Comment. Math. Helvetici 58 (1983) 573–598 0010-2571/83/040573-26\$01.50 + 0.20/0

The integral homology of SL_2 and PSL_2 of euclidean imaginary quadratic integers

JOACHIM SCHWERMER and KAREN VOGTMANN

Introduction

One way to study the cohomology of a group Γ of finite virtual cohomological dimension is to find a finite dimensional contractible space X on which Γ acts properly (such a space X always exists by [18], 1–7), and to then analyze the action. In this paper we want to consider the arithmetic groups $SL_2(\mathcal{O})$ and $PSL_2(\mathcal{O}) = SL_2(\mathcal{O})/\pm I$ where $\mathcal{O}$ is the ring of integers in an imaginary quadratic number field k; the classical choice of X in this case is hyperbolic three-space H, i.e., the associated symmetric space $SL_2(\mathbb{C})/SU(2)$. As early as 1892 Bianchi [2] exhibited fundamental domains for the action of $PSL_2(\mathcal{O})$ on H for some small values of the discriminant. The space H has also turned out to be very useful in studying the relation between automorphic forms associated to $SL_2(\mathcal{O})$ and the cohomology of $SL_2(\mathcal{O})$ (cf. [12], [10]), and in studying the topology of certain hyperbolic 3-manifolds (cf. [25]).

However, this choice of X is inconvenient for actual explicit computations of the cohomology of $\Gamma = (P)SL_2(\mathcal{O})$ with integral coefficients because the dimension of H is three, whereas the virtual cohomological dimension of Γ is two, indicating that it may be possible for Γ to act properly on a contractible space of dimension two; in addition, the quotient $\Gamma \backslash H$ is not compact. A more useful space X for our purposes is given by work of Mendoza [14], which we recall in §3; using Minkowski's reduction theory (cf. §2), he constructs a Γ-invariant 2-dimensional deformation retract $I(k)$ of H such that the quotient of $I(k)$ by any subgroup of Γ of finite index is compact; $I(k)$ is endowed with a natural CW structure such that the action of Γ is cellular and the quotient $\Gamma \backslash I(k)$ is a finite CW-complex.

The main object of this paper is to show how this construction can be used to completely determine the integral homology groups of $PSL_2(\mathcal{O})$. This is done by analyzing a spectral sequence which relates the homology of $PSL_2(\mathcal{O})$ to the homology of the quotient space $PSL_2(\mathcal{O}) \backslash I(k)$ and the homology of the stabilizers of the cells (cf. [5], VII). We will confine our computations to the cases where $\mathcal{O}$ is a euclidean ring, i.e., $\mathcal{O} = \mathcal{O}_{-d}$ is the ring of integers in $k = \mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 3, 7$ and 11. We will write out in detail the case $d = 2$ (cf. §5), which contains all the essential features in the computations. In the other cases, we indicate briefly any necessary modifications in the analysis of the spectral sequence and list the results.

The complex $I(k)$ and spectral sequence may be used to compute homology and cohomology groups for the groups $SL_2(\mathcal{O})$, $GL_2(\mathcal{O})$ and $PGL_2(\mathcal{O})$ as well as $PSL_2(\mathcal{O})$. The homology of these groups with coefficients in the Steinberg module has particular interest in algebraic K-theory ([15]). We indicate here how to do this computation for $SL_2(\mathcal{O}_{-2})$ and list the results in the other euclidean cases (cf. §6).

We conclude the paper with some observations on torsion classes in the cohomology of subgroups of finite index in $SL_2(\mathcal{O})$ which are not detected by the torsion in the stabilizers of cells in $I(k)$.

Both authors would like to thank the Institute for Advanced Study in Princeton for its hospitality during the spring of 1981 when this work was begun, and the Max Planck Institut in Bonn where this paper was finished. We would also like to thank J. Smillie, E. Mendoza and F. Grunewald for helpful discussions on this material.

Notation

(1) $\mathbb{Z}/m$ denotes the cyclic group $\mathbb{Z}/m\mathbb{Z}$, $m \in \mathbb{N}$

(2) If a is an element of $SL_2(\mathbb{C})$, we denote by $\bar{a}$ the matrix whose entries are complex conjugates of the entries of a.

(3) For a group Γ, Γ^{ab} denotes the abelianization of Γ, i.e., the quotient of Γ by its commutator subgroup.

§1. A spectral sequence

1.1 Let Γ be a group which acts cellularly on a contractible CW-complex X of dimension $\dim X = n$. If σ is a cell of X we let Γ_σ denote the stabilizer of σ in Γ, i.e. $\Gamma_\sigma = \{\gamma \in \Gamma \mid \gamma\sigma = \sigma\}$. If Σ (resp. Σ_p) is a set of representatives for Γ-orbits of cells (resp. p-cells) of X, then there exists a natural spectral sequence (cf. [5], VII, [18], p. 93ff.) whose E^1-term is given by the homology groups $H_*(\Gamma_\sigma, \mathbb{Z}_\sigma)$, $\sigma \in \Sigma$, and which converges to the homology $H_*(\Gamma, \mathbb{Z})$ of Γ with trivial coefficients $\mathbb{Z}$. (Here $\mathbb{Z}_\sigma$ denotes the Γ_σ-module $\mathbb{Z}$ given by the homomorphism $\varepsilon: \Gamma_\sigma \to \{\pm 1\}$ where $\varepsilon(\gamma) = 1$ (resp. -1) if γ fixes (resp. reverses) the orientation of σ.) This spectral sequence can be constructed as follows: Let $C_q(X)$ be the group of cellular q-chains of X. If X_q denotes the q-skeleton of X, one has $C_q(X) =$

$H_q(X_q, X_{q-1}; \mathbb{Z})$. Then the action of Γ on X gives rise to a natural action of the group algebra $\mathbb{Z}[\Gamma]$ on the cellular q-chains $C_q(X)$. Let E_* be a $\mathbb{Z}[\Gamma]$-free resolution of $\mathbb{Z}$. Then we can form the double complex $C_*(X)\otimes_{\mathbb{Z}[\Gamma]}E_*$. Since X is contractible, C_* is exact except at $C_0(X)$, where $\operatorname{cok}(\partial_1 : C_1(X) \to C_0(X)) \cong \mathbb{Z}$. Since E_p is free, $C_*(X)\otimes_{\mathbb{Z}[\Gamma]}E_p$ is still exact, except that $\operatorname{cok}(\partial_1 \otimes 1) \cong \mathbb{Z}\otimes_{\mathbb{Z}[\Gamma]}E_p$. Thus the horizontal filtration of the double complex gives us a spectral sequence with

$$E^1_{pq} = \begin{cases} 0 & q > 0 \\ \mathbb{Z}\otimes_\Gamma E_p & q = 0 \end{cases} \tag{1}$$

The differential d^1 is given by $d^1 = 1 \otimes \partial_{E_*}$, where ∂_{E_*} is the boundary map for E_*. Thus the spectral sequence converges to the homology $H_*(\Gamma, \mathbb{Z})$ with trivial $\mathbb{Z}$-coefficients, i.e., $E^2 = E^\infty = H_*(\Gamma, \mathbb{Z})$.

The vertical filtration of the double complex $C_*(X)\otimes_{\mathbb{Z}[\Gamma]}E_*$ gives us

$$E^1_{pq} = H_q(\Gamma, C_p(X)) \tag{2}$$

with the differential d^1 induced by $\partial_{C_*(X)}$. We note that $C_p(X)$ can be identified with the direct sum $\oplus_\sigma \mathbb{Z}[\Gamma]\otimes_{\mathbb{Z}[\Gamma_\sigma]}\mathbb{Z}_\sigma$, where the sum is taken over $\sigma \in \Sigma_p$. We assume from now on that Γ_σ fixes σ pointwise for every cell σ. In this case the orbit space $\Gamma\backslash X$ inherits a CW-structure. One can then orient each cell of X in such a way that the Γ-action preserves orientations. In particular, it follows the action of Γ_σ on $\mathbb{Z}$ is trivial. Thus we get

$$\begin{aligned} E_p \otimes_{\mathbb{Z}[\Gamma]} C_q(X) &= \bigoplus_{\sigma\in\Sigma_q} (E_p \otimes_{\mathbb{Z}[\Gamma]} \mathbb{Z}[\Gamma]\otimes_{\mathbb{Z}[\Gamma_\sigma]}\mathbb{Z}) \\ &= \bigoplus_{\sigma\in\Sigma_q} (E_p \otimes_{\mathbb{Z}[\Gamma_\sigma]}\mathbb{Z}) \end{aligned} \tag{3}$$

and the spectral sequence has

$$E^1_{pq} = H_q(\Gamma, C_p(X)) \cong \bigoplus_{\sigma\in\Sigma_p} H_q(\Gamma_\sigma, \mathbb{Z}). \tag{4}$$

The differential d^1 in E^1 can be described in terms of the homology of the stabilizers as follows. If $\sigma \in \Sigma$ is a p-cell in the orbit complex $\Gamma\backslash X$, and $\partial\sigma = \Sigma_\tau \pm g_\tau \tau$, where $g_\tau \in \Gamma$ and $\tau \in \Sigma$ are $(p-1)$-cells in $\Gamma\backslash X$, then there are maps

$$(i_{\sigma\tau})_* : H_*(\Gamma_\sigma, \mathbb{Z}) \to H_*(\Gamma_{g_\tau\tau}, \mathbb{Z}) \tag{5}$$

induced by the inclusion $\Gamma_\sigma \to \Gamma_{g_\tau \tau}$, resp.

$$(g_\tau)_* : H_*(\Gamma_{g_\tau \tau}, \mathbb{Z}) \to H_*(\Gamma_\tau, \mathbb{Z}), \tag{6}$$

induced by the conjugation isomorphism $a \mapsto g_\tau^{-1} a g_\tau$. The restriction of the differential $d^1 : E^1_{p,*} \to E^1_{p-1,*}$ to $H_*(\Gamma_\sigma, \mathbb{Z})$ is then given by

$$d^1_{|H_*(\Gamma_\sigma,\mathbb{Z})} = \Sigma (g_\tau)_* \circ (i_{\sigma\tau})_* \tag{7}$$

where we sum over all $\tau \in \Sigma$ which occur in $\partial\sigma$. In other words, the d^1-maps are boundary maps 'twisted' by the identifications of $\partial\sigma$ in the orbit complex $\Gamma \backslash X$.

§2. Reduction theory

Let k be an imaginary quadratic number field and $\mathcal{O}$ its ring of integers. In this section we recall briefly reduction theory for the action of $PSL_2(\mathcal{O})$ on the Poincaré upper half space $H = \{(z, \zeta) \in \mathbb{C} \times \mathbb{R} \mid \zeta > 0\}$. The version we use is based on the notion 'distance from a cusp' and is a special case of Harder's results [11]. These were inspired by ideas of Siegel obtained in the case SL_2 (resp. SL_n) over the ring of integers of a totally real number field [20]. In fact, the proofs given by Siegel there can be easily generalized to the case we are considering.

2.1 We denote by $\bar{H} = \{(z, \zeta) \in \mathbb{C} \times \mathbb{R} \mid \zeta \geq 0\} \cup \{(\infty, \infty)\}$ the extended upper half space. The usual action of $SL_2(k)$ on H extends then to one on $\bar{H}$. An element $g = \begin{pmatrix} a b \\ c d \end{pmatrix}$ of $SL_2(k)$ acts on $\bar{H}$ by

$$g(z, \zeta) = (z', \zeta') \tag{1}$$

where

$$z' = \frac{(az+b)(\bar{c}\bar{z}+\bar{d}) + a\bar{c}\zeta^2}{(cz+d)(\bar{c}\bar{z}+\bar{d}) + c\bar{c}\zeta^2}$$

resp.

$$\zeta' = \frac{\zeta}{(cz+d)(\bar{c}\bar{z}+\bar{d}) + c\bar{c}\zeta^2}.$$

We call an element $(z, \zeta) \in \bar{H}$ a *cusp* if either $(z, \zeta) = (\infty, \infty)$ or $\zeta = 0$ and $z \in k$. The set of cusps will be identified with $k \cup \{\infty\}$, the projective space over k. Let $\lambda = \alpha/\beta$, $\alpha, \beta \in k$, be a cusp; in the case $\lambda = \infty$ we write $\lambda = 1/0$. The distance from a point $(z, \zeta) \in H$ to the cusp λ is then defined by

$$n_\lambda(z, \zeta) = \frac{|z-\lambda|^2 + \zeta^2}{\zeta N_\lambda}, \qquad \lambda \neq \infty \tag{2}$$

(and $n_\infty(z, \zeta) = 1/\zeta$) where N_λ is the norm of the fractional ideal (α, β) (in $\mathcal{O}$) divided by $\beta\bar{\beta}$; note that N_λ depends only on λ and not on the choice of α and β.

Remarks. (1) Note that each level set of the smooth function $n_\lambda : H \to \mathbb{R}_+$ is a horosphere at λ. For $\lambda = \infty$, $n_\infty^{-1}(r)$, $r > 0$, is a horizontal plane of height $1/r$.

(2) The definition is intuitively motivated by the following alternative description. We can identify H with the set $\underline{H}$ of binary positive definite hermitian forms on $\mathbb{C}^2$ with determinant equal to one; a cusp λ is identified with the line L_λ in $\mathbb{C}^2$ with slope λ. The distance from a point (z, ζ) to a cusp λ is then the area of a fundamental domain for $L_\lambda \cap \mathcal{O}^2$ in $\mathbb{C}^2$ measured using the hermitian form in $\underline{H}$ corresponding to (z, ζ), and normalized so that $n_\infty(0, 1) = 1$.

The distance function to a cusp has the following invariance property which follows easily from the reinterpretation in remark (2): Let (z, ζ) be a point in H and λ a cusp of k; then for any $g \in SL_2(\mathcal{O})$

$$n_{g\lambda}(g(z, \zeta)) = n_\lambda(z, \zeta) \tag{3}$$

The main results in reduction theory can then be formulated in terms of the n_λ as follows (cf. [11], §1).

(i) For a given point $(z, \zeta) \in H$ and a fixed constant C there are only finitely many cusps μ such that $n_\mu(z, \zeta) \leq C$.

(ii) There is a constant C_1 (depending only on k) such that for every point (z, ζ) in H there exists at least one cusp λ of k such that $n_\lambda(z, \zeta) \leq C_1$.

(iii) There exists a constant C_2 with the following property: For each $(z, \zeta) \in H$ there is at most one cusp μ of k such that $n_\mu(z, \zeta) < C_2$ i.e. if $n_\mu(z, \zeta) < C_2$ and $n_{\mu'}(z, \zeta) < C_2$ then $\mu = \mu'$.

For later use we fix such a constant C_2. Elementary proofs of (i)–(iii) which also yield actual values for C_1 and C_2 are given in [14], (§1).

2.2 *Fundamental domain for $PSL_2(\mathcal{O})$.* We conclude this section by reviewing the construction of a strict fundamental domain for the action of $PSL_2(\mathcal{O}) = SL_2(\mathcal{O})/\pm I$ on H. To each cusp λ of k we associate the set

$$H(\lambda) = \{(z, \zeta) \in H \mid n_\lambda(z, \zeta) \leq n_\mu(z, \zeta) \text{ for all cusps } \mu \neq \lambda\};$$

this is called the minimal set of λ. By property 2.1 (iii) of the distance function, the set $H(\lambda)$ is non-empty. Moreover, each $H(\lambda)$ is a closed subset of H, and we denote its boundary by $I(\lambda)$. The main facts in reduction theory imply that the sets $H(\lambda)$ make up a locally finite closed covering of H. Note that the minimal sets transform under an element of $PSL_2(\mathcal{O})$ in the following way

$$gH(\lambda) = H(g\lambda), \qquad g \in PSL_2(\mathcal{O}). \tag{1}$$

We can now begin to construct the fundamental domain. One knows ([20], p. 242) that there are exactly h_k $PSL_2(\mathcal{O})$-orbits of cusps of k, where $h = h_k$ denotes the class number of k. Let $\lambda_1, \ldots, \lambda_h$ be representatives of these $PSL_2(\mathcal{O})$-orbits. Furthermore, denote by Γ_{λ_i}, $i = 1, \ldots, h$, the isotropy group of λ_i in $PSL_2(\mathcal{O})$, and let $T(\lambda_i)$ be a fundamental domain for the action of Γ_{λ_i} on H. Now let $F_i = H(\lambda_i) \cap T(\lambda_i)$, $i = 1, \ldots, h_k$. Then

$$F = \bigcup_{i=1}^{h} F_i \tag{2}$$

is a fundamental domain for the action of $PSL_2(\mathcal{O})$ on H. We refer to Siegel's notes [20], p. 261–269, for a detailed proof.

Finally, we have as a consequence the following compactness criterion (cf. [20], p. 270): Let $r_1, \ldots, r_h$ be positive real numbers. Then the set

$$F(r_1, \ldots, r_h) = \{(z, \zeta) \in F \mid n_{\lambda_i}(z, \zeta) \geq r_i \text{ for all } 1 \leq i \leq h\} \tag{3}$$

is compact.

§3. The minimal incidence set

We now review Mendoza's construction of a contractible CW-complex $I(k)$ with a $PSL_2(\mathcal{O})$-action. $I(k)$ is a 2-dimensional closed subspace of H with $PSL_2(\mathcal{O})$-equivariant deformation retraction from H to $I(k)$. The quotient $\Gamma \backslash I(k)$ by a subgroup Γ of finite index of $PSL_2(\mathcal{O})$ is a compact finite CW complex, with cell structure inherited naturally from $I(k)$. In [14] he uses extensively the main facts in reduction theory.

3.1 The minimal incidence set $I(k)$ of the given imaginary quadratic number field k is defined as the set of all points (z, ζ) in H which lie in the minimal sets of (at least) two different cusps, i.e. $I(k)$ is formally given as

$$I(k) = \bigcup_{\substack{(\lambda,\mu) \\ \lambda,\mu \text{ distinct cusps}}} H(\lambda) \cap H(\mu). \tag{1}$$

By 2.2(1) one sees that $I(k)$ is stable under the action of $PSL_2(\mathcal{O})$. Moreover, $I(k)$ is closed since the sets $H(\lambda) \cap H(\mu)$ form a locally finite closed covering of $I(k)$.

Remark. If one does the same construction on the upper half plane (substitute the real variable x for the complex variable z, and the projective space $P_1(\mathbb{Q})$ for the cusps of k) one obtains the tree for $SL_2(\mathbb{Z})$ studied by Serre [19], p. 52.

3.2 THEOREM (Mendoza [14]). (i) *The minimal incidence set $I(k)$ in H associated to an imaginary quadratic field k is a closed subspace of the symmetric space H, such that $I(k)$ is invariant under the proper action of $PSL_2(\mathcal{O})$ and the quotient $PSL_2(\mathcal{O})\backslash I(k)$ is compact. Moreover, $I(k)$ is a $PSL_2(\mathcal{O})$-equivariant deformation retract of H, and hence connected and contractible.*

(ii) *The set $I(k)$ is naturally endowed with the structure of a 2-dimensional locally finite regular CW-complex. The action of $PSL_2(\mathcal{O})$ on $I(k)$ is cellular, and so, $PSL_2(\mathcal{O})\backslash I(k)$ is a finite CW-complex.*

Since the thesis [14] of Mendoza is not easily at hand everywhere we sketch his proof with his kind permission.

Ad(i). We observed already that $I(k)$ is closed. To show that $PSL_2(\mathcal{O})\backslash I(k)$ is compact it suffices to exhibit a compact set $K \subset H$ such that $I(k) = \Gamma \cdot K$. We take $K = I(k) \cap F$ where F is the fundamental domain described in 2.2. This set is non-empty, closed, and by 2.1(iii) contained in $F' = F \cap \{(z, \zeta) \in H \mid n_{\lambda_i}(z, \zeta) \geq C_2\}$, where $\lambda_r, \ldots, \lambda_h$ are representatives of $PSL_2(\mathcal{O})$-orbits of cusps. The compactness criterion (cf. 2.2.(3)) implies that F' is compact. To show that $I(k)$ is a deformation retract of H it suffices to prove that for each cusp λ the boundary $I(\lambda)$ is a deformation retract of $H(\lambda)$, since the sets $H(\lambda)$ make up a locally finite closed covering of $I(k)$ as pointed out before.

This latter assertion follows from the fact that the distance function n_λ is a Morse function without critical points in $H(\lambda) - I(\lambda)$. (The retraction is perpendicular to the level sets of n_λ, i.e. we retract along a geodesic. For $\lambda \neq \infty$ this is a vertical semicircle, for $\lambda = \infty$ a vertical half line). The $PSL_2(\mathcal{O})$-equivariance is a direct consequence of 2.1.(3).

Ad(ii). The main ingredient here is to prove that each set $H(\lambda)\cap H(\mu)$ with λ, μ distinct cusps can be obtained by intersecting a finite number of hemispheres with center in the plane $\zeta=0$ and vertical half planes. Hence $H(\lambda)\cap H(\mu)$ is in a natural way a regular cell. For the rather technical complete proof we refer to [14], p. 26–38.

Remarks. (1) The theorem is clearly true for any subgroup Γ in $PSL_2(\mathcal{O})$ of finite index. Furthermore, for any such Γ one can refine the natural cell structure such that the stabilizers Γ_x in Γ remain the same for all points x in an open cell.

(2) Note that the dimension of $I(k)$ is exactly the same as the virtual cohomological dimension of Γ.

3.3 If the class number of k is one, the reduction theory described in §2 coincides with classical reduction theory as initiated by Bianchi [2] and pursued by Humbert [13] (for an account of the latter one see Swan [24]). Therefore, in this case, the minimal incidence set $I(k)$ can be obtained as the translation by Γ of the 'bottom'-boundary $B(k)$ of the classical fundamental domains determined by Bianchi and Humbert. Indeed, for someone familiar with the geometry of the examples in [2] it is not too difficult to work out separately in each case of class number one a fundamental domain for the action of $PSL_2(\mathcal{O})$ on H which is good enough for actual cohomological computations. Part of this is done by Flöge in his unpublished thesis [8], where he also constructs fundamental domains in some cases of class number two. He has used this to give group theoretical descriptions of $PSL_2(\mathcal{O}_{-d})$ for $d=1, 2, 3, 5, 6, 7, 10, 11$ as amalgamated products of suitable subgroups or as HNN-extensions of such products. But we preferred to use Mendoza's conceptual and systematic approach in order to have some general foundations for later considerations in [9], [26].

§4. Finite subgroups of $PSL_2(\mathcal{O})$

In this section we list the finite groups which may occur as subgroups of $PSL_2(\mathcal{O})$, together with their integral homology. These homology groups appear in the spectral sequence described in §1 which we will use to compute the homology of $PSL_2(\mathcal{O})$.

4.1 The only finite subgroups which occur in $SL_2(\mathbb{C})$ are the binary polyhedral groups (see, e.g. [22], §4.4). Let x be an element of $SL_2(\mathcal{O})\leq SL_2(\mathbb{C})$ of finite order n. Then x has eigenvalues ρ and $\bar{\rho}$, where ρ is a primitive nth root of unity. Since $\operatorname{tr} x=\rho+\bar{\rho}$ is in $\mathcal{O}\cap\mathbb{R}=\mathbb{Z}$, we must have $n=1, 2, 3, 4$ or 6. Thus the only finite subgroups of $SL_2(\mathcal{O})$ which can occur are cyclic of the above orders, the quaternion group, binary tetrahedral group and binary octahedral group. Since the stabilizer of any point in H is finite and contains $\pm I$, the possible stabilizers in $PSL_2(\mathcal{O})$ are the cyclic groups of orders two and three, the Klein four-group $D_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, the symmetric group S_3 and the alternating group A_4. The integral homology of these groups can be computed by elementary means (see, e.g. [5]) and is given in the following tables:

4.2 LEMMA. *The integral homology of the finite groups $G = \mathbb{Z}/n$, $(n > 0)$, D_2, S_3, A_4 respectively is given as follows (for simplicity we neglect to write the trivial coefficients $\mathbb{Z}$)*

$$H_q(\mathbb{Z}/n) = \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z}/n & q \text{ odd} \\ 0 & q \text{ even}, q>0 \end{cases} \tag{1}$$

$$H_q(D_2) = \begin{cases} \mathbb{Z} & q=0 \\ (\mathbb{Z}/2)^{(q+3)/2} & q \text{ odd} \\ (\mathbb{Z}/2)^{q/2} & q \text{ even}, q>0 \end{cases} \tag{2}$$

$$H_q(S_3) = \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z}/2 & q \equiv 1\ (4) \\ 0 & q \equiv 2\ (4) \\ \mathbb{Z}/6 & q \equiv 3\ (4) \\ 0 & q \equiv 0\ (4),\ q>0. \end{cases} \tag{3}$$

$$H_q(A_4) = \begin{cases} \mathbb{Z} & q=0 \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/3 & q=6k+1 \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/2 & q=6k+2 \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/6 & q=6k+3 \\ (\mathbb{Z}/2)^k & q=6k+4 \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/6 & q=6k+5 \\ (\mathbb{Z}/2)^k & q=6(k+1) \end{cases} \tag{4}$$

In order to compute in the spectral sequence, we must also determine the maps induced on homology by the inclusions of cyclic groups into D_2, S_3 and A_4; these are given in the following series of lemmas.

4.3 LEMMA. (1) *Any inclusion* $i:\mathbb{Z}/2\to S_3$ *induces an injection on homology.*

(2) *An inclusion* $i:\mathbb{Z}/3\to S_3$ *induces an injection on homology in degrees congruent to* 3 *mod* 4 (*and is otherwise zero*).

Proof. One can directly compute the Leray spectral sequence of the extension

$$1\to\mathbb{Z}/3\to S_3\to\mathbb{Z}/2\to 1$$

The action of $\mathbb{Z}/2$ on $\mathbb{Z}/3$ is the non-trivial action, and the result is that $E^2_{pq}=0$ for $p,\ q>0$, $E^2_{p,0}=H_p(\mathbb{Z}/2)$ and $E^2_{0,q}=\mathbb{Z}/3$ in dimensions congruent to 3 mod 4, zero otherwise. The diagrams

$$\begin{tikzcd} & & \mathbb{Z}/2 \arrow[r, phantom, "="] \arrow[d, "i"'] & \mathbb{Z}/2 \arrow[d] & \\ 1 \arrow[r] & \mathbb{Z}/3 \arrow[r] & S_3 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 1 \end{tikzcd}$$

and

$$\begin{tikzcd} & \mathbb{Z}/3 \arrow[r, phantom, "="] \arrow[d] & \mathbb{Z}/3 \arrow[d] & & \\ 1 \arrow[r] & \mathbb{Z}/3 \arrow[r, "i"] & S_3 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 1 \end{tikzcd}$$

induce maps of $H_*(\mathbb{Z}/2)$ onto the bottom row of the spectral sequence, and of $H_*(\mathbb{Z}/3)$ onto the left-hand column of the spectral sequence, $H_q(\mathbb{Z}/3)\to H_0(\mathbb{Z}/2,\ H_q(\mathbb{Z}/3))$. Since all differentials are zero $(\mathrm{Hom}\,(\mathbb{Z}/2,\mathbb{Z}/3)=0)$, these maps induce maps on the abutment $(H_*(S_3))$ as claimed.

4.4 LEMMA. *Any inclusion* $i:\mathbb{Z}/2\to D_2$ *induces an injection on homology in all dimensions.*

Proof. This is clear from the trivial extension

$$1\to\mathbb{Z}/2\xrightarrow{i} D_2\to\mathbb{Z}/2\to 1.$$

4.5 LEMMA. (1) *An inclusion* $i:\mathbb{Z}/3\to A_4$ *induces injections on homology in all dimensions.*

(2) *An inclusion* $i:\mathbb{Z}/2\to A_4$ *induces injections on homology in dimensions greater than* 1, *and is zero on* H_1.

Proof. We consider the spectral sequence of the extension

$$1\to D_2\to A_4\to\mathbb{Z}/3\to 1. \tag{3}$$

The homology of $\mathbb{Z}/3$ appears in the bottom row of the E^2-term, and every differential which originates in this row must be zero, since it lands in the 2-torsion group $H_p(\mathbb{Z}/3; H_q(D_2))$ for some $q>0$. Thus $E^2_{p,0}=E^\infty_{p,0}$, and the homology of $\mathbb{Z}/3$ injects into the homology of A_4.

For a map $i:\mathbb{Z}/2\to A_4$, factor through the (unique) 2-sylow subgroup D_2:

$$\begin{tikzcd} & \mathbb{Z}/2 \arrow[d, "\alpha"'] \arrow[dr, "i"] & & & \\ 1 \arrow[r] & D_2 \arrow[r, "\beta"] & A_4 \arrow[r] & \mathbb{Z}/3 \arrow[r] & 1 \end{tikzcd} \tag{4}$$

Then the homology of $\mathbb{Z}/2$ maps to the left-hand column of the spectral sequence by the maps

$$H_q(\mathbb{Z}/2) \xrightarrow{\alpha_*} H_q(D_2) \xrightarrow{\pi} H_0(\mathbb{Z}/3; H_q(D_2)). \tag{5}$$

The action of $\mathbb{Z}/3$ on D_2 is non-trivial, and one can compute that the composition $\pi\circ\alpha_*$ in (5) is injective for $q\geq 2$, and zero for $q=1$; also $E^2_{pq}=0$ for p and $q>0$, so all the differentials are zero, and $E^2=E^\infty$. Thus the injectivity properties extend to the abutment, and i_* is injective for $q>1$.

§5. Integral homology of $PSL_2(\mathcal{O}_{-d})$

We denote by $\mathcal{O}_{-d}$ the ring of integers of the imaginary quadratic number field $k=\mathbb{Q}(\sqrt{-d})$, $d\in\mathbb{N}$, d squarefree. In this section we will give a reasonably detailed explanation of the calculations for the integral homology of $PSL_2(\mathcal{O}_{-d})$ for $d=2$, then list the results of our computations for the other cases where $\mathcal{O}_{-d}$ is a euclidean ring, i.e. $d=1,3,7,11$.

For simplicity, we leave out the coefficients $\mathbb{Z}$ when we mean trivial $\mathbb{Z}$ coefficients.

5.1 We begin with a notion of fundamental domain which takes the cell structure on $I(k)$ into account. A finite sub-complex F of $I(k)$ is called a *fundamental cellular domain* for $PSL_2(\mathcal{O})=\Gamma$ if $I(k)=\Gamma\cdot F$ and if points in open induced 2-cells are not $PSL_2(\mathcal{O})$-equivalent. If we denote by "$\sim$" the cellular equivalence relation on F induced by identification of 0-cells or 1-cells, then it follows easily that $\sim\backslash F$ and $PSL_2(\mathcal{O})\backslash I(k)$ are isomorphic CW-complexes.

5.2 *The case $d=2$.*

Let $\omega=\sqrt{-2}$; then ω and 1 generate $\mathcal{O}_{-2}$ as a free $\mathbb{Z}$-module (lattice) in

$\mathbb{Q}(\sqrt{-2}) = k$. A fundamental cellular domain for the action of $PSL_2(\mathcal{O}_{-2})$ on the complex $I(k)$ is the area on the unit hemisphere centered at $(0, 0) \subseteq \mathbb{C} \times \mathbb{R}^+$ lying above the rectangle in $\mathbb{C}$ with vertices $\pm(\sqrt{2}/2)i$ and $\frac{1}{2} \pm (\sqrt{2}/2)i$ (cf. 4.2.5 in [14]). We label the vertices of this two-cell as follows:

$$P_1 = \left(-\frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2}\right), \quad P_2 = \left(\frac{1}{2} - \frac{\sqrt{2}}{2}i, \frac{1}{2}\right), \quad P_3 = \left(\frac{1}{2} + \frac{\sqrt{2}}{2}i, \frac{1}{2}\right), \quad P_4 = \left(\frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2}\right).$$

If we let Γ_i denote the stabilizer in $PSL_2(\mathcal{O}_{-2})$ of P_i, and Γ_{ij} the stabilizer of the one-cell P_iP_j, then we have specific descriptions of these stabilizers as follows: Let $a = \begin{pmatrix} 1 & \omega \\ \omega & -1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then:

$$\begin{array}{ll} \Gamma_{12} = \langle a \rangle \cong \mathbb{Z}/2; & \Gamma_1 = \langle a, c \rangle \cong D_2 \\ \Gamma_{23} = \langle b \rangle \cong \mathbb{Z}/3; & \Gamma_2 = \langle a, b \rangle \cong A_4 \\ \Gamma_{34} = \langle \bar{a} \rangle \cong \mathbb{Z}/2; & \Gamma_3 = \langle \bar{a}, b \rangle \cong A_4 \\ \Gamma_{41} = \langle c \rangle \cong \mathbb{Z}/2; & \Gamma_4 = \langle \bar{a}, c \rangle \cong D_2 \end{array} \tag{2}$$

Note that the stabilizer of the only 2-cell is trivial. In pictorial form, the fundamental domain and stabilizers are

$$\begin{tikzcd} P_4 \arrow[r, "\langle \bar{a} \rangle = \mathbb{Z}/2"] \arrow[d, dash, "\mathbb{Z}/2 = \langle c \rangle"'] & P_3 \arrow[d, dash, "\langle b \rangle = \mathbb{Z}/3"] \\ P_1 \arrow[r, "\langle a \rangle = \mathbb{Z}/2"'] & P_2 \end{tikzcd}$$

The top and bottom edges of the rectangle are identified by the element $g = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$ of $PSL_2(\mathcal{O}_{-2})$: $gP_1P_2 = P_4P_3$. These are the only identifications, so the quotient by $PSL_2(\mathcal{O}_{-2})$ is a cylinder.

We now feed this information into the spectral sequence described in §1, with

$$E^1_{pq} = \bigoplus_{p\text{-cells } \sigma} H_q(\Gamma_\sigma) \Rightarrow H_{p+q}(PSL_2(\mathcal{O}_{-2})).$$

There are only three non-zero columns, corresponding to the $0-$, $1-$ and $2-$ cells of the complex. In fact, since the stabilizer of the 2-cell is trivial, the third column is zero except that $E^1_{2,0}=\mathbb{Z}$. For $q>0$, the qth row is

$$H_q(\Gamma_1)\oplus H_q(\Gamma_2) \xleftarrow{d^1} H_q(\Gamma_{12})\oplus H_q(\Gamma_{23})\oplus H_q(\Gamma_{14}) \tag{3}$$

i.e.

$$\begin{aligned} H_q(D_2)\oplus H_q(A_4) &\xleftarrow{d^1} H_q(\mathbb{Z}/2)\oplus H_q(\mathbb{Z}/3)\oplus H_q(\mathbb{Z}/2) \\ (-i_*a+g_*c-i_*c,\ i_*a+g_*b-i_*b) &\xleftarrow{d^1} (a,b,c) \end{aligned} \tag{4}$$

where the maps i_* are induced by inclusion, and g_* by conjugation by g. In particular, $g_*:H_q(\Gamma_{14})\to H_q(\Gamma_1)$ is induced by the map from Γ_{14} to Γ_1 sending $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ to $\begin{pmatrix}\omega & -1\\ -1 & -\omega\end{pmatrix}$, and $g_*:H_q(\Gamma_{23})\to H_q(\Gamma_2)$ is induced by the map from Γ_{23} to Γ_2 sending $\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}$ to $\begin{pmatrix}1-\omega & \omega+1\\ 1 & \omega\end{pmatrix}$.

The bottom row of the E^1-term is just a $\mathbb{Z}$-chain complex giving the homology of the quotient; thus $E^2_{0,0}=E^2_{1,0}=\mathbb{Z}$, and $E^2_{p,0}=0$ for $p\geq 2$. Since the third column has disappeared entirely by E^2, we have $E^2=E^\infty$. Note that for q even and bigger than zero 4.2.(1), implies that $H_q(\mathbb{Z}/2)=H_q(\mathbb{Z}/3)=0$, so $E^\infty_{1,q}=E^1_{1,q}=0$ and $E^\infty_{0,q}=E^1_{0,q}=H_q(D_2)\oplus H_q(A_4)$. For q odd, we must calculate using the explicit description of the d^1-map given in 1.1.(7). For example, to calculate $g_*:H_q(\Gamma_{14})\to H_q(\Gamma_1)$, we write a resolution for $\Gamma_{14}=\mathbb{Z}/2=\langle t\rangle$, a resolution for $\Gamma_1=D_2$ and calculate the chain map induced by the map $g:\Gamma_{14}\to\Gamma_1$ given by $t\mapsto ac$:

$$\begin{tikzcd}[column sep=large] {} \arrow[r, "t-1"] & \mathbb{Z}[\mathbb{Z}/2] \arrow[r, "t+1"] \arrow[d] & \mathbb{Z}[\mathbb{Z}/2] \arrow[r, "t-1"] \arrow[d] & \mathbb{Z}[\mathbb{Z}/2] \arrow[r] \arrow[d, "g"] & 0 \\ {} \arrow[r, "\partial"'] & (\mathbb{Z}[D_2])^3 \arrow[r, "{\begin{pmatrix}a+1 & 1-c & 0\\ 0 & a-1 & c+1\end{pmatrix}}"'] & \mathbb{Z}[D_2]^2 \arrow[r, "{(a-1,\ c-1)}"'] & \mathbb{Z}[D_2] \arrow[r] & 0. \end{tikzcd} \tag{5}$$

With $\mathbb{Z}$-coefficients, this becomes

$$\begin{tikzcd}[column sep=large] \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}"'] & \mathbb{Z} \arrow[r, "2"] \arrow[d, "{\begin{pmatrix}1\\1\\1\end{pmatrix}}"'] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "{\begin{pmatrix}1\\1\end{pmatrix}}"'] & \mathbb{Z} \arrow[r] \arrow[d, "(1)"'] & 0 \\ \mathbb{Z}^5 \arrow[r, "{\begin{pmatrix}2&0&0&0&0\\0&0&2&0&0\\0&0&2&0&0\\0&0&0&0&2\end{pmatrix}}"'] & \mathbb{Z}^4 \arrow[r, "{\begin{pmatrix}0&0&0&0\\0&2&-2&0\\0&0&0&0\end{pmatrix}}"'] & \mathbb{Z}^3 \arrow[r, "{\begin{pmatrix}2&0&0\\0&0&2\end{pmatrix}}"'] & \mathbb{Z}^2 \arrow[r, "{(0\ \ 0)}"'] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd} \tag{6}$$

The effect on homology is now easily calculated, the map $g_*: \mathbb{Z}/2 \to (\mathbb{Z}/2)^{(q+3)/2}$ sends x to $(x, x, \ldots, x)$. In a similar manner, we make use of lemmas 4.3.–4.5. to calculate the other maps i_* and g_*. The end result is that the d^1-map is given by

$$\begin{array}{lcl} (\mathbb{Z}/2)^{(q+3)/2} \quad \oplus (\mathbb{Z}/2)^k \oplus (\mathbb{Z}/3) & \xleftarrow{d^1} & \mathbb{Z}/2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2 \\ (x-z, x, \ldots, x, x-x) \oplus (z, 0) & \longleftarrow & (z, \quad y, \quad x) \end{array} \tag{7}$$

We now have $E^2_{1q} = \operatorname{Ker} d_1 = \mathbb{Z}/3$ and $E^2_{0,q} = \operatorname{coker} \ d_1 = (H_q(D_2) \oplus H_q(A_4))/$ $(\operatorname{im} d_1 \cong (\mathbb{Z}/2 \oplus \mathbb{Z}/2))$. Since $E^2 = E^\infty$, the spectral sequence has E^∞-term

$$\begin{array}{l|ll} q & & \\ & \vdots & \vdots \\ q \text{ even} & H_q(D_2) \oplus H_q(A_4) & 0 \\ q \text{ odd} & (H_q(D_2) \oplus H_q(A_4))/(\mathbb{Z}/2 \oplus \mathbb{Z}/2) & \mathbb{Z}/3 \\ & \vdots & \vdots \\ & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & 0 \\ & \mathbb{Z}/2 \oplus \mathbb{Z}/3 & \mathbb{Z}/2 \oplus \mathbb{Z}/3 \\ & \mathbb{Z} & \mathbb{Z} \\ \hline & & \qquad\qquad p. \end{array} \tag{8}$$

We can now state the theorem.

5.3 THEOREM. *The integral homology of $PSL_2(\mathcal{O}_{-2})$ is given by*

$$H_q(PSL_2(\mathcal{O}_{-2})) \cong \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z} \oplus \mathbb{Z}/6 & q=1 \\ \mathbb{Z}/4 \oplus \mathbb{Z}/6 & q=2 \\ (\mathbb{Z}/2)^{2q/3} \oplus \mathbb{Z}/3 & q \equiv 0(3), q>0 \\ (\mathbb{Z}/2)^{2(q-1)/3} \oplus \mathbb{Z}/3 & q \equiv 1(3), q>1 \\ (\mathbb{Z}/2)^{2(q+1)/3} \oplus \mathbb{Z}/3 & q \equiv 2(3), q>2 \end{cases}$$

For $q \neq 2$ these results follow directly from the above computation 5.2.(8) of the E^∞-term of the spectral sequence together with the descriptions of the homology of A_4 and D_2 given in §4. For $q=2$, the spectral sequence gives us an exact sequence

$$1 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to H_2(PSL_s(\mathcal{O}_{-2})) \to \mathbb{Z}/2 \oplus \mathbb{Z}/3 \to 1.$$

To resolve the ambiguity in the 2-torsion, we consider our spectral sequence with $\mathbb{Z}/2$-coefficients; we find that for $0 \le q \le 2$,

$$E^1 = \begin{array}{c|llll} q & (\mathbb{Z}/2)^3 \oplus \mathbb{Z}/2 & \leftarrow \mathbb{Z}/2 \oplus 0 \oplus \mathbb{Z}/2 & & \\ & (c-a, c, 0, a) & \leftarrow\!\shortmid (a, 0, c) & & \\ & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \leftarrow \mathbb{Z}/2 \oplus 0 \oplus \mathbb{Z}/2 & & \\ & (c-a, 0) & \leftarrow\!\shortmid (a, 0, c) & & \\ & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \leftarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \leftarrow \mathbb{Z}/2 & \\ & (-a, a) & \leftarrow\!\shortmid (a, b, c) & & \\ \hline & & & & p \end{array}$$

giving

$$E^\infty = E^2 = \begin{array}{c|lll} q & \vdots & \vdots & \\ & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & 0 & \\ & \mathbb{Z}/2 & \mathbb{Z}/2 & \\ & \mathbb{Z}/2 & \mathbb{Z}/2 & 0 \\ \hline & & & p \end{array}$$

Thus $H_2(PSL_2(\mathcal{O}_{-2}); \mathbb{Z}/2) \cong (\mathbb{Z}/2)^3$. By the universal coefficient theorem, this is isomorphic to $H_2(PSL_2(\mathcal{O}_{-2})) \otimes \mathbb{Z}/2 \oplus \operatorname{Tor}(H_1(PSL_2(\mathcal{O}_{-2})), \mathbb{Z}/2) = H_2(PSL_2(\mathcal{O}_{-2})) \otimes \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Therefore $H_2(PSL_2(\mathcal{O}_{-2})) \otimes \mathbb{Z}/2 \cong (\mathbb{Z}/2)^2$, so $H_2(PSL_2(\mathcal{O}_{-2})) \cong \mathbb{Z}/6 \oplus \mathbb{Z}/4$.

We will now list the theorems for the other euclidean cases, indicating briefly any necessary modifications in the computations. We denote by $S = \{(z, \zeta) \in H \mid |z|^2 + \zeta^2 = 1\}$ the hemisphere with center $(0, 0)$ and radius 1. For points in S we will sometimes give only the first coordinate.

5.4 *The case* $d = 1$. A fundamental cellular domain for the action of $PSL_2(\mathcal{O}_{-1})$ on the complex $I(\mathbb{Q}(\sqrt{-1}))$ is the set $F = \{(z, \zeta) \in S \mid 0 \le \operatorname{Re} z \le \frac{1}{2}, 0 \le \operatorname{Im} z \le \frac{1}{2}\}$; the vertices are the points $P_1 = (0, 1)$, $P_2 = (\frac{1}{2}, \sqrt{3}/2)$, $P_3 = (\frac{1}{2} + \frac{1}{2}i, \sqrt{2})$, $P_4 = (\frac{1}{2}i, \sqrt{3}/2)$. There are no further identifications (cf. [14], 4.1.9 or [2], §12). Let

$$a = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} -1 & i \\ i & 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

then the cellular domain and the stabilizers of the cells are given in pictorial form as follows:

$$\begin{tikzcd}
\langle c, d\rangle \cong S_3 & P_4 \arrow[r, dash, "\langle a\rangle \cong \mathbb{Z}/3"] \arrow[d, dash, "\langle d\rangle = \mathbb{Z}/2"'] & P_3 \arrow[d, dash, "\langle b\rangle = \mathbb{Z}/3"] & \langle b, c\rangle \cong A_4 \\
\langle a, d\rangle \cong D_2 & P_1 \arrow[r, dash, "\langle a\rangle \cong \mathbb{Z}/2"'] & P_2 & \langle a, b\rangle \cong S_3
\end{tikzcd}$$

By 4.3–4.5 the inclusion maps $\Gamma_{ij} \to \Gamma_i$, $1 \le i, j \le 4$, of the stabilizers induce injections on homology, except for $\Gamma_{34} \to \Gamma_4$ and $\Gamma_{23} \to \Gamma_2$. These induce (cf. 4.2.) injections on homology of degree n if $n \equiv 3 \bmod 4$, and otherwise induce the zero map.

5.5 THEOREM. *The integral homology of $PSL_2(\mathcal{O}_{-1})$ is given by*

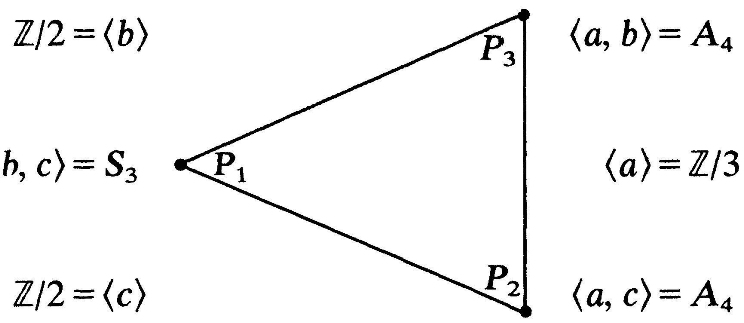
$$H_q(PSL_2\mathcal{O}_{-1}) \cong \begin{cases} \mathbb{Z} & q = 0 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^2 & q = 12k+1 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3 & q = 12k+2 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^4 \oplus \mathbb{Z}/3 & q = 12k+3 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^2 & q = 12k+4 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^5 & q = 12k+5 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^4 \oplus \mathbb{Z}/3 & q = 12k+6 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^6 \oplus \mathbb{Z}/3 & q = 12k+7 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^6 & q = 12k+8 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^7 & q = 12k+9 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^6 \oplus \mathbb{Z}/3 & q = 12k+10 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^{10} \oplus \mathbb{Z}/3 & q = 12k+11 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^8 & q = 12(k+1) \end{cases}$$

where $k \in \mathbb{N}$.

5.6 *The case $d = 3$.* A fundamental cellular domain for the action of $PSL_2(\mathcal{O}_{-3})$ on the complex $I(\mathbb{Q}(\sqrt{-3}))$ is a subset of the unit hemisphere S; the following picture contains all the information we need. The exact coordinates of the points P_i can be easily worked out if the reader so desires. (cf. [14], 4.2.3 or [2], §13). Let

$$a=\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix},\quad b=\begin{pmatrix}0 & \omega\\ \omega^2 & 0\end{pmatrix},\quad c=\begin{pmatrix}0 & \omega^2\\ \omega & 0\end{pmatrix} \tag{1}$$

where $\omega=(\frac{1}{2})(1+\sqrt{3}\,i)$. We have in pictorial form



(2)

The map $d_1:E^1_{1,q}\to E^1_{0,q}$ is injective for $q\geq 2$; for $q=1$, $d_1:H_1(\mathbb{Z}/2)\oplus H_1(\mathbb{Z}/3)\oplus H_1(\mathbb{Z}/2)\to H_1(S_3)\oplus H_1(A_4)\oplus H_1(A_4)$ sends (a,b,c) to $(-a-c,-b,b)$, and for $q=0$, the d_1-maps are the boundary maps for the integral homology of the (contractible) fundamental domain.

5.7 THEOREM. *The integral homology of $PSL_2(\mathcal{O}_{-3})$ is given by*

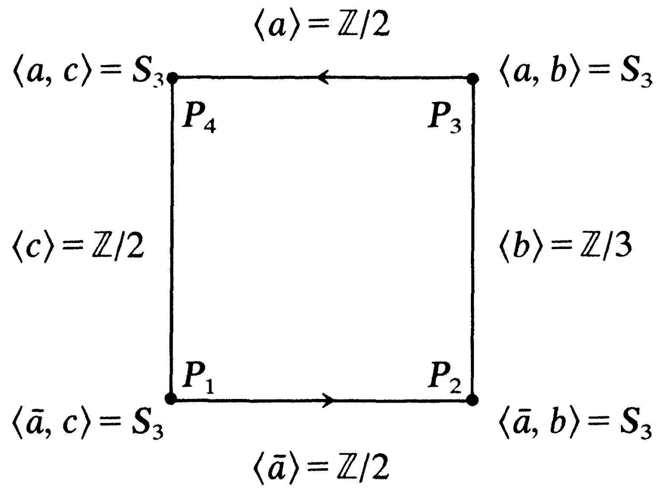
$$H_q(PSL_2(\mathcal{O}_{-3}))=\begin{cases}\mathbb{Z} & q=0\\ (\mathbb{Z}/2)^{4k} & q=12k>0\\ \mathbb{Z}/3\oplus(\mathbb{Z}/2)^{4k} & q=12k+1\\ \mathbb{Z}/4\oplus\mathbb{Z}/2\oplus(\mathbb{Z}/2)^{4k} & q=12k+2\\ \mathbb{Z}/6\oplus\mathbb{Z}/3\oplus(\mathbb{Z}/2)^{4k} & q=12k+3\\ (\mathbb{Z}/2)^{4k} & q=12k+4\\ \mathbb{Z}/6\oplus(\mathbb{Z}/2)^2\oplus(\mathbb{Z}/2)^{4k} & q=12k+5\\ (\mathbb{Z}/2)^4\oplus(\mathbb{Z}/2)^{4k} & q=12k+6\\ \mathbb{Z}/6\oplus\mathbb{Z}/3\oplus(\mathbb{Z}/2)^{4k} & q=12k+7\\ (\mathbb{Z}/2)^4\oplus(\mathbb{Z}/2)^{4k} & q=12k+8\\ \mathbb{Z}/6\oplus(\mathbb{Z}/2)^2\oplus(\mathbb{Z}/2)^{4k} & q=12k+9\\ (\mathbb{Z}/2)^2\oplus(\mathbb{Z}/2)^{4k} & q=12k+10\\ (\mathbb{Z}/2)^4\oplus\mathbb{Z}/6\oplus\mathbb{Z}/3\oplus(\mathbb{Z}/2)^{4k} & q=12k+11\end{cases}$$

where $k\in\mathbb{N}$.

5.8 *The case* $d=7$. The fundamental cellular domain for the action of $PSL_2(\mathcal{O}_{-7})$ on the complex $I(\mathbb{Q}(\sqrt{-7}))$ is again a subset of the hemisphere S; the domain and the stabilizers of the cells are given in pictorial form as follows, if we let

$$a=\begin{pmatrix}1 & -\bar{\omega}\\ \omega & -1\end{pmatrix},\quad b=\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix},\quad c=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

where $\omega=(\frac{1}{2})(1+\sqrt{-7})$. (cf. [14], 4.2.11. or [2], §16).



The element $g=\begin{pmatrix}0 & -1\\ 1 & \omega-1\end{pmatrix}$ sends P_1P_2 to P_3P_4, so the quotient of $I(\mathbb{Q}(\sqrt{-7}))$ by $PSL_2(\mathcal{O}_{-7})$ is topologically a Möbius band. The homology of each stabilizer is periodic of period 2 or 4 by 4.2. so we need only compute the induced maps on H_1 and H_3 to calculate the $E^2=E^\infty$-term of the spectral sequence. The result is:

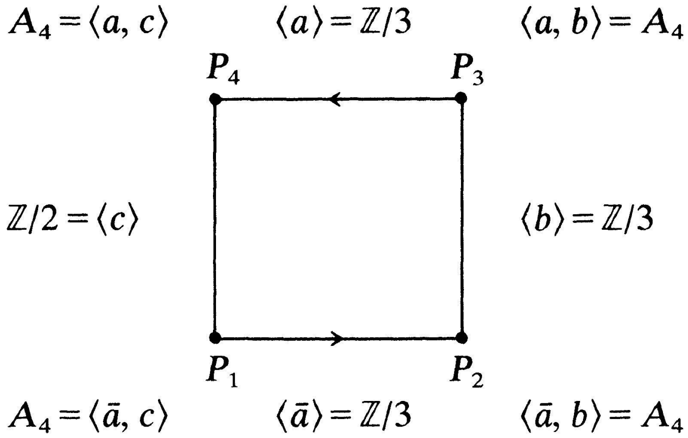
5.9 THEOREM. *The integral homology of* $PSL_2(\mathcal{O}_{-7})$ *is given by*

$$H_q(PSL_2(\mathcal{O}_{-7}))\cong\begin{cases}\mathbb{Z} & q=0\\ \mathbb{Z}\oplus\mathbb{Z}/2 & q=1\\ \mathbb{Z}/6 & q\equiv 2,3(4)\\ \mathbb{Z}/2 & q\equiv 0,1(4)\quad q\geq 4\end{cases}$$

5.10 *The case* $d=11$. From the topological point of view the situation is quite similar to the case $d=7$. Let $\omega=(\frac{1}{2})(1+\sqrt{-11})$, and put

$$a=\begin{pmatrix}-2 & \omega\\ -\bar{\omega} & 1\end{pmatrix},\quad b=\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix},\quad c=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

The fundamental cellular domain for the action of $PSL_2(\mathcal{O}_{-11})$ on the complex $I(\mathbb{Q}(\sqrt{-11}))$ is again a subset of the hemisphere S; the domain and the stabilizers of the cells can be visualized as follows: (cf. [14], 4.2.12 or [2], §16).



The element $g = \begin{pmatrix} -\omega & 1 \\ -1 & 0 \end{pmatrix}$ sends P_1P_2 to P_3P_4, and there are no further identifications, so the quotient $PSL_2(\mathcal{O}_{-11})\backslash I(\mathbb{Q}(\sqrt{-11}))$ is topologically a Möbius band. The d^1-maps in odd dimensions $q \geq 3$ are $(x, y) \mapsto (x-y, y-x)$ on 3-torsion and injective on 2-torsion; for $q = 1$, the d^1-map is the same on 3-torsion but zero on $H_1(\mathbb{Z}/2)$. There is ambiguity in the 2-torsion of $H_2(PSL_2(\mathcal{O}_{-11}))$ which can be resolved, as in the case $d = 2$, by looking at the spectral sequence with $\mathbb{Z}/2$-coefficients. The result is:

5.11 THEOREM. *The integral homology of $PSL_2(\mathcal{O}_{-11})$ is given by*

$$H_q(PSL_2(\mathcal{O}_{-11})) \cong \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z} \oplus \mathbb{Z}/3 & q=1 \\ \mathbb{Z}/6 \oplus \mathbb{Z}/4 & q=2 \\ (\mathbb{Z}/2)^{k+1} \oplus \mathbb{Z}/3 & q=3k-1>2 \\ (\mathbb{Z}/2)^{k} \oplus \mathbb{Z}/3 & q=3k>0 \\ (\mathbb{Z}/2)^{k-1} \oplus \mathbb{Z}/3 & q=3k+1>1 \end{cases}$$

where $k \in \mathbb{N}$.

§6. Integral cohomology of $SL_2(\mathcal{O}_{-d})$ and homology with Steinberg coefficients

The complex $I(k)$ and spectral sequence in §1 (resp. a cohomological analogue of it) may be used to compute homology and cohomology groups for the groups

$SL_2(\mathcal{O}_{-d})$, $PGL_2(\mathcal{O}_{-d})$ and $GL_2(\mathcal{O}_{-d})$ as well as $PSL_2(\mathcal{O}_{-d})$. The homology of $SL_2(\mathcal{O}_{-d})$ with coefficients in the Steinberg module $\mathrm{St}\,(2)_{-d}$ of $(\mathbb{Q}(\sqrt{-d}))^2$ has particular interest for algebraic K-theory, (cf. [15]); we indicate here how to do this computation for $d=2$ and list the results for the other euclidean cases. Since homology with Steinberg coefficients is dual to cohomology in degree >2 (see 6.4.) we begin by computing the integral cohomology of $SL_2(\mathcal{O}_{-d})$.

6.1 As noted in §5, the groups which may possibly appear as finite subgroups of $SL_2(\mathcal{O}_{-d})$ are the cyclic groups of orders 2, 3, 4 and 6, the quaternion group Q, the binary octahedral group D and the binary tetrahedral group Te. Any finite subgroup of $SL_2(\mathbb{C})$ acts freely on the maximal compact subgroup $SU_2 \subset SL_2(\mathbb{C})$, which is a 3-sphere. Such a subgroup must therefore have periodic cohomology of period dividing 4, so we need only compute four cohomology groups to obtain each of the following cohomologies:

$$H^q(Q)=\begin{cases}\mathbb{Z} & q=0\\ 0 & q\equiv 1(4)\\ \mathbb{Z}/2\oplus\mathbb{Z}/2 & q\equiv 2(4)\\ 0 & q\equiv 3(4)\\ \mathbb{Z}/8 & q\equiv 0(4),\, q>0\end{cases} \tag{1}$$

$$H^q(D)=\begin{cases}\mathbb{Z} & q=0\\ 0 & q\equiv 1(4)\\ \mathbb{Z}/4 & q\equiv 2(4)\\ 0 & q\equiv 3(4)\\ \mathbb{Z}/12 & q\equiv 0(4),\, q>0\end{cases} \tag{2}$$

$$H^q(Te)=\begin{cases}\mathbb{Z} & q=0\\ 0 & q\equiv 1(4)\\ \mathbb{Z}/3 & q\equiv 2(4)\\ 0 & q\equiv 3(4)\\ \mathbb{Z}/24 & q\equiv 0(4),\, q>0\end{cases} \tag{3}$$

$$H^q(\mathbb{Z}/n)=\begin{cases}\mathbb{Z} & q=0\\ 0 & q \text{ odd}\\ \mathbb{Z}/n & q \text{ even},\, q>0\end{cases} \tag{4}$$

6.2 We now specialize to the case $d=2$, and look at the spectral sequence of

§ 1 in cohomology (cf. [5], VII), we have

$$E_1^{pq} = \bigoplus_{\substack{\text{orbits of}\\ p\text{-cells } \sigma_p}} H^q(\Gamma_{\sigma_p}) \Rightarrow H^{p+q}(SL_2(\mathcal{O}_{-2})). \tag{1}$$

Note that the stabilizer of the 2-cell in the fundamental cellular domain for the action of $SL_2(\mathcal{O}_{-2})$ on $I(\mathbb{Q}(\sqrt{-2}))$ is now $\mathbb{Z}/2$, and not trivial as it was for $PSL_2(\mathcal{O}_{-2})$. The E_1-term and d_1-maps are as follows:

$$\begin{array}{c|ccccc}
q & & & & & \\
4 & \mathbb{Z}/8\oplus\mathbb{Z}/24 & \longrightarrow & \mathbb{Z}/4\oplus\mathbb{Z}/6\oplus\mathbb{Z}/4 & \longrightarrow & \mathbb{Z}/2\\
 & (x, y) & \longmapsto & (y-x, 0, 0) & & \\
3 & 0 & & 0 & & 0\\
2 & (\mathbb{Z}/2)^2\oplus\mathbb{Z}/3 & \longrightarrow & \mathbb{Z}/4\oplus\mathbb{Z}/6\oplus\mathbb{Z}/4 & \longrightarrow & \mathbb{Z}/2\\
 & (x_1, x_2, y) & \longmapsto & (2x_2, 0, 2x_1-2x_2) & & \\
1 & 0 & & 0 & & 0\\
0 & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}\\
 & (x, y) & \longmapsto & (y-x, 0, 0) & & \\
 & & & (a, b, c) & \longmapsto & (c-b)\\
\hline
 & 1 & & 2 & & 3 \quad p
\end{array} \tag{2}$$

The result of these computations is

6.3 THEOREM. *The integral cohomology of $SL_2(\mathcal{O}_{-2})$ is given by*

$$H^q(SL_2(\mathcal{O}_{-2})) \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z} & q=1 \\ \mathbb{Z}/3 & q\equiv 2(4) \\ \mathbb{Z}/2\oplus\mathbb{Z}/6 & q\equiv 3(4) \\ \mathbb{Z}/2\oplus\mathbb{Z}/24 & q\equiv 0(4),\ q>0 \\ \mathbb{Z}/12 & q\equiv 1(4),\ q>1 \end{cases}$$

6.4 To obtain the homology of $SL_2(\mathcal{O}_{-2})$ with coefficients in the Steinberg module $\mathrm{St}\,(2)_{-2}$, we use Farrell–Tate cohomology theory $\hat{H}^*$ for $SL_2(\mathcal{O}_{-2})$ (cf. [7] or [5], chap. X). By a general result of Borel–Serre ([4], 11.4) the group $SL_2(\mathcal{O}_{-2})$ is a virtual duality group of dimension 2 (in the sense of [3] or [6], §3) whose dualizing module is the Steinberg module $\mathrm{St}\,(2)_{-2} = H^2(SL_2(\mathcal{O}_{-2}), \mathbb{Z}[SL_2(\mathcal{O}_{-2})])$ with its natural $SL_2(\mathcal{O}_{-2})$-action (cf. [4], 11.4. and 8.6). Thus, for $q>2$, there is an isomorphism ([6], 11.7.):

$$H_q(SL_2(\mathcal{O}_{-2}), \mathrm{St}\,(2)_{-2}) \cong \hat{H}^{1-q}(SL_2(\mathcal{O}_{-2})). \tag{1}$$

For $0 \leq q \leq 2$, there is an exact sequence relating Farrell–Tate cohomology, the regular cohomology and homology with Steinberg coefficients (cf. [6], 11.8.); for simplicity we abbreviate $\hat{H}^* = \hat{H}^*(SL_2(\mathcal{O}_{-2}))$ resp. $H^* = H^*(SL_2(\mathcal{O}_{-2}))$:

$$\begin{aligned} 0 &\to \hat{H}^{-1} \to H_2(SL_2(\mathcal{O}_{-2}), \mathrm{St}\,(2)_{-2}) \\ &\to H^0 \xrightarrow{\alpha_0} \hat{H}^0 \to H_1(SL_2(\mathcal{O}_{-2}), \mathrm{St}\,(2)_{-2}) \\ &\to H^1 \xrightarrow{\alpha_1} \hat{H}^1 \to H_0(SL_2(\mathcal{O}_{-2}), \mathrm{St}\,(2)_{-2}) \to H^2 \xrightarrow{\alpha_2} \hat{H}^2 \to 0. \end{aligned} \tag{2}$$

The spectral sequence 6.2(1) can be used to compute $\hat{H}^*(SL_2(\mathcal{O}_{-2}))$; we have ([5], X, 4.1.)

$$\hat{E}_1^{p,q} = \bigoplus_{\sigma \in \Sigma_p} \hat{H}^q(\Gamma_\sigma) \Rightarrow \hat{H}^{p+q}(SL_2(\mathcal{O}_{-2})). \tag{3}$$

Note that for the finite groups Γ_σ, the Farrell–Tate groups $\hat{H}^q(\Gamma_\sigma)$ coincide with the standard Tate cohomology groups. The maps $\alpha_i : H^i \to \hat{H}^i$, in (2) are then induced by the maps on the cohomology of the stabilizers Γ_σ in the spectral sequences; these are the standard maps from cohomology to Tate cohomology, i.e. for a finite group Γ_σ of order $|\Gamma_\sigma|$ the map $H^i(\Gamma_\sigma) \to \hat{H}^i(\Gamma_\sigma)$ is an isomorphism for $i>0$ and $H^0(\Gamma_\sigma) \to \hat{H}^0(\Gamma_\sigma)$ is the morphism $\mathbb{Z} \to \mathbb{Z}/|\Gamma_\sigma|$. We use these remarks to prove

6.5 THEOREM *The cohomology of $SL_2(\mathcal{O}_{-2})$ with coefficients in the Steinberg module* St (2)$_{-2}$ *is given by*

$$H_q(SL_2(\mathcal{O}_{-2}), \mathrm{St}\,(2)_{-2}) = \begin{cases} 0 & q=0 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & q=1 \\ \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/6 & q=2 \\ \mathbb{Z}/3 & q \equiv 3(4) \\ \mathbb{Z}/12 & q \equiv 0(4), q>0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/24 & q \equiv 1(4), q>1 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/6 & q \equiv 2(4), q>2 \end{cases}$$

Proof. For $q \geq 3$, 6.4.(1) says that $H_q(SL_2(\mathcal{O}_{-2}), \mathrm{St}\,(2)_{-2}) \cong \hat{H}^{1-q}(SL_2(\mathcal{O}_{-2}))$; since $\hat{H}^*(SL_2(\mathcal{O}_{-2}))$ is periodic of period 4, and $\hat{H}^i(SL_2(\mathcal{O}_{-2})) \cong H^i(SL_2(\mathcal{O}_{-2}))$ for $i \geq 3$ (cf. [6], 11.4), the result follows from our calculation of $H^*(SL_2(\mathcal{O}_{-2}))$ in Theorem 6.3. For $0 \leq q \leq 2$, the result follows from the calculations of the maps α_i in 6.4.(2) as outlined above.

We now state the results of our computations of $H_q(SL_2(\mathcal{O}_{-d}), \mathrm{St}\,(2)_{-d})$ in the other euclidean cases:

6.6 THEOREM. *The cohomology of* $SL_2(\mathcal{O}_{-d})$ *for* $d = 1, 3, 7, 11$ *with coefficients in the Steinberg module* $\mathrm{St}\,(2)_{-d}$ *is given by*

$$H_q(SL_2(\mathcal{O}_{-1}), \mathrm{St}\,(2)_{-1}) = \begin{cases} 0 & q = 0 \\ \mathbb{Z}/4 & q = 1 \\ \mathbb{Z} \oplus \mathbb{Z}/6 & q = 2 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & q \equiv 3(4) \\ 0 & q \equiv 0(4),\ q > 0 \\ \mathbb{Z}/12 \oplus \mathbb{Z}/8 & q \equiv 1(4),\ q > 1 \\ \mathbb{Z}/6 & q \equiv 2(4),\ q > 2 \end{cases} \tag{1}$$

$$H_q(SL_2(\mathcal{O}_{-3}), \mathrm{St}\,(2)_{-3}) = \begin{cases} 0 & q = 0 \\ \mathbb{Z}/6 & q = 1 \\ \mathbb{Z} \oplus \mathbb{Z}/4 & q = 2 \\ \mathbb{Z}/3 & q \equiv 3(4) \\ 0 & q \equiv 0(4),\ q \geq 4 \\ \mathbb{Z}/24 \oplus \mathbb{Z}/6 & q \equiv 1(4),\ q \geq 5 \\ \mathbb{Z}/4 & q \equiv 2(4),\ q \geq 6 \end{cases} \tag{2}$$

$$H_q(SL_2(\mathcal{O}_{-7}), \mathrm{St}\,(2)_{-7}) = \begin{cases} 0 & q = 0 \\ \mathbb{Z} & q = 1 \\ \mathbb{Z} \oplus \mathbb{Z}/12 & q = 2 \\ \mathbb{Z}/4 & q \equiv 3(4) \\ \mathbb{Z}/4 & q \equiv 0(4),\ q \geq 4 \\ \mathbb{Z}/12 & q \equiv 1(4),\ q \geq 5 \\ \mathbb{Z}/12 & q \equiv 2(4),\ q \geq 6 \end{cases} \tag{3}$$

$$H_q(SL_2(\mathcal{O}_{-11}), \mathrm{St}\,(2)_{-11}) = \begin{cases} 0 & q=0 \\ \mathbb{Z}\oplus\mathbb{Z}/2 & q=1 \\ \mathbb{Z}\oplus\mathbb{Z}/12\oplus\mathbb{Z}/2 & q=2 \\ \mathbb{Z}/3 & q\equiv 3(4) \\ \mathbb{Z}/6 & q\equiv 0(4), q\geq 4 \\ \mathbb{Z}/24\oplus\mathbb{Z}/2 & q\equiv 1(4), q\geq 5 \\ \mathbb{Z}/12\oplus\mathbb{Z}/2 & q\equiv 2(4), q\geq 6 \end{cases} \tag{4}$$

Remark. Up to 2-torsion $H_q(SL_2(\mathcal{O}_{-1}), \mathrm{St}\,(2)_{-1})$ was determined by Staffeldt ([23], Thm. IV.1.3.).

6.7 *Torsion classes in $H^2(PSL_2(\mathcal{O}_{-d}))$.* We conclude this paper with some observations concerning torsion classes in $H^*(PSL_2(\mathcal{O}_{-d}))$. There is a natural map between the usual cohomology and the Farrell cohomology of a subgroup Γ of finite index in $PSL_2(\mathcal{O}_{-d})$

$$H^*(\Gamma) \to \hat{H}^*(\Gamma)$$

which is an isomorphism for $q > vcd(\Gamma)$, and one has $\hat{H}^*(\Gamma) = 0$ if Γ is torsionfree. It is shown in [6], §15 that a great deal of information about $\hat{H}^*(\Gamma)$ can be extracted from the finite subgroups of Γ. The arguments there are of a general nature. As pointed out in §4 there is only 2- and 3-torsion in $\hat{H}^*(PSL_2(\mathcal{O}_{-d}))$, and in the euclidean cases $d = 1, 2, 3, 7, 11$ we considered this is also true for the usual cohomology $H^*(PSL_2(\mathcal{O}_{-d}))$ (cf. §5).

We will give now some examples of subgroups Γ of small index in $PSL_2(\mathcal{O}_{-d})$, $d = 1, 3$, where one has torsion classes in the low-dimensional cohomology of Γ, whose order p is different from 2 and 3. It would be of great interest to have an arithmetic explanation for these phenomena. For more details on this subject see [9].

(1) The group $PSL_2(\mathcal{O}_{-3})$ is generated by the matrices

$$a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}$$

where $\omega = -1/2 + i\sqrt{3}/2$. In [9] it will be shown that there are seven conjugacy classes of subgroups of index 12 in $PSL_2(\mathcal{O}_{-3})$. One of these classes can be represented by the torsionfree group

$$\Gamma_2 = \langle x, y \mid xy\,x^{-1}y\,xy^{-1}x^{-1}yx^{-1}y^{-1} = 1\rangle$$

where $x=a$ resp. $y=bcb$. We note that the manifold $\Gamma_2\backslash H$ is homeomorphic to the complement of the figure-eight-knot in the three sphere S^3 (cf. [16], [17]); in particular one has $H_1(\Gamma_2)=\Gamma_2^{ab}\cong\mathbb{Z}$.

Another class can be represented by

$$\Gamma_7=\langle u, v \mid uv\, uv\, uv^2\, uv^{-1}uv^2=1\rangle$$

where $u=a^2$, $v=abc\ aba\ c^{-1}bc^{-1}ba^{-1}$. One sees easily that

$$H_1(\Gamma_7)=\Gamma_7^{ab}\cong\mathbb{Z}\oplus\mathbb{Z}/5$$

and uv is indeed an element of order 5 in Γ_7^{ab}. This implies that there is a torsion class of order 5 in $H^2(\Gamma_7)$.

(2) For a given prime ideal f of degree 1 in the ring of integers $\mathcal{O}_{-1}$ of $\mathbb{Q}(\sqrt{-1})$ we consider the group

$$\Gamma_0(f)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in PSL_2(\mathcal{O}_{-1}) \mid c\in f\right\}$$

Denote by $p=N(f)$ the norm of f. Then machine computation (cf. [9]) shows, for example, if $p=101$ that

$$\Gamma_0(f)^{ab}\cong\mathbb{Z}/4\oplus\mathbb{Z}/25\oplus\mathbb{Z}/17.$$

There are other examples of this type due to Grunewald.

REFERENCES

[1] Alperin, R., *Homology of $SL_2(\mathbb{Z}[\omega])$*. Comment. Math. Helvetici **55** (1980), 364–377.

[2] Bianchi, L., *Sui gruppi di sostituzioni lineari con coefficienti appartenenti a corpi quadratici imaginari.* Math. Annalen **40** (1892), 332–412.

[3] Bieri, R. and Eckmann, B., *Groups with homological duality generalizing Poincaré duality,* Invent. Math. **20** (1973), 103–124.

[4] Borel, A. and Serre, J. P., *Corners and arithmetic groups,* Comment. Math. Helvetici **48** (1973) 436–491.

[5] Brown, K. S., *Cohomology of groups,* Graduate Texts in Maths., vol **87,** Berlin–Heidelberg–New York 1982.

[6] Brown, K. S., *Groups of virtually finite dimension.* In: Homological group theory, ed. C.T.C. Wall. London Math. Soc. Lect. Note Series **36,** Cambridge 1979, 27–70.

[7] Farrell, F. T., *An extension of Tate cohomology to a class of infinite groups.* J. Pure Appl. Algebra **10** (1977), 153–161.

[8] Flöge, D., *Zur Struktur der PSL_2 über einigen imaginär-quadratischen Zahlringen.* Math. Zeitschrift. **183** (1983), 255–279.

[9] Grunewald, F. and Schwermer, J., *Subgroups of small index in Bianchigroups,* in preparation.

[10] GRUNEWALD, F. and SCHWERMER, J., *A non-vanishing theorem for the cuspidal cohomology of SL_2 over imaginary quadratic integers*, Math. Annalen **258** (1981), 183–200.
[11] HARDER, G., *A Gauss-Bonnet formula for discrete arithmetically defined groups.* Ann. scient. Ec. Norm. Sup., (4) **4** (1971), 409–455.
[12] HARDER, G., *Period integrals of cohomology classes which are represented by Eisenstein series.* In: Automorphic forms, Representation theory and arithmetic (Bombay colloq. 1979), Berlin–Heidelberg–New York 1981, 41–115.
[13] HUMBERT, G., *Sur la reduction des formes d'Hermite dans un corps quadratique imaginaire.* C.R. Acad. Sc. Paris, **161** (1915), 190–196.
[14] MENDOZA, E., *Cohomology of PGL_2 over imaginary quadratic integers.* Bonner Math. Schriften, no. 128, Bonn 1980.
[15] QUILLEN, D., *Finite generation of the groups K_i of rings of algebraic integers.* In: Algebraic K-Theory I, ed. H. Bass, Lect. Notes in Maths. **341,** Berlin–Heidelberg–New York. 1973.
[16] RILEY, R., *A quadratic parabolic group.* Math. Proc. Camb. Phil. Soc. **77** (1975), 281–288.
[17] SCHWERMER, J., *A note on link complements and arithmetic groups*, Math. Annalen **249** (1980), 107–110.
[18] SERRE, J. P., *Cohomologie des groupes discrets.* In: Prospects in Mathematics, Annals of Math. Studies **70,** Princeton 1971, 77–169.
[19] SERRE, J. P., *Arbres, amalgames et SL_2*, Astérisque **46,** Paris 1977.
[20] SIEGEL, C. L., *On Advanced Analytic Number theory.* Lectures on maths. and physics, vol. **23,** Tata Institute, Bombay 1961.
[21] SOULÉ, C., *The cohomology of $SL_3(\mathbb{Z})$.* Topology **17** (1978), 1–22.
[22] SPRINGER, T. A., *Invariant Theory*, Lect. Notes in Maths. **585,** Berlin–Heidelberg–New York 1977.
[23] STAFFELDT, R., *Reduction theory and K_3 of the Gaussian integers*, Duke Math. Journal **46** (1979), 773–798.
[24] SWAN, R., *Generators and relations for certain special linear groups*, Advances in Math. **6** (1971), 1–78.
[25] THURSTON, W. P., *Three dimensional manifolds, Kleinian groups and hyperbolic geometry.* Bull. A.M.S. (N.S.) **6** (1982), 357–381.
[26] VOGTMANN, K., *On SL_2 of non-euclidean imaginary quadratic integers*, in preparation.

Mathematisches Institut
Universität Bonn
Wegelerstrasse 10
D-5300 Bonn 1

Dept of Mathematics
Columbia University
New York. N.Y. 10027
USA

Received November 3, 1982.