+1)/2$ such closed geodesics.*

Remark. The indices of the closed geodesics in (2.1) lie in the interval $[0, 2(n-1)]$. Using the methods developed in [BTZ1] and [BTZ2] one easily obtains stability properties of these closed geodesics.

Proof. Since M is not simply connected, we have $d(M) \le \pi/2\sqrt{\delta}$ by a result of Shiohama, see [Sh], Proposition 2.1. If $\pi_1(M) = \mathbb{Z}_2$, $\frac{1}{4} \le \delta \le K \le 1$, and $d(M) = \pi/2\sqrt{\delta}$, then a result of Sakai [Sa], p. 428, implies that $\tilde{M}$ is isometric to S^n or $P^k\mathbb{C}$ with their standard metrics. Since the theorem is obvious for such spaces we can assume $d(M) < \pi/2\sqrt{\delta} \le \pi$.

Let v be any unit tangent vector of M and let $c_v(t)$ be the geodesic determined by $\dot{c}_v(0) = v$. There is a first $t(v) > 0$ such that $c_v \mid [-s, s]$ is not minimizing for any $s > t(v)$. We have $2t(v) \le d(M)$, and hence $t(v) < \pi/4\sqrt{\delta} \le \pi/2$. Since there is no conjugate point along $c_v \mid [-t(v), t(v)]$, there is a second geodesic segment d_v such that $d_v(-t(v)) = c_v(-t(v))$ and $d_v(t(v)) = c_v(t(v))$. $(c_v \mid [-t(v), t(v)]) *$ $(d_v^{-1} \mid [-t(v), t(v)])$ is not null-homotopic since $i(\tilde{M}) \ge \pi$. c is uniquely determined since $\pi_1(M) = \mathbb{Z}_2$. Hence $t(v)$ and $\dot{d}_v(-t(v))$ depend continuously on v and satisfy the equation

$$d^2(c_v(t(v)), \exp 2t(v)\dot{d}_v(-t(v))) = 0.$$

Since there are no conjugate points for $t < \pi$, $t(v)$ depends differentiably on v by the implicit function theorem.

Suppose v is a critical point of the function t. Given two vectors $X \perp \dot{c}_v(-t(v))$ and $Y \perp \dot{c}_v(t(v))$ there exists a variation c_s of c_v through geodesics such that $c_0 = c_v$ and

$$\left.\frac{d}{ds}\right|_{s=0} c_s(t(v)) = Y, \qquad \left.\frac{d}{ds}\right|_{s=0} c_s(-t(v)) = X.$$

Set $v(s) = \dot{c}_s(0)/\|\dot{c}_s(0)\|$. Then by the chain rule and the first variation formula we have

$$\begin{aligned}
0&=\frac{1}{2}\frac{d}{ds}\bigg|_{s=0} t(v(s))=\frac{d}{ds}\bigg|_{s=0} L(c_s[-t(v(s)),t(v(s))])\\
&=\frac{d}{ds}\bigg|_{s=0} L(d_s[-t(v(s)),t(v(s))])\\
&=\left\langle\frac{d}{ds}\bigg|_{s=0} d_s(t(v(s))),\dot d_v(t(v))\right\rangle-\left\langle\frac{d}{ds}\bigg|_{s=0} d_s(-t(v(s))),\dot d_v(-t(v))\right\rangle\\
&=\langle Y,\dot d_v(t(v))\rangle-\langle X,\dot d_v(-t(v))\rangle.
\end{aligned}$$

Hence a critical point of t corresponds to a closed geodesic on M of length $4t(v)$. Such a closed geodesic does not have self-intersections since $4t(v)<2\pi$.

The definition of t implies $t(v)=t(-v)$. Hence we obtain a differentiable function on T_1M/θ, where $\theta v=-v$, and the critical points correspond to closed geodesics. Notice though that each closed geodesic c gives rise to a circle of critical points $\dot c(t+\alpha)$, $0\le\alpha\le 1$. Lusternik–Schnirelmann theory implies that there exist at least as many critical circles as there are homology classes $h_1,\dots,h_s$ in $H_*(T_1M/\theta,\mathbb{Z}_2)$ such that $\xi_i\cap h_i=h_{i-1}$, $i=2,3,\dots,s$, for some cohomology classes $\xi_i\in H^*(T_1M/\theta,\mathbb{Z}_2)$, $*>0$, with the property that ξ_i vanishes on every sufficiently small neighborhood of a critical circle $\dot c(t+\alpha)$, $0\le\alpha\le 1$, see [BTZ2], (1.2) and (1.3). We now show that there exist $g(n)$ such homology classes. Since the unit tangent bundles with respect to g and the constant curvature 1 metric g_0 are θ equivariantly diffeomorphic, the computation can be done for the case $M=(P^n\mathbb{R},g_0)$.

The geodesics on $\tilde M=(S^n,\tilde g_0)$ are the great circles. Each unit tangent vector $v\in T_1\tilde M$ determines a unique parametrized great circle γ_v with $\dot\gamma_v(0)=v$. This identifies $T_1\tilde M$ and the space G of all parametrized great circles. We have an O(2) action on G defined by $\psi\gamma(t)=\gamma(\psi t)$ for $\psi\in O(2)$. Let $\theta=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ and $\phi=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\in O(2)$. Then T_1M corresponds to G/ϕ and T_1M/θ to G/Γ, where $\Gamma\simeq\mathbb{Z}_2\times\mathbb{Z}_2$ is the subgroup of O(2) generated by θ and ϕ. Let $\bar G=G/O(2)$ be the space of unparametrized great circles on S^n. $H_*(\bar G,\mathbb{Z}_2)$ has a basis of $n(n+1)/2$ homology classes $[a,b]$, $0\le a\le b\le n-1$, of dimension $a+b$. Denote by (a,b) the corresponding dual basis of $H^*(\bar G,\mathbb{Z}_2)$. Then $(0,1)$ is the first Stiefel–Whitney class of the S^0 bundle $G/SO(2)\to\bar G$ and (1.1) is the second Stiefel–Whitney class of the S^1 bundle $G/\theta\to\bar G$. $(0,1)^{2n-2s-2}\cup(1,1)^s=(n-1,n-1)$ and $(0,1)^{2n-2s-1}=0$, and hence $\bar G$ has a chain of $g(n)$ subordinate homology classes, and each chain of subordinate homology classes has length $\le g(n)$, see [Kl], p. 49.

The second Stiefel–Whitney class of the S^1 bundle $p: G/\Gamma \to \bar{G}$ is zero since it it twice the second Stiefel–Whitney class of $G/\theta \to \bar{G}$. The Gysin sequence of p then implies that p^* is injective and p_* is surjective. Choose a class $h_{g(n)}$ such that $p_*(h_{g(n)}) = [n-1, n-1]$. From the naturality of cup and cap products it follows that the classes h_i, $1 \leq i \leq g(n)$, inductively defined by

$$h_{g(n)-1} = p^*(1, 1) \cap h_{g(n)}, \ldots, h_1 = p^*((1, 1)^s \cup (0, 1)^{2n-s-2}) \cap h_{g(n)}$$

are non-zero. Hence they are a chain of $g(n)$ subordinate homology classes in $H_*(G/\Gamma) = H_*(T_1M/\theta)$.

We now show that $p^*(0, 1)$ and $p^*(1, 1)$ vanish on a sufficiently small neighborhood U of a critical circle in T_1M/θ. If U is a tabular neighborhood of the critical circle, then it has the homotopy type of a circle and hence $p^*(1, 1) \mid U = 0$. $p^*(0, 1) \mid U$ vanishes if U is sufficiently small since $p^*(0, 1)$ is the Stiefel–Whitney class of the bundle $q: T_1M \to T_1M/\theta$, and $q^{-1}(U) \to U$ is trivial if U is sufficiently small. This finishes the proof of the existence of $g(n)$ closed geodesics.

Suppose now that all closed geodesics of length $\leq 2\pi$ are non-degenerate. We first want to show that this implies that all critical points of $t: T_1M \to \mathbb{R}$ are non-degenerate critical circles. Let c be a closed geodesic such that $v = \dot{c}(0)$ is a critical point of t. T_1M can be viewed as a set of geodesic biangles as in the beginning of the proof and hence $T_1M \subset \Lambda$. Since $E \mid T_1M = \frac{1}{2}t^2$ the Hessian of t is proportional to $H \mid TT_1M$. The tangent space of T_1M in Λ consists of piecewise Jacobi fields with breaks at 0 and $\frac{1}{2}$, and since $\dim T_1M = 2n-1$, it coincides with the set of all such Jacobi fields. $V(c)$ is the direct sum of these Jacobi fields and the set of vector fields vanishing at 0 and $\frac{1}{2}$. This direct sum is orthogonal with respect to H. Hence the nullspace of $E \mid T_1M$ coincides with the nullspace of H. Therefore all critical circles of $t: T_1M \to \mathbb{R}$ and hence also all critical circles of $t: T_1M/\theta \to \mathbb{R}$ are non-degenerate. The local homology of a critical circle vanishes in dimension $\neq \operatorname{ind}(c)$, $\operatorname{ind}(c)+1$ and is equal to $\mathbb{Z}_2$ in dimension $\operatorname{ind}(c)$ and $\operatorname{ind}(c+1$. The Morse inequalities for such functions now imply that there are at least $\frac{1}{2}\sum b_i(G/\Gamma, \mathbb{Z}_2)$ critical circles. But the Gysin sequence of $p: G/\Gamma \to \bar{G}$ implies $\sum b_i(G/\Gamma, \mathbb{Z}_2) = n(n+1)$ since the second Stiefel–Whitney class of p vanishes. $\square$

As we noted in the above proof, it follows by results of Shiohama [Sh] and Sakai [Sa] that $d(M) < \pi/2\sqrt{\delta} \leq \pi$ if $\pi_1(M) = \mathbb{Z}_2$, $\frac{1}{4} \leq \delta \leq K \leq 1$, and $\tilde{M}$ not a symmetric space of rank one. Then the proof of Lemma 4.1 in [Sh] applies and shows that there is a closed geodesic c of length $2d(M)$ through p and q if $d(p, q) = d(M)$. c is not null-homotopic since $L(c) < 2\pi$. Clearly c has maximal perimeter in the set of all geodesic biangles with minimizing sides. Note that this also follows from the above proof: the first part of the proof only uses $\pi_1(M)=\mathbb{Z}_2$ and the curvature restrictions; c corresponds to a maximum of t. As in the proof of (1.5) it follows that $\operatorname{ind}_0(c)=2(n-1)$.

Since there is no conjugate point along a geodesic segment of length $<\pi$, there exists a closed geodesic d of length $2i(M)$ by Lemma (5.6) in [CE]. $L(d)\geq\pi$ by (1.2). d is not null-homotopic since $L(d)<2\pi$. Hence d is a shortest curve in its homotopy class and therefore $\operatorname{ind}(d)=0$. $L(c)\geq L(d)$ since $i(M)\leq d(M)$. $L(c)=L(d)$ implies $i(M)=d(M)$. By (5.6) in [CE] this implies that all geodesics are closed of length $2i(M)$. Since $\pi_1(M)=\mathbb{Z}_2$ it follows that the universal covering space of M is a Wiedersehen manifold and the generalized Blaschke conjecture, recently proved by Berger, Kazdan, Weinstein, and Yang, implies that K is constant, see [Bs].

Summarizing the above we obtain

2.2. THEOREM. *Suppose that $\pi_1(M^n)=\mathbb{Z}_2$ and $\frac{1}{4}\leq\delta\leq K\leq 1$. Then there exist two closed geodesics c and d which are not null-homotopic, have no self-intersections and satisfy*

$$\pi\leq 2i(M)=L(d)\leq L(c)=2d(M)\leq\pi/\sqrt{\delta}$$

$$\operatorname{ind}(d)=0 \quad \textit{and} \quad \operatorname{ind}_0(c)=2(n-1).$$

c has maximal perimeter in the set of all biangles with minimizing sides, and $L(d)=L(c)$ implies that K is constant. □

3. The Morse condition

To prove the existence of more than one closed geodesic on S^n, M. Morse [Mo], p. 354, introduced the following condition: Let g_0 be the metric on S^n of constant curvature 1. Then the metric g on S^n satisfies the *Morse condition* if

$$g_0<g<4g_0$$

This condition immediately implies that the critical levels of certain homology classes consisting of circles lie in $(2\pi, 4\pi)$. Hence the closed geodesics on which they remain hanging cannot be iterates of each other. But this does not prove the existence of geometrically different closed geodesics since these closed geodesics could all be iterates of one short closed geodesic. In [Al] Alber stated the theorem that $g_0\leq g<4g_0$ and $0<K\leq 1$ if n even or $\frac{1}{4}<K\leq 1$ if n odd implies the

existence of $g(n)$ closed geodesics without self-intersections and with lengths in $[2\pi, 4\pi)$. Under these conditions there are no closed geodesics of length $<2\pi$ by (1.2). But the topological part of his proof turned out to be incorrect. Correct proofs have been given in [BTZ2], [An], and [Hi]. In [BTZ2] it was also proved that $\frac{1}{4} \leq K \leq 1$ implies the existence of $g(n)$ closed geodesics, i.e. the Morse condition is not needed for $\frac{1}{4}$-pinched manifolds. In this chapter we give some further theorems involving the Morse condition.

Let

$$d_p = \max_{q \in M} d(p, q)$$

3.1. LEMMA [Sh]. *If $d_p > \pi$ for every $p \in M$ and $K \geq \frac{1}{4}$, then the length of every closed geodesic is $> 2\pi$.*

Proof. Let c be a closed geodesic with $L(c) \leq 2\pi$ and let $p = c(0)$. Since $d_p > \pi$, there exists a point $q \in M$ with $d(p, q) > \pi$. Let d be a minimal geodesic from p to q. Then we have a generalized triangle whose one side consists of c and the two minimal sides are d. This is a generalized triangle in the sense of the Toponogov comparison theorem since $L(c) \leq 2\pi$ and $2L(d) > 2\pi \geq L(c)$. Since the angles of one of the minimal sides with c is $\leq \pi/2$, it follows from Toponogov's theorem that $d(p, q) < \pi$, a contradiction. □

Remark. A theorem of Berger and Grove–Shiohama [GS] states that $K \geq \frac{1}{4}$ and $d_p > \pi$ for some $p \in M$ implies that M is homeomorphic to S^n.

As a first consequence we obtain the following theorem.

3.2. THEOREM. *If $d_p > \pi$ for every $p \in M$ and $K \geq \frac{1}{4}$, then there exist at least $n-1$ closed geodesics with lengths in $(2\pi, 4\pi]$.*

Proof. Since a closed geodesic c with $L(c) > 4\pi$ has index $\geq 2(n-1)$, it follows as in the proof of (3.3) in [BTZ2] that there exist $n-1$ closed geodesics of length $\leq 4\pi$. By (3.1) every closed geodesic has length $> 2\pi$ and hence these $n-1$ closed geodesics are geometrically different. □

Remark. It does not follow that these closed geodesics have no self-intersections. Notice also that $d_p > \pi$ for every $p \in M$ follows from one half of the Morse condition, namely $g > g_0$.

3.3. THEOREM. *If g is a metric on S^n with $g_0 < g < 4g_0$ and $K \geq \frac{1}{4}$, then there exist $g(n)$ closed geodesics with lengths in $(2\pi, 4\pi)$.*

Proof. $g > g_0$ implies that $d_p > \pi$ for every $p \in M$ and hence all closed geodesics have lengths $> 2\pi$. $g < 4g_0$ implies that the homology classes considered in [BTZ2] thm. (2.4), have critical levels $< 4\pi$ and hence the theorem follows from the methods in [BTZ2]. □

Finally we observe that on $P^n\mathbb{R}$ the Morse condition suffices without any curvature assumptions.

3.4. THEOREM. *Let g be a metric on $P^n\mathbb{R}$ and g_0 the metric on $P^n\mathbb{R}$ with constant curvature 1. If $g_0 < g < 9g_0$, then there exists at least $g(n)$ closed geodesics which are not null-homotopic and with lengths in $(\pi, 3\pi)$.*

Proof. $g > g_0$ implies that every closed curve which is not null-homotopic, and hence every closed geodesic which is not null-homotopic, has length $> \pi$. We denote by $\bar{\Lambda}_*$ the unparametrized closed curves on $P^n\mathbb{R}$ which are not null-homotopic, by $\bar{\Lambda}_*^\alpha$ those curves in $\bar{\Lambda}_*$ whose energy with respect to the g metric is $\leq \alpha$, and by $\bar{\Lambda}_{*,0}^\alpha$ those in $\bar{\Lambda}_*$ whose energy in the g_0 metric is $\leq \alpha$. Then $g_0 < g < 9g_0$ implies that we have the following inclusions

$$\bar{\Lambda}_{*,0}^{\pi^2/2} \subset \bar{\Lambda}_*^{9\pi^2/2} \subset \bar{\Lambda}_{*,0}^{9\pi^2/2-}. \tag{$*$}$$

Every curve in $\bar{\Lambda}_*$ has odd multiplicity. The closed geodesics in $\bar{\Lambda}_{*,0}^{9\pi^2/2-}$ consist of the ones of minimal length, i.e., the great circles. Hence $\bar{\Lambda}_{*,0}^{9\pi^2/2-}$ is homotopy equivalent to $G(2, n-1)$, the space of unoriented two planes in $\mathbb{R}^{n+1}$. Thus $\bar{\Lambda}_{*,0}^{9\pi^2/2}$ and $\bar{\Lambda}_{*,0}^{\pi^2/2}$ have $g(n)$ subordinate homology classes and the inclusions in ($*$) together with naturality of cap products implies that $\bar{\Lambda}_*^{9\pi^2/2}$ also has $g(n)$ subordinate homology classes. Standard Lusternik–Schnirelmann theory now implies that there exist $g(n)$ closed geodesics in $\bar{\Lambda}_*^{9\pi^2/2}$, i.e. $g(n)$ closed geodesics with lengths $< 3\pi$. Since every closed geodesic has length $> \pi$ and since two fold iterates do not lie in $\bar{\Lambda}_*$, these closed geodesics are geometrically different. □

4. Closed geodesics on convex surfaces

In this chapter we study metrics of positive curvature on S^2. As is well-known, such metrics can be realized by embeddings into Euclidean space E^3. We first improve Theorem (1.7) on the gap in the length spectrum. We get the following result.

4.1. THEOREM. *Suppose M is diffeomorphic to S^2 and $\frac{1}{9} < \delta \leq K \leq 1$. Then there does not exist any prime closed geodesic with length in $(2\pi/\sqrt{\delta}, 6\pi)$.*

Remark. This implies that there exists no closed geodesic with length in $(4\pi/\sqrt{\delta}, 6\pi)$ if $\delta > \frac{4}{9}$, since a closed geodesic of length $<6\pi$ which is not prime is a twofold cover of a closed geodesic without self-intersections and hence has length $\leq 4\pi/\sqrt{\delta}$ by [To].

Proof. We say that a prime closed geodesic $c:[0,1]\to M$ has k self-intersections if $k=\sum_{x\in \mathrm{im}(c)}(\#\{0\leq t\leq 1 \mid c(t)=x\}-1)$. If a prime closed geodesic has only one self-intersection, it is the boundary of a convex polygon and hence has length $\leq 2\pi/\sqrt{\delta}$ by [To]. If c has more than one self-intersection one has the following cases: Either there exist $0\leq t_1<t_1'<t_2<t_2'\leq 1$ with $c(t_i)=c(t_i')$, or there exist $0\leq t_1<t_2<t_3<t_1'<t_2'<t_3'<1$ with $c(t_i)=c(t_i')$. In the first case c consists of at least two geodesic loops and a biangle, in the second case of at least three biangles. In either case $L(c)\geq 6\pi$ by (1.2). □

Remark. It follows from arguments as in the proof that there do not exist prime closed geodesics with 1, 2, or 3 self-intersections if $\delta > \frac{1}{4}$, $\frac{1}{9}$, or $\frac{4}{9}$ respectively. Furthermore, if c is a prime closed geodesic with k self-intersections, then $L(c)\leq (k+2)\pi/\sqrt{\delta}$ since the complement of c consists of $k+2$ regions with convex polygons as boundary.

Let α_0 be the energy of a shortest closed geodesic. One can ask whether there exists a homotopy class $h\in\pi_\alpha(M)$ with $\alpha_\Lambda(h)=\alpha_0$. In general this is false as the closed geodesic on the equator of an hour glass shows. We can prove:

4.2. THEOREM. *Suppose M is diffeomorphic to S^2 and $\frac{1}{4}\leq K\leq 1$. If h is a generator of $\pi_2(M)$, then $\alpha_0=\alpha_\Lambda(h)$. Any shortest closed geodesic on M has no self-intersections and index* 1.

Proof. By (1.6) we can assume that a shortest closed geodesic has length $<4\pi$ and hence no self-intersections. c has index ≥ 1 since a parallel orthogonal vectorfield X along c satisfies $H(X,X)<0$.

M is the union of two closed balls B_+ and B_- such that $B_+\cap B_-=\mathrm{im}\,(c)$. B_+ and B_- are both locally convex since c is a geodesic. Let X be the parallel vector field pointing into B_+. The map $f:S^1\times[-\varepsilon,\varepsilon]\to M$, $(e^{2\pi it},s)\to \exp_{c(t)}(s\cdot X(t))$, $\varepsilon>0$ sufficiently small, defines a variation of c such that the closed curves f_τ defined by $f_\tau(t)=f(t,\tau)$ lie in B_+ for $\tau>0$ and in B_- for $\tau<0$, and $E(f_\tau)<E(c)$ for $\tau\neq 0$. One can now easily extend f to a variation $\tilde{f}:S^1\times[-1,+1]\to M$ with $E(\tilde{f}_\tau)<E(c)$ for all $\tau\neq 0$ and such that $\tilde{f}_\tau$ lies in B_+ for $\tau>0$ and in B_- for $\tau<0$ and $\tilde{f}_1$, $\tilde{f}_{-1}$ are point curves in B_+ resp. B_-: deform $f_{\pm\varepsilon}$ into geodesic polygons in $B_\pm$ and then apply the negative gradient flow of the energy in a finite dimensional approximation, see [Mi], §16. The curves stay in $B_\pm$ since $B_\pm$ is locally convex and one eventually obtains point curves since there exist no closed geodesics of energy $<\alpha_0$. $\tilde{f}$ can be viewed as a map $g: I^2/\partial I^2 = S^2 \to M$ such that g_Λ (up to homotopy) consists of the curves $\tilde{f}_\tau$. g has degree ± 1 since the inverse image of $c(t)$ consists only of one point. Hence g is a generator of $\pi_2(M)$, and by changing X into $-X$ if necessary, g is in the homotopy class h. Hence $\alpha_\Lambda(h) \leq \alpha_0$ and since α_0 is the energy of a shortest closed geodesic we obtain $\alpha_\Lambda(h) = \alpha_0$.

If $\operatorname{ind}(c) > 1$ one can apply Lemma 2 in [CG] to a finite dimensional approximation to show that $\alpha_\Lambda(h) < \alpha_0$. This is a contradiction. Hence $\operatorname{ind}(c) = 1$. □

Remarks. (a) If one can show that a shortest closed geodesic on a convex surface has no self-intersections the above proof would apply and show that $\alpha_0 = \alpha_\Lambda(h)$ and $\operatorname{ind}(c) = 1$.

(b) One can also show that for an arbitrary metric on S^2 a closed geodesic without self-intersections which is shortest among all closed geodesics without self-intersections has index ≤ 1. This follows as in the above proof if one replaces the negative gradient flow by the Lusternik–Schnirelmann deformation, which leaves the set of closed curves without self-intersections invariant, see [LS] and [Ba].

5. Closed geodesics on convex hypersurfaces

5.1. THEOREM. *Let M be a convex hypersurface in E^{n+1} which contains a ball of radius r and is contained in a ball of radius R. Assume that $2r > R$. Then there are at least $g(n)$ closed geodesics on M with lengths in the interval* $[2\pi r, 2\pi R]$.

Remark. The projection of a convex hypersurface onto a convex hypersurface inside its interior is length-decreasing. Hence the assumption $2r > R$ implies the Morse condition. In addition, we have the estimate of C. Croke [Cr], Theorem 1.5: the closed geodesics on a convex hypersurface containing a ball of radius r are of length $\geq 2\pi r$; equality occurring if and only if the hypersurface is tangential to the ball of radius r along a great circle. Under the hypothesis of the theorem we can prove that M has at least $g(n)$ closed geodesics with lengths in the interval $[2\pi r, 2\pi R)$ unless the ball of radius R is tangential to the hypersurface along a great circle.

Note that the proof below cannot be used to decide whether the $g(n)$ closed geodesics have self-intersections or not.

Proof. The intersections of M with two-planes define a map from the space of parameterized circles on a sphere into the space $P(M)$ (in the notation of

[BTZ2]). The lengths of these curves are $\leq 2\pi R$ since they are contained in the convex hull of circles on the sphere of radius R, and their lengths are $<2\pi R$ unless M is tangential to the sphere of radius R along a great circle. As in Theorem (2.4) in [BTZ2] this gives rise to $g(n)$ subordinate homology classes in $(\bar{P}^{2\pi^2R^2}, (\bar{V}\cap\bar{P})^{2\pi^2R^2})$. The theorem of Croke quoted in the remark above now implies, together with Lemma (1.5)(ii) in [BTZ2], that there are $g(n)$ closed geodesics on M with lengths in $[2\pi r, 2\pi R]$. $\square$

REFERENCES

[Al] S. I. ALBER, *The Topology of Functional Manifolds and the Calculus of Variations in the Large.* Uspehi Mat. Nauk (N.S.) *25* (1970), 57–122 [Russian]; Russ. Math. Surveys *25* (1970), 51–117.

[An] D. V. ANOSOV, *Some Homotopies in Spaces of Closed Curves.* Izvestija Akad. Nauk. SSSR *44* (1980), 1219–1254 [Russian].

[Ba] W. BALLMANN, *Der Satz von Lusternik und Schnirelmann.* Bonner Math. Schriften *102* (1978), 1–25.

[BTZ1] W. BALLMANN, G. THORBERGSSON and W. ZILLER, *Closed Geodesics on Positively Curved Manifolds.* Ann. of Math. *116* (1982), 213–247.

[BTZ2] W. BALLMANN, G. THORBERGSSON and W. ZILLER, *Existence of Closed Geodesics on Positively Curved Manifolds.* J. Diff. Geom., to appear.

[Be] M. BERGER, *Sur les variétés à courbure positive de diamètre minimum.* Comm. Math. Helv. *35* (1961), 28–34.

[Bs] A. L. BESSE, *Manifolds all of whose Geodesics are Closed.* Ergebnisse der Mathematik und ihrer Grenzgebiete 93, Springer-Verlag, Berlin–Heidelberg–New York 1978.

[Bi] G. D. BIRKHOFF, *Dynamical Systems.* American Mathematical Society Colloquium Publications, Volume IX. American Mathematical Society, New York 1927.

[CE] J. CHEEGER and D. G. EBIN, *Comparison Theorems in Riemannian Geometry.* North-Holland Publishing Company, Amsterdam 1975.

[CG] J. CHEEGER and D. GROMOLL, *On the Injectivity Radius of* 1/4*-pinched Manifolds.* J. Diff. Geom. *15* (1980), 437–442.

[Cr] C. CROKE, *Poincare's Problem and the Length of the Shortest Closed Geodesic on a Convex Hypersurface.* Preprint, Philadelphia 1982.

[GS] K. GROVE and K. SHIOHAMA, *A generalized sphere theorem.* Ann. Math. *106* (1977), 201–211.

[Hi] N. B. HINGSTON, *Equivariant Morse Theory and Closed Geodesics.* Thesis, Harvard, 1981.

[Ka] H. KARCHER, *Anwendungen der Alexandrowschen Winkelvergleichssatze.* Manuscripta Math. *2* (1970), 77–102.

[Kl] W. KLINGENBERG, *Lectures on Closed Geodesics.* Grundlehren der Mathematischen Wissenschaften 230, Springer-Verlag, Berlin–Heidelberg–New York 1978.

[KS] W. KLINGENBERG and T. SAKAI, *Injectivity radius estimate for* 1/4*-pinched manifolds.* Archiv d. Math. *34* (1980), 371–376.

[LF] L. LUSTERNIK and A. I. FET, *Variational Problems on Closed Manifolds.* Dokl. Akad. Nauk SSR *81* (1951), 17–18. [Russian.]

[LS] L. LUSTERNIK and L. SCHNIRELMANN, *Sur le problème de trois géodésiques fermées sur les surfaces de genre 0.* C.R. Acad. Sci. Paris *189* (1929), 269–271.

[Mi] J. MILNOR, *Morse Theory.* Annals of Mathematics Studies 51, Princeton University Press, Princeton 1963.

[Mo] M. MORSE, *The Calculus of Variations in the Large.* Amer. Math. Soc. Colloq. Publ. 18, Amer. Math. Soc., Providence, R.I. 1934.

[Sa] T. SAKAI, *On the diameter of some Riemannian manifolds.* Archiv. d. Math. *30* (1978), 427–434.

[Sh] K. SHIOHAMA, *The Diameter of δ-Pinched Manifolds.* J. Diff. Geom. *5* (1971), 61–74.

[Su] M. SUGIMOTO, *On Riemannian Manifolds with a Certain Closed Geodesic.* Tôhoku Math. J. *22* (1970), 56–64.

[Th] G. THORBERGSSON, *Non-Hyperbolic Closed Geodesics.* Math. Scand. *44* (1979), 135–148.

[To] V. A. TOPONOGOV, *Computation of the Length of a Closed Geodesic on a Convex Surface.* Dokl. Akad. Nauk SSSR *124* (1959), 282–284. [Russian.]

[Ts] Y. TSUKAMOTO, *Closed Geodesics on Certain Riemannian Manifolds of Positive Curvature.* Tôhoku Math. J. *18* (1966), 138–143.

Mathematisches Institut
Wegelerstr. 10
53 Bonn 1
West Germany

Department of Mathematics
University of Pennsylvania
Philadelphia, Pennsylvania 19104
U.S.A.

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