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On the homotopy groups of a finite dimensional space

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The purpose of this note is to prove the following.

THEOREM 1. *Let X be a 1-connected space and p a prime number such that*
(i) $H_n(X;\mathbb{Z}/p)\neq 0$ *for some* $n>0$, *and*
(ii) $H_n(X;\mathbb{Z}/p)=0$ *for all n sufficiently large.*
Then for infinitely many n, $\pi_n X$ contains a subgroup of order p. □

Thirty years ago, J.-P. Serre conjectured such a result for $p=2$ [3, page 219]. He arrived at this conjecture after having proved the 2-primary part of the following result.

THEOREM 2. *Let X and p be as in Theorem 1. Moreover, assume that $H_*(X;\mathbb{Z})$ is of finite type. Then for infinitely many n, $\pi_n X$ contains an element whose order either equals p or is infinite.* □

Serre's proof in this case used, among other things, Poincaré series and methods of analytic number theory. Later Y. Umeda, [5], showed that these methods could be modified to work for odd primes as well.

Notice that Theorem 1 represents an improvement over Theorem 2 in two respects. First, of course, it establishes the existence of torsion in $\pi_* X$ in infinitely many dimensions and, second, it does so without the hypothesis of finite type.

The key ingredient in our proof is the following recent result of Haynes Miller, [1].

THEOREM 3. *Let X and p be as in Theorem 1. Let $B=B\mathbb{Z}/p$, the classifying space for the group $\mathbb{Z}/p$. Then the space of pointed maps from B to X is weakly contractible; that is, $\pi_n(\mathrm{map}_*(B,X))=0$ for all $n\geq 0$.* □

Of course, in this theorem, B may also be regarded as the Eilenberg–MacLane space $K(\mathbb{Z}/p,1)$ or, in the case when $p=2$, as the infinite real projective space RP^∞. We should add that we have not stated Miller's result in its most general form. However, for our purposes the statement above is sufficient.

Theorem 3 indicates a remarkable property of the iterated loop spaces, $\Omega^n X$, of such a space X. In more detail, notice that if $\mathrm{map}_*(B, X)$ is weakly contractible then so is its iterated loop space $\Omega^n(\mathrm{map}_*(B, X))$. This latter space, however, is easily seen to be homeomorphic to $\mathrm{map}_*(B, \Omega^n X)$. Hence Theorem 3 implies that for all $n \geq 0$, the space $\mathrm{map}_*(B, \Omega^n X)$ is weakly contractible, or equivalently, for all $n \geq 0$, every map from B to $\Omega^n X$ is null homotopic.

To begin the proof, let X and p satisfy the hypothesis of Theorem 1. Without loss of generality, we may assume that X has been localized at p. Notice that the conditions on X do not rule out the possibility that some of the groups $\pi_n X$ may contain rational vector spaces.

Our first goal is to establish that for infinitely many n, the mod p homotopy groups $\pi_n(X; \mathbb{Z}/p) \neq 0$. Recall that these groups are defined for $n \geq 2$, as

$$\pi_n(X; \mathbb{Z}/p) = \pi_0(\mathrm{map}_*(S^{n-1} \cup_p e^n, X)).$$

They are related to the ordinary homotopy groups of X by a short exact sequence

$$0 \to \pi_n X \otimes \mathbb{Z}/p \to \pi_n(X; \mathbb{Z}/p) \to \mathrm{Tor}\,(\pi_{n-1} X, \mathbb{Z}/p) \to 0.$$

For more details, see [2].

Suppose that at most a finite number of the mod p homotopy groups of X are nontrivial. Then by condition (i) and the mod p Hurewicz theorem we can choose a largest integer, say m, such that $\pi_m(X; \mathbb{Z}/p) \neq 0$.

What does this supposition imply about the ordinary homotopy groups of X? By the universal coefficient sequence, mentioned earlier, it follows that there are just two possibilities; either

Case 1. $\pi_m X \otimes \mathbb{Z}/p \neq 0$, or

Case 2. $\pi_m X \otimes \mathbb{Z}/p = 0$ and $\mathrm{Tor}\,(\pi_{m-1} X; \mathbb{Z}/p) \neq 0$

Moreover in both cases, if $\pi = \pi_n X$, then $\pi \otimes \mathbb{Z}/p = 0$ if $n > m$ and $\mathrm{Tor}\,(\pi, \mathbb{Z}/p) = 0$ if $n \geq m$.

The second case is the easier to handle. In it we see that $\mathbb{Z}/p$ is a subgroup of $\pi_{m-1} X = \pi_1 \Omega^{m-2} X$. Hence there is an essential map

$$f_1 : K(\mathbb{Z}/p, 1) \to K(\pi_{m-1} X, 1)$$

Consider the obstructions to lifting this map up the Postnikov tower of $\Omega^{m-2}X$ to a map

$$f_\infty: K(\mathbb{Z}/p, 1) \to \Omega^{m-2}X$$

These obstructions take values in $\tilde{H}^*(K(\mathbb{Z}/p, 1); \pi)$ where $\pi = \pi_n X$ and $n > m$. By the universal coefficient theorem for cohomology [4, page 246], these obstruction groups are trivial since $\pi \otimes \mathbb{Z}/p = \mathrm{Tor}(\pi, \mathbb{Z}/p) = 0$.

Hence Case 2 implies the existence of the essential map, f_∞, which in turn contradicts Theorem 3. That leaves us with Case 1. In it, we see that $\mathbb{Z}_{(p)}$ is a subgroup of $\pi_m X = \pi_2 \Omega^{m-2}X$. More precisely we see that there is a monomorphism

$$g: \mathbb{Z}_{(p)} \to \pi_m X$$

which, when tensored with $\mathbb{Z}/p$, is still injective. This, in turn, implies that the following composition

$$g_2: K(\mathbb{Z}/p, 1) \to K(\mathbb{Z}_{(p)}, 2) \to K(\pi_m X, 2)$$

is essential. Here the first map represents a generator of $H^2(K(\mathbb{Z}/p, 1); \mathbb{Z}_{(p)}) = \mathbb{Z}/p$, and the second map is determined by g.

Let $\Omega^{m-2}X\langle 1\rangle$ denote the 1-connective cover of $\Omega^{m-2}X$. The map g_2 can be taken to be a map into the first stage of the Postnikov tower for this cover. The obstructions to lifting g_2 up to a map

$$g_\infty: K(\mathbb{Z}/p, 1) \to \Omega^{m-2}X\langle 1\rangle,$$

are zero for the same reasons as before. Thus g_∞ exists and is essential. The composition of g_∞ with the covering projection back into $\Omega^{m-2}X$ would likewise be essential. Once again we have reached a contradiction of Theorem 3. We therefore conclude that $\pi_n(X; \mathbb{Z}/p) \neq 0$ for infinitely many n. Notice that Theorem 2 is an immediate consequence of this fact.

To complete the proof of Theorem 1, suppose that $\mathrm{Tor}(\pi_n X, \mathbb{Z}/p) \neq 0$ for at most a finite number of n. Then we may choose $m > 0$ large enough so that

(i) $\mathrm{Tor}(\pi_q \Omega^m X, \mathbb{Z}/p) = 0$ for all $q > 0$, and
(ii) $\pi_2 \Omega^m X \otimes \mathbb{Z}/p \neq 0$.

These conditions on $\pi_2\Omega^m X$, in particular, are the same as those in the case just considered. Hence, as before there is a commutative diagram of essential maps

$$\begin{tikzcd}
K(\mathbb{Z}_{(p)}, 2) \arrow[rr, "h"] & & K(\pi_2\Omega^m X, 2) \\
& K(\mathbb{Z}/p, 1) \arrow[ul] \arrow[ur, "j"'] &
\end{tikzcd}$$

This time, however, we will consider the lifting problem for h, rather than working directly with map j.

We want to lift h up through the Postnikov tower for $\Omega^m X\langle 1\rangle$. At the n-th stage this involves the diagram

$$\begin{tikzcd}
& E_{n+1} \arrow[d] & \\
K(\mathbb{Z}_{(p)}, 2) \arrow[ur, dashed, "h_{n+1}"] \arrow[r, "h_n"] & E_n \arrow[r, "k"] & K(\pi', q)
\end{tikzcd}$$

where h_n is some lift of h. As usual, the next lift, h_{n+1}, exists if and only if the composition kh_n is trivial. With this in mind, note that under rationalization the k-invariant (and hence kh_n) is taken to zero. This follows because $\Omega^m X\langle 1\rangle$ is an H-space. On the other hand, since π' is torsion-free, a simple calculation shows that

$$H^*(K(\mathbb{Z}_{(p)}, 2), \pi') \to H^*(K(\mathbb{Z}_{(p)}, 2), \pi' \otimes Q)$$

is injective. We conclude that kh_n must therefore represent the zero class in the first group. Thus kh_n is null homotopic and there is a solution, h_{n+1}, to the lifting problem.

In summary, the map h has been shown to lift to a map into $\Omega^m X\langle 1\rangle$. Composing this lift with maps previously considered we obtain an essential map $K(\mathbb{Z}/p, 1) \to \Omega^m X$. This third and final contradiction of Theorem 3, completes the proof of Theorem 1.

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